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Biperiodic Fibonacci ve Lucas Sayılarını İçeren Gaussian Quaternionlar

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ÖZ

Bu çalışmada, biperiodic Fibonacci ve Lucas sayılarının, biperiodic Fibonacci ve Lucas Gaussian quaternionlar olarak isimlendirilen yeni bir tipi tanımlanmıştır. Çalışma içerisinde, negabiperiodic Fibonacci ve Lucas Gaussian quaternionlarla biperiodic Fibonacci ve Lucas Gaussian quaternionlar arasındaki ilişkiden de bahsedilmiştir. Ayrıca, bu sayılar için Binet formülü, dizinin genelleştirme fonksiyonu, d'Ocagne eşitliği, Catalan eşitliği, Cassini eşitliği, like-Tagiuri eşitliği, Honberger eşitliği ve bazı toplam formülleri verilmiştir. Bi-periodic Fibonacci ve Lucas Gaussian quaternionların bazı cebirsel özellikleri ele alınmıştır.

Gaussian Quaternions Including Biperiodic Fibonacci and Lucas Numbers

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ABSTRACT

In this study, we define a new type of biperiodic Fibonacci and Lucas numbers which are called biperiodic Fibonacci and Lucas Gaussian quaternions. We also give the relationship between negabiperiodic Fibonacci and Lucas Gaussian quaternions and biperiodic Fibonacci and Lucas Gaussian quaternions. Moreover, Binet's formula, generating function, d'Ocagne's identity, Catalan's identity, Cassini's identity, like-Tagiuri's identity, Honberger's identity and some formulas for these new type numbers are obtained. Some algebraic proporties of biperiodic Fibonacci and Lucas Gaussian quaternions which are connected between Gaussian quaternions and biperiodic Fibonacci and Lucas numbers are investigated.

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1. Introduction

Fibonacci sequence has pleased science lovers alike for centuries with their interesting aspect. Pisa has not even guess that the number sequences would be so adventurous with the rabbit problem. The Fibonacci numbers are found in most fields of mathematics. They also occur in variety of other fields such as computer sciences, physics, finance, architecture, geostatics, art, color image processing and music. There have been many works in literature about this special number sequence. There are many generalizations on this sequence some of which can be seen in (Horadam, 1961; Pand, 1968; George, 1969; Pethe and Phadte, 1992; Falcon and Plaza, 2007; Gökbaş, 2021).

The biperiodic Fibonacci sequence is also winthin generalized Fibonacci sequences. This sequence was defined by Edson and Yayenie (Edson and Yayenie, 2009). For $0 \leq n \in \mathbb{N}$ and $0 \neq a, b \in \mathbb{R}$, the bi-periodic Fibonacci sequence is

$$F_n = \begin{cases} aF_{n-1} + F_{n-2}, & \text{if } n \text{ is even}, \quad n \geq 2 \\ bF_{n-1} + F_{n-2}, & \text{if } n \text{ is odd}, \quad n \geq 2 \end{cases}$$

with initial conditions $F_0 = 0,\ F_1 = 1$. They also investigated the generating function for biperiodic Fibonacci sequence as

$$f(t) = \frac{t(1 + at - t^2)}{1 - (ab + 2)t^2 + 4t^4}.$$

From the round-down function definition($\lfloor \alpha \rfloor$), they obtained the Binet's formula of the biperiodic Fibonacci sequence as

$$PF_n = \frac{a^{1-\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\gamma^n - \delta^n}{\gamma - \delta} \right)$$

where $\lfloor \alpha \rfloor$ is the floor function of a, $\varepsilon(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ is the parity function, γ and δ given by

$$\gamma = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}, \delta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}, \gamma + \delta = ab \text{ and } \gamma\delta = -ab$$

are the roots of the equation

$$x^2 - abx - ab = 0.$$

The biperiodic Fibonacci number with negative subscripted is given by

$$PF_{-n} = (-1)^{n+1} PF_n.$$

D'Ocagne's, Catalan's, Cassini's identities and some related summation formulas were given by them (Edson and Yayenie, 2009). Also, (Bilgici, 2014), for any $n \geq 0$ and $0 \neq a, b \in \mathbb{R}$, the biperiodic Lucas sequence is

$$L_n = \begin{cases} bL_{n-1} + L_{n-2}, & \text{if } n \text{ is even}, \quad n \geq 2 \\ aL_{n-1} + L_{n-2}, & \text{if } n \text{ is odd}, \quad n \geq 2 \end{cases}$$

with initial conditions $L_0 = 2, L_1 = a$. The generating function for biperiodic Lucas sequence as

$$l(t) = \frac{at(1 + bt - t^2) + 2(1 - (1 + ab)t^2 + at^3)}{1 - (ab + 2)t^2 + t^4}.$$

The Binet's formula of the biperiodic Lucas sequence as

$$PL_n = \frac{a^{\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} (\gamma^n + \delta^n)$$

where $\lfloor \alpha \rfloor$ is the floor function of a, $\varepsilon(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ is the parity function, γ and δ given by

$$\gamma = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}, \delta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}, \gamma + \delta = ab \text{ and } \gamma\delta = -ab$$

are the roots of the equation

$$x^2 - abx - ab = 0.$$

The biperiodic Lucas number with negative subscripted is given by

$$PL_{-n} = (-1)^n PL_n.$$

In applied and theoretical sciences, quaternions have growing interest and they are also good at representing rotations in three-dimensional space. Also it has applications in areas such as super string theory, projective geometry, topology, and Jordan algebras (Adler, 1994; Ward, 1997; Baez, 2001).

