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On the Algebra of Interval Vectors

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Abstract
In this study, we examine some important subspaces by showing that the set of n-dimensional interval vectors is a quasilinear space. By defining the concept of dimensions in these spaces, we show that the set of n-dimensional interval vectors is actually a (n_r, n_s)-dimensional quasilinear space and any quasilinear space is $(n_r, 0_s)$-dimensional if and only if it is n-dimensional linear space. We also give examples of $(2_r, 0_s)$ and $(0_r, 2_s)$-dimensional subspaces. We define the concept of dimension in a quasilinear space with natural number pairs. Further, we define an inner product on some spaces and talk about them as inner product quasilinear spaces. Further, we show that some of them have Hilbert quasilinear space structure.

Keywords: *Quasilinear space; interval vectors; inner product quasilinear space; Hilbert quasilinear space.*

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1. Introduction

Interval analysis is one of the main areas developed to determine the solutions of many problems in a certain compact interval. Modeling such situations sometimes emerges as a linear interval equation system and the solutions of such systems are often difficult. One of the main studies on the solutions of this type of equations is given by [1]. Another important fundamental work is [2]. Further, linear programming problems with incomplete information also appear as a system of linear interval equations, and some of the important studies on the solution of such problems were given by J.Rohn [3, 4]. The existence of the solution of linear interval equation systems or the determination of the properties of the solution set is also a difficult process, and the results obtained in [5, 6] are also important studies for this purpose. Moreover, [7] is another important work that examines the solubility of equations of this type based on some specific conditions. Since the solutions of such equations appear as interval vectors, it is important to know the properties of n-dimensional interval vectors and the algebraic structure of the set formed by these types of vectors. But, we know that the set of n-dimensional interval vectors is not a vector space. We can see this immediately for 1-dimensional interval vectors. The reverse of the shuffle between intervals may

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not be available. First of all, let's specify that the interval $[0,0]=\{0\}$ is the unit element of the addition operation between intervals. But, we know that it is not possible to find an interval $[\underline{x},\overline{x}]$ such that $[1,0]+[\underline{x},\overline{x}]=[0,0]$. Although, the set of interval vectors does not have a vector space structure, it has an algebraic structure that we call quasilinear space, which is a generalization of vector spaces.

The concept of a quasilinear space on the field of real numbers was first introduced by Aseev in [8]. In this study, the normed quasilinear space and the finite quasilinear operator definitions defined between these types of spaces are also given and some properties are examined. However, in this study, the definition of subspace has not been characterized and there is no such definition as a quasilinear space or a quasi-stretch. Moreover, whether it is a generalization of a definition such as the linear dependence or independence of a subset in quasilinear space is not given in Aseev's pioneering work. In fact, the definition of these concepts is extremely vital for the establishment of a healthy quasilinear algebra. In our [9–12] referenced articles, we tried to eliminate some of these shortcomings in quasilinear algebra. Then we also introduced the concept of inner product in quasilinear spaces, and thus we were able to define the concept of Hilbert quasilinear space definition [13–16]. The introduction of these concepts also provides us with the opportunity to make many applications. For example, in [17, 18] we gave examples of how quasilinear spaces can be used in signal processing. In addition, normed and Hilbert quasilinear space examples of some fuzzy number sets are given in [19] and their properties are examined. A recent study on qasilinear spaces is the concept of quasi-algebra its details can be found in [20, 21].

In this study, we examine some important subspaces by showing that the set of n-dimensional interval vectors is a quasilinear space. By defining the concept of dimensions in these spaces, we show that the set of n-dimensional interval vectors is actually a (n_r,n_s)-dimensional quasilinear space and any quasilinear space is $(n_r,0_s)$-dimensional if and only if it is n-dimensional linear space. We also give examples of $(2_r,0_s)$ and $(0_r,2_s)$-dimensional subspaces. We define the concept of dimension in a quasilinear space with natural number pairs. Further, we define an inner product on some spaces and talk about them as inner product quasilinear spaces.

2. Preliminaries

Let us give basic facts on interval vectors from [22]. The term interval will mean closed interval $x=[\underline{x},\overline{x}]$ in this work and the left and right endpoints of x will be denoted by $\underline{x}$ and $\overline{x}$, respectively. We say that x is *degenerate* if $\underline{x}=\overline{x}$. The *width* of x is defined and denoted by $w(x)=\overline{x}-\underline{x}$ and the *absolute value* of x, denoted $|x|$, is the maximum of the absolute value of its endpoints: $|x|=|[\underline{x},\overline{x}]|=\max\{|\underline{x}|,|\overline{x}|\}$. The *midpoint* of x is given by $m(x)=\frac{1