The quaternion algebra

$$Q = \left\{ \sum_{k=0}^{3} a_k e_k : a_k \in \mathbb{R} \right\}$$

is a four dimensional non-commuttive vector space over $\mathbb{R}$ and the basis satisfy the following multiplication rules:

$$e_k^2 = -1, k \in \{1,2,3\},$$

$e_1 e_2 = -e_2 e_1 = e_3$, $e_2 e_3 = -e_3 e_2 = e_1$ and $e_3 e_1 = -e_1 e_3 = e_2$.

e_0 can be identified with real number 1. There are some studies on varied types of sequences over quaternion algebra (Horadam, 1963; Ramirez, 2015; Çimen and İpek, 2016). Harman, called Gaussian numbers, gave an supplementation of Fibonacci numbers into the complex plane and generalized the methods by Horadam (Harman, 1981)

$$GF_n = F_n + iF_{n-1}.$$

2. The Gaussian Quaternions biperiodic Fibonacci and Lucas

In the following sections, the biperiodic Fibonacci and Lucas Gaussian quaternions will be defined. In this section, a variety of algebraic proporties of both the bicomplex quaternions and the biperiodic Fibonacci and Lucas Gaussian quaternions and the negabiperiodic Fibonacci and Lucas Gaussian quaternions are presented in a unified manner. Some identities will be given for biperiodic Fibonacci and Lucas Gaussian quaternions such as Binet's formula, generating function formula, d'Ocagne's, Catalan's, Cassini's, Honsberger's, like-Tagiuri's identities and some formulas.

Definition 1: For $n \geq 3$, the biperiodic Fibonacci Gaussian quaternions PGF_n are defined by the recurrence relation

$$PGF_n = \sum_{k=0}^{3} PF_{n-k} e_k$$

where PF_n is the *nth* biperiodic Fibonacci number. The biperiodic Fibonacci Gaussian quaternions starting from $n = 0$ can be written as

$PGF_0 = 0e_0 + 1e_1 - ae_2 + (1+ab)e_3$, $PGF_1 = 1e_0 + 0e_1 + 1e_2 - ae_3$ and $PGF_2 = ae_0 + 1e_1 + 0e_2 + 1e_3, \ldots$

$$PGF_n = PGF_{n-1} + PGF_{n-2}$$

is a recurrence relationship in biperiodic Fibonacci Gaussian quaternions.

Definition 2: For $n \geq 1$, the negabiperiodic Fibonacci Gaussian quaternions PGF_{-n} are defined by the recurrence relation

$$PGF_{-n} = \sum_{k=0}^{3} (-1)^{n+k+1} PF_{n+k} e_k$$

where PF_n is the *nth* biperiodic Fibonacci number.

$PGF_{-n} = PF_{-n}e_0 + PF_{-n-1}e_1 + PF_{-n-2}e_2 + PF_{-n-3}e_3$

$PGF_{-n} = PF_{-n}e_0 + PF_{-(n+1)}e_1 + PF_{-(n+2)}e_2 + PF_{-(n+3)}e_3$

$PGF_{-n} = (-1)^{n+1}[PF_n e_0 - PF_{n+1}e_1 + PF_{n+2}e_2 - PF_{n+3}e_3]$

is obtained when the equality is arranged.

Definition 3: For $n \geq 3$, the biperiodic Lucas Gaussian quaternions PLF_n are defined by the recurrence relation

$$PGL_n = \sum_{k=0}^{3} PL_{n-k} e_k$$

where PL_n is the *nth* biperiodic Lucas number. The biperiodic Lucas quaternions starting from $n = 0$ can be written as

$PGL_0 = 2e_0 - ae_1 + (2 + ab)e_2 - (3a + a^2b)e_3$, $PGL_1 = ae_0 + 2e_1 - ae_2 + (2 + ab)e_3$ and $PGL_2 = (2 + ab)e_0 + ae_1 + 2e_2 - ae_3, \ldots$

$$PGL_n = PGL_{n-1} + PGL_{n-2}$$

is a recurrence relationship in biperiodic Lucas quaternions.

Definition 4: For $n \geq 1$, the negabiperiodic Lucas Gaussian quaternions PGL_n are defined by the recurrence relation

$$PGL_{-n} = \sum_{k=0}^{3} (-1)^{n+k+1} PL_{n+k} e_k$$

where PL_n is the *nth* biperiodic Lucas number.

$PGL_{-n} = PL_{-n}e_0 + PL_{-n-1}e_1 + PL_{-n-2}e_2 + PL_{-n-3}e_3$

$PGL_{-n} = PL_{-n}e_0 + PL_{-(n+1)}e_1 + PL_{-(n+2)}e_2 + PL_{-(n+3)}e_3$

$PGL_{-n} = (-1)^n[PL_n e_0 - PL_{n+1}e_1 + PL_{n+2}e_2 - PL_{n+3}e_3]$

is obtained when the equality is arranged.

Theorem 5: (Generating Function Formula) Let PGF_n and PGL_n be the biperiodic Fibonacci Gaussian quaternion and Lucas Gaussian quaternion, respectively. Generating function for these numbers is as follows

$$H(t) = \frac{PGF_0 + (PGF_1 - bPGF_0)t + (a-b)M(t)}{(1 - bt - t^2)}$$

where

$$M(t) = th(t)e_0 + t^2 h(t)e_1 + t^3\left(h(t) + \frac{1}{t}\right)e_2 + t^4\left(h(t) + \frac{1}{t}\right)e_3,$$

$$h(t) = \sum_{n=1}^{\infty} PF_{2n-1}t^{2n-1} = \frac{t - t^3}{1 - (ab+2)t^2 + t^4}$$

and PF_n is the *nth* biperiodic Fibonacci number.

$$H(t) = \frac{PGL_0 + (PGL_1 - aPGL_0)t + (b-a)M(t)}{(1 - at - t^2)}$$

where

$$M(t) = th(t)e_0 + t^2 h(t)e_1 + t^3\left(h(t) - \frac{a}{t}\right)e_2 + t^4\left(h(t) - \frac{a}{t}\right)e_3,$$

$$h(t) = \sum_{n=1}^{\infty} PL_{2n-1}t^{2n-1} = \frac{a(t + t^3)}{1 - (ab+2)t^2 + t^4}$$

and PL_n is the *nth* biperiodic Lucas number.

Proof: Let $H(t)$ be the generating function for biperiodic Fibonacci Gaussian quaternions as

$$H(t) = \sum_{n=0}^{\propto} PGF_n t^n.$$

Using $H(t)$, $btH(t)$ and $t^2H(t)$, we get the following equations $btH(t) = \sum_{n=0}^{\propto} bPGF_n t^{n+1}$, $t^2H(t) = \sum_{n=0}^{\propto} PGF_n t^{n+2}$. Since $PF_{2n-1} = bPF_{2n-2} + PF_{2n-3}$, $PF_{2n} = aPF_{2n-1} + PF_{2n-2}$ and $\sum_{n=1}^{\infty} PF_{2n-1}t^{2n-1} = \frac{t-t^3}{1-(ab+2)t^2+t^4}$, After the necessary calculations, the generating function for biperiodic Fibonacci Gaussian quaternions is obtained as

$$\begin{aligned}(1 - bt - t^2)H(t) &= PGF_0 + (PGF_1 - bPGF_0)t + \sum_{n=2}^{\infty}(PGF_n - bPGF_{n-1} - PGF_{n-2})t^n \\ &= PGF_0 + (PGF_1 - bPGF_0)t + t(a-b)(\textstyle\sum_{n=1}^{\infty} f_{2n-1}t^{2n-1})e_0 \\ &\quad + t^2(a-b)(\textstyle\sum_{n=1}^{\infty} f_{2n-1}t^{2n-1})e_1 + t^3(a-b)(\textstyle\sum_{n=0}^{\infty} f_{2n-1}t^{2n-1})e_2 \\ &\quad + t^4(a-b)(\textstyle\sum_{n=0}^{\infty} f_{2n-1}t^{2n-1})e_3.\end{aligned}$$

$$\begin{aligned}(1 - bt - t^2)H(t) &= PGF_0 + (PGF_1 - bPGF_0)t + t(a-b)h(t)e_0 \\ &\quad + t^2(a-b)h(t)e_1 + t^2(a-b)(th(t)+1)e_2 \\ &\quad + t^3(a-b)(th(t)+1)e_3.\end{aligned}$$

$$H(t) = \frac{PGF_0 + (PGF_1 - bPGF_0)t + (a-b)M(t)}{(1 - bt - t^2)}.$$

Similarly, the generating function for biperiodic Lucas Gaussian quaternions is obtained.

Theorem 6: (Binet's Formula) The Binet's formula for the biperiodic Fibonacci Gaussian quaternion and Lucas Gaussian quaternion PGF_n and PGL_n are

$$PGF_n = \begin{cases} \dfrac{\gamma^*\gamma^n - \delta^*\delta^n}{(ab)^{\lfloor \frac{n}{2} \rfloor}(\gamma-\delta)}, n \text{ is even} \\ \dfrac{\gamma^{**}\gamma^n - \delta^{**}\delta^n}{(ab)^{\lfloor \frac{n}{2} \rfloor}(\gamma-\delta)}, n \text{ is odd} \end{cases}$$

and

$$PGL_n = \begin{cases} \dfrac{\gamma^{**}\gamma^n + \delta^{**}\delta^n}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}, n \text{ is even} \\ \dfrac{\gamma^*\gamma^n + \delta^*\delta^n}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}, n \text{ is odd} \end{cases}$$

where

$\gamma^* = \sum_{i=0}^{3}\left(\frac{a^{\varepsilon(i+1)}}{(ab)^{\lfloor \frac{i}{2} \rfloor}}\right)\gamma^{-i}e_i$, $\delta^* = \sum_{i=0}^{3}\left(\frac{a^{\varepsilon(i+1)}}{(ab)^{\lfloor \frac{i}{2} \rfloor}}\right)\delta^{-i}e_i$, $\gamma^{**} = \sum_{i=0}^{3}\left(\frac{a^{\varepsilon(i)}}{(ab)^{\lfloor \frac{i+1}{2} \rfloor}}\right)\gamma^{-i}e_i$ and $\delta^{**} = \sum_{i=0}^{3}\left(\frac{a^{\varepsilon(i)}}{(ab)^{\lfloor \frac{i+1}{2} \rfloor}}\right)\delta^{-i}e_i$.

Proof: By using the definition of the biperiodic Fibonacci Gaussian quaternion and Lucas Gaussian quaternion, we get the desired result.

Theorem 7: (Catalan's Identity) For $0 \leq p, k \in \mathbb{Z}$, with $p \geq k$, we have

$$PGF_{2(p-k)}PGF_{2(p+k)} - PGF_{2p}^2 = \frac{PF_{2k}}{\gamma(\gamma-\delta)(ab)^k}[\gamma^*\delta^*\delta^{2k} - \delta^*\gamma^*\gamma^{2k}]$$

and

$$PGL_{2(p-k)}PGL_{2(p+k)} - PGL_{2p}^2 = \frac{(\gamma-\delta)PF_{2k}}{a(ab)^{2p-k}}[\delta^{**}\gamma^{**}\gamma^{2k} - \gamma^{**}\delta^{**}\delta^{2k}].$$

Proof:

$$PGF_{2(p-k)}PGF_{2(p+k)} - PGF_{2p}^2 = \left(\frac{\gamma^*\gamma^{2(p-k)} - \delta^*\delta^{2(p-k)}}{(ab)^{\lfloor \frac{2(p-k)}{2} \rfloor}(\gamma-\delta)}\right)\left(\frac{\gamma^*\gamma^{2(p+k)} - \delta^*\delta^{2(p+k)}}{(ab)^{\lfloor \frac{2(p+k)}{2} \rfloor}(\gamma-\delta)}\right) - \left(\frac{\gamma^*\gamma^{2p} - \delta^*\delta^{2p}}{(ab)^{\lfloor \frac{2p}{2} \rfloor}(\gamma-\delta)}\right)^2$$

$$= \frac{1}{(ab)^{2p}(\gamma-\delta)^2}[\gamma^*\delta^*((\gamma\delta)^{2p} - \gamma^{2p-2k}\delta^{2p-2k}) + \delta^*\gamma^*((\gamma\delta)^{2p} - \gamma^{2p+2k}\delta^{2p+2k})]$$

$$= \frac{1}{(ab)^{2p}(\gamma-\beta)^2}(\gamma\delta)^{2p}\left[\gamma^*\delta^*\left(1 - \frac{\delta^{2k}}{\gamma^{2k}}\right) + \delta^*\gamma^*\left(1 - \frac{\gamma^{2k}}{\delta^{2k}}\right)\right]$$

$$= \frac{PF_{2k}}{\gamma(\gamma-\delta)(ab)^k}[\gamma^*\delta^*\delta^{2k} - \delta^*\gamma^*\gamma^{2k}].$$

Similarly, we prove that

$$PGL_{2(p-k)}PGL_{2(p+k)} - PGL_{2p}^2 = \frac{(\gamma-\delta)PF_{2k}}{a(ab)^{2p-k}}[\delta^{**}\gamma^{**}\gamma^{2k} - \gamma^{**}\delta^{**}\delta^{2k}].$$

Theorem 8: (Cassini's Identity) For $p \geq 1$, we have

$$PGF_{2(p-1)}PGF_{2(p+1)} - PGF_{2p}^2 = \frac{1}{(\gamma-\delta)(ab)}[\gamma^*\delta^*\delta^2 - \delta^*\gamma^*\gamma^2]$$

and

$$PGL_{2(p-1)}PGL_{2(p+1)} - PGL_{2p}^2 = \frac{(\gamma-\delta)}{(ab)^{2p-1}}[\delta^{**}\gamma^{**}\gamma^2 - \gamma^{**}\delta^{**}\delta^2].$$

Proof: Since Cassini's formula is a special case of Catalan's formula, the proof is seen by taking $k = 1$.

Theorem 9: (d'Ocagne's Identity) For $0 \leq p, k \in \mathbb{Z}$, with $k \geq p$, we have

$$PGF_{2k}PGF_{2(p+1)} - PGF_{2(k+1)}PGF_{2p} = \frac{1}{(ab)^{p+k}(\gamma-\delta)}[\gamma^*\delta^*\gamma^{2k-2p} - \delta^*\gamma^*\delta^{2k-2p}]$$

and

$$PGL_{2k}PGL_{2(p+1)} - PGL_{2(k+1)}PGL_{2p} = \frac{(\gamma-\delta)}{(ab)^{p+k}}[\delta^{**}\gamma^{**}\delta^{2k-2p} - \gamma^{**}\delta^{**}\gamma^{2k-2p}].$$

Proof:

$$\frac{(\gamma^*\gamma^{2k}-\delta^*\delta^{2k})}{(ab)^k(\gamma-\delta)}\frac{(\gamma^*\gamma^{2(p+1)}-\delta^*\delta^{2(p+1)})}{(ab)^{p+1}(\gamma-\delta)} - \frac{(\gamma^*\gamma^{2(k+1)}-\delta^*\delta^{2(k+1)})}{(ab)^{k+1}(\gamma-\delta)}\frac{(\gamma^*\gamma^{2p}-\delta^*\delta^{2p})}{(ab)^p(\gamma-\delta)}$$

$$= \frac{1}{(ab)^{k+p+1}(\gamma-\delta)^2}[\gamma^*\delta^*(\gamma^{2k+2}\delta^{2p} - \gamma^{2k}\delta^{2p+2}) + \delta^*\gamma^*(\gamma^{2p}\delta^{2k+2} - \gamma^{2p+2}\delta^{2k})]$$

$$= \frac{1}{(ab)^{k+p+1}(\gamma-\delta)^2}[\gamma^*\delta^*(\gamma^{2k-2p+2} - \gamma^{2k-2p}\delta^2) + \delta^*\gamma^*(\delta^{2k-2p+2} - \gamma^2\delta^{2k-2p})]$$

$$= \frac{1}{(ab)^{k+p+1}(\gamma-\delta)^2}[\gamma^*\delta^*\gamma^{2k-2p}(\gamma^2-\delta^2) - \delta^*\gamma^*\delta^{2k-2p}(\gamma^2-\delta^2)]$$

$$= \frac{1}{(ab)^{k+p}(\gamma-\delta)}[\gamma^*\delta^*\gamma^{2k-2p} - \delta^*\gamma^*\delta^{2k-2p}].$$

D'Ocagne's formula for the biperiodic Lucas Gaussian quaternion can be demonstrated similarly.

Theorem 10: (like-Tagiuri's Identity) For $p \geq 1$ and nonnegative even integer *p* such that $k \leq p$, we have

$$PGF_{p+k}PGF_{p-k} - PGF_p^2 = \begin{cases} \frac{(-1)^p PF_k(ab)^{\lfloor\frac{k}{2}\rfloor}}{a^{1-\varepsilon(k)}(\delta-\gamma)(-ab)^k}(\gamma^*\delta^*\gamma^k - \delta^*\gamma^*\delta^k), p \text{ is even} \\ \frac{(-1)^p(ab)PF_k(ab)^{\lfloor\frac{k}{2}\rfloor}}{a^{1-\varepsilon(k)}(\delta-\gamma)(-ab)^k}(\gamma^{**}\delta^{**}\gamma^k - \delta^{**}\gamma^{**}\delta^k), p \text{ is odd} \end{cases}$$

and

$$PGL_{p+r}PGL_{p-k} - PGL_p^2 = \begin{cases} \frac{PF_k(ab)^{\lfloor\frac{k}{2}\rfloor-1}(\gamma-\delta)}{a^{1-\varepsilon(k)}(-ab)^k}(\gamma^{**}\delta^{**}\gamma^k - \delta^{**}\gamma^{**}\delta^k), p \text{ is even} \\ \frac{PF_k(ab)^{\lfloor\frac{k}{2}\rfloor-1}(\gamma-\delta)}{a^{1-\varepsilon(k)}(-ab)^{k+1}}(\gamma^*\delta^*\gamma^k - \delta^*\gamma^*\delta^k), p \text{ is odd} \end{cases}$$

Proof: If *p* is even

$$PGF_{p+k}PGF_{p-k} - PGF_p^2 = \left(\frac{\gamma^*\gamma^{p+k} - \delta^*\delta^{p+k}}{(ab)^{\lfloor\frac{p+k}{2}\rfloor}(\gamma-\delta)}\right)\left(\frac{\gamma^*\gamma^{p-k} - \delta^*\delta^{p-k}}{(ab)^{\lfloor\frac{p-k}{2}\rfloor}(\gamma-\delta)}\right) - \left(\frac{\gamma^*\gamma^p - \delta^*\delta^p}{(ab)^{\lfloor\frac{p}{2}\rfloor}(\gamma-\delta)}\right)^2$$

$$= \frac{1}{(ab)^p(\alpha-\delta)^2}(\alpha^*\alpha^{p+k} - \delta^*\delta^{p+k})(\alpha^*\alpha^{p-k} - \delta^*\delta^{p-k}) - (\alpha^*\alpha^p - \delta^*\delta^p)^2$$

$$= \frac{(\gamma\delta)^p}{(ab)^p(\gamma-\delta)^2(\gamma\delta)^k}\left(\gamma^*\delta^*(\frac{\delta^k-\gamma^k}{\delta^k}) - \delta^*\gamma^*\left(\frac{\delta^k-\gamma^k}{\gamma^k}\right)\right)$$

$$= \frac{(-1)^p PF_k (ab)^{\left\lfloor\frac{k}{2}\right\rfloor}}{a^{1-\varepsilon(k)}(\delta-\gamma)(-ab)^k}(\gamma^*\delta^*\gamma^k - \delta^*\gamma^*\delta^k).$$

If p is odd

$$PGF_{p+k}PGF_{p-k} - PGF_p^2 = \left(\frac{\gamma^{**}\gamma^{p+k} - \delta^{**}\delta^{p+k}}{(ab)^{\left\lfloor\frac{p+k}{2}\right\rfloor}(\gamma-\delta)}\right)\left(\frac{\gamma^{**}\gamma^{p-k} - \delta^{**}\delta^{p-k}}{(ab)^{\left\lfloor\frac{p-k}{2}\right\rfloor}(\gamma-\delta)}\right) - \left(\frac{\gamma^{**}\gamma^p - \delta^{**}\delta^p}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor}(\gamma-\delta)}\right)^2$$

$$= \frac{1}{(ab)^{p-1}(\gamma-\delta)^2}(\gamma^{**}\gamma^{p+k} - \delta^{**}\delta^{p+k})(\gamma^{**}\gamma^{p-k} - \delta^{**}\delta^{p-k}) - (\gamma^{**}\gamma^p - \delta^*\delta^p)^2$$

$$= \frac{(\gamma\delta)^p}{(ab)^p(\gamma-\beta)^2(\gamma\delta)^k}\left(\gamma^{**}\delta^{**}(\frac{\delta^k-\gamma^k}{\delta^k}) - \delta^{**}\gamma^{**}\left(\frac{\delta^k-\gamma^k}{\gamma^k}\right)\right)$$

$$= \frac{(-1)^p (ab) PF_k (ab)^{\left\lfloor\frac{k}{2}\right\rfloor}}{a^{1-\varepsilon(k)}(\delta-\gamma)(-ab)^k}(\gamma^{**}\delta^{**}\gamma^k - \delta^{**}\gamma^{**}\delta^k).$$

The other part of the proof is computed following the same way.

Theorem 11: (Honsberger's Identity) For $p \geq 1$ and nonnegative even integer k such that $k \leq p$, we have

$$PGF_pPGF_k + PGF_{p+1}PGF_{k+1}$$

$$= \begin{cases} \dfrac{\gamma^{p+k}(\gamma^{2*} + \gamma^2\gamma^{2**}) + \delta^{p+k}(\delta^{2*} + \delta^2\delta^{2**}) - (-ab)^p(\gamma^*\delta^* + \gamma^{**}\delta^{**})PL_{k-p}}{(\gamma-\delta)^2}, p, k \text{ are even} \\ \dfrac{\gamma^{p+k}(\gamma^{2**} + \gamma^2\gamma^{2*}) + \delta^{p+k}(\delta^{2**} + \delta^2\delta^{2*}) - (-ab)^p(\gamma^*\delta^* + \gamma^{**}\delta^{**})PL_{k-p}}{(\gamma-\delta)^2}, p, k \text{ are odd} \end{cases}$$

$$PGL_pPGL_k + PGL_{p+1}PGL_{k+1}$$

$$= \begin{cases} \dfrac{\gamma^{p+k}(\gamma^{2**} + \gamma^2\gamma^{2*}) + \delta^{p+k}(\delta^{2**} + \delta^2\delta^{2*}) + (-1)^p(\gamma^*\delta^* + \gamma^{**}\delta^{**})PL_{k-p}}{(ab)^{\frac{p+k}{2}}}, p, k \text{ are even} \\ \dfrac{\gamma^{p+k}(\gamma^{2*} + \gamma^2\gamma^{2**}) + \delta^{p+k}(\delta^{2*} + \delta^2\delta^{2**}) - (-1)^p(\gamma^*\delta^* + \gamma^{**}\delta^{**})PL_{k-p}}{(ab)^{\frac{p+k}{2}}}, p, k \text{ are odd} \end{cases}$$

Proof: If p and k are even

$$PGF_pPGF_k + PGF_{p+1}PGF_{k+1}$$

$$= \frac{(\alpha^*\alpha^p - \delta^*\delta^p)}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor}(\alpha-\delta)}\frac{(\alpha^*\alpha^k - \delta^*\delta^k)}{(ab)^{\left\lfloor\frac{k}{2}\right\rfloor}(\alpha-\delta)} + \frac{(\alpha^{**}\alpha^{p+1} - \delta^{**}\delta^{p+1})}{(ab)^{\left\lfloor\frac{p+1}{2}\right\rfloor}(\alpha-\delta)}\frac{(\alpha^{**}\alpha^{k+1} - \delta^{**}\delta^{k+1})}{(ab)^{\left\lfloor\frac{k+1}{2}\right\rfloor}(\alpha-\delta)}$$

$$= \frac{\gamma^{p+k}(\gamma^{2*} + \gamma^2\gamma^{2**}) + \delta^{p+k}(\delta^{2*} + \delta^2\delta^{2**}) - (-1)^p(\gamma^{k-p} + \delta^{k-p})(\gamma^*\delta^* + \gamma^{**}\delta^{**})}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor + \left\lfloor\frac{k}{2}\right\rfloor}(\gamma-\delta)^2}$$

$$=\frac{\gamma^{p+k}(\gamma^{2*}+\gamma^{2}\gamma^{2**})+\delta^{p+k}(\delta^{2*}+\delta^{2}\delta^{2**})-(-ab)^{p}(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})PL_{k-p}}{(\gamma-\delta)^{2}}$$

If *p* and *k* are odd

$$PGF_{p}PGF_{k}+PGF_{p+1}PGF_{k+1}$$

$$=\frac{(\gamma^{**}\gamma^{p}-\delta^{**}\delta^{p})}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor}(\gamma-\delta)}\frac{(\gamma^{**}\gamma^{k}-\delta^{**}\delta^{k})}{(ab)^{\left\lfloor\frac{k}{2}\right\rfloor}(\gamma-\delta)}+\frac{(\gamma^{*}\gamma^{p+1}-\delta^{*}\delta^{p+1})}{(ab)^{\left\lfloor\frac{p+1}{2}\right\rfloor}(\gamma-\delta)}\frac{(\gamma^{*}\gamma^{k+1}-\delta^{*}\delta^{k+1})}{(ab)^{\left\lfloor\frac{k+1}{2}\right\rfloor}(\gamma-\delta)}$$

$$=\frac{\gamma^{p+k}(\gamma^{2**}+\gamma^{2}\gamma^{2*})+\delta^{p+k}(\delta^{2**}+\delta^{2}\delta^{2*})-(-1)^{p}(\gamma^{k-p}+\delta^{k-p})(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor+\left\lfloor\frac{k}{2}\right\rfloor}(\gamma-\delta)^{2}}$$

$$=\frac{\gamma^{p+k}(\gamma^{2**}+\gamma^{2}\gamma^{2*})+\delta^{p+k}(\delta^{2**}+\delta^{2}\delta^{2*})-(-ab)^{p}(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})PL_{k-p}}{(\gamma-\delta)^{2}}$$

If *p* and *k* are even

$$PGL_{p}PGL_{k}+PGL_{p+1}PGL_{k+1}$$

$$=\frac{(\gamma^{**}\gamma^{p}+\delta^{**}\delta^{p})}{(ab)^{\left\lfloor\frac{p+1}{2}\right\rfloor}}\frac{(\gamma^{**}\gamma^{k}+\delta^{**}\delta^{k})}{(ab)^{\left\lfloor\frac{k+1}{2}\right\rfloor}}+\frac{(\gamma^{*}\gamma^{p+1}-\delta^{*}\delta^{p+1})}{(ab)^{\left\lfloor\frac{p+2}{2}\right\rfloor}}\frac{(\gamma^{*}\gamma^{k+1}-\delta^{*}\delta^{k+1})}{(ab)^{\left\lfloor\frac{k+2}{2}\right\rfloor}}$$

$$=\frac{\gamma^{p+k}(\gamma^{2**}+\gamma^{2}\gamma^{2*})+\delta^{p+k}(\delta^{2**}+\delta^{2}\delta^{2*})+(-1)^{p}(\gamma^{-p}+\delta^{k-p})(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor+\left\lfloor\frac{k}{2}\right\rfloor}}$$

$$=\frac{\gamma^{p+k}(\gamma^{2**}+\gamma^{2}\gamma^{2*})+\delta^{p+k}(\delta^{2**}+\delta^{2}\delta^{2*})+(-1)^{p}(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})PL_{k-p}}{(ab)^{\frac{p+k}{2}}}$$

If *p* and *k* are odd

$$PGL_{p}PGL_{k}+PGL_{p+1}PGL_{k+1}$$

$$=\frac{(\gamma^{*}\gamma^{p}-\delta^{*}\delta^{p})}{(ab)^{\left\lfloor\frac{p+1}{2}\right\rfloor}}\frac{(\gamma^{*}\gamma^{k}-\delta^{*}\delta^{k})}{(ab)^{\left\lfloor\frac{p+1}{2}\right\rfloor}}+\frac{(\gamma^{**}\gamma^{p+1}+\delta^{**}\delta^{p+1})}{(ab)^{\left\lfloor\frac{p+2}{2}\right\rfloor}}\frac{(\gamma^{**}\gamma^{k+1}+\delta^{**}\delta^{k+1})}{(ab)^{\left\lfloor\frac{k+2}{2}\right\rfloor}}$$

$$=\frac{\gamma^{p+k}(\gamma^{2*}+\gamma^{2}\gamma^{2**})+\delta^{p+k}(\delta^{2*}+\delta^{2}\delta^{2**})-(-1)^{p}(\gamma^{k-p}+\delta^{k-p})(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})}{(ab)^{\left\lfloor\frac{p}{2}\right\rfloor+\left\lfloor\frac{k}{2}\right\rfloor}}$$

$$=\frac{\gamma^{p+k}(\gamma^{2*}+\gamma^{2}\gamma^{2**})+\delta^{p+k}(\delta^{2*}+\delta^{2}\delta^{2**})-(-1)^{p}(\gamma^{*}\delta^{*}+\gamma^{**}\delta^{**})PL_{k-p}}{(ab)^{\frac{p+k}{2}}}$$

The other part of the proof is computed following the same way.

Theorem 12: The connections between biperiodic Fibonacci Gaussian quaternion PGF_a and biperiodic Lucas Gaussian quaternion PGL_a are as fallows

$$PGF_{a-1}+PGF_{a+1}=PGL_{a}$$
$$PGF_{a+2}-PGF_{a-2}=(\gamma^{2}-\delta^{2})PGL_{a}$$
$$PGL_{a-1}+PGL_{a+1}=(\gamma-\delta)^{2}PGF_{a}$$
$$PGL_{a+2}-PGL_{a-2}=-(\gamma-\delta)^{2}PGF_{a}$$

Proof: If a is even

$$PGF_{a-1}+PGF_{a+1} = \left(\frac{\gamma^*\gamma^{a-1}-\delta^*\delta^{a-1}}{(ab)^{\lfloor\frac{a-1}{2}\rfloor}(\gamma-\delta)}\right)+\left(\frac{\gamma^*\gamma^{a+1}-\delta^*\delta^{a+1}}{(ab)^{\lfloor\frac{a+1}{2}\rfloor}(\gamma-\delta)}\right)$$

$$=\left(\frac{\gamma^*\gamma^{a-1}-\delta^*\delta^{a-1}}{(ab)^{\frac{a}{2}-1}(\gamma-\delta)}\right)+\left(\frac{\gamma^*\gamma^{a+1}-\delta^*\delta^{a+1}}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)$$

$$=\left(\frac{(-\gamma\delta)(\gamma^*\gamma^{a-1}-\delta^*\delta^{a-1})}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)+\left(\frac{\gamma^*\gamma^{a+1}-\delta^*\delta^{a+1}}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)$$

$$=\left(\frac{\gamma^*\gamma^{a}(\gamma-\delta)+\delta^*\delta^{a}(\gamma-\delta)}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)$$

$$=\left(\frac{\gamma^*\gamma^{a}+\delta^*\delta^{a}}{(ab)^{\lfloor\frac{a+1}{2}\rfloor}}\right)=PGL_a$$

If a is odd

$$PGF_{a-1}+PGF_{a+1} = \left(\frac{\gamma^{**}\gamma^{a-1}-\delta^{**}\delta^{a-1}}{(ab)^{\lfloor\frac{a-1}{2}\rfloor}(\gamma-\delta)}\right)+\left(\frac{\gamma^{**}\gamma^{a+1}-\delta^{**}\delta^{a+1}}{(ab)^{\lfloor\frac{a+1}{2}\rfloor}(\gamma-\delta)}\right)$$

$$=\left(\frac{\gamma^{**}\gamma^{a-1}-\delta^{**}\delta^{a-1}}{(ab)^{\frac{a}{2}-1}(\gamma-\delta)}\right)+\left(\frac{\gamma^{**}\gamma^{a+1}-\delta^{**}\delta^{a+1}}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)$$

$$=\left(\frac{(-\gamma\delta)(\gamma^{**}\gamma^{a-1}-\delta^{**}\delta^{a-1})}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)+\left(\frac{\gamma^{**}\gamma^{a+1}-\delta^{**}\delta^{a+1}}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)$$

$$=\left(\frac{\gamma^{**}\gamma^{a}(\gamma-\delta)+\delta^{**}\delta^{a}(\gamma-\delta)}{(ab)^{\frac{a}{2}}(\gamma-\delta)}\right)$$

$$=\left(\frac{\gamma^{**}\gamma^{a}+\delta^{**}\delta^{a}}{(ab)^{\lfloor\frac{a+1}{2}\rfloor}}\right)=PGL_a$$

The other part of the proof is computed following the same way.

3. Conclusion

This study shows the biperiodic Fibonacci and Lucas Gaussian quaternions. We acquire these new quaternions which were not defined in the literature before. We generate Binet's formula, generating function formula and the relationships between for these quaternions. Also we give d'Ocagne's, Catalan's, Cassini's, Honsberger's and like-Tagiuri's identities. Since this study comprises some new outcomes, it conduces to literature by providing requisite information regarding the bicomplex quaternions. The main contribution of this research is that one can get a great number of distinct quaternion sequence by providing the initial values in the bi-periodic Fibonacci and Lucas sequences. For further studies, we plan to find some proporties for the bi-periodic Fibonacci and Lucas quaternions.

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Conflicts of Interest

The author declare no conflict of interest.

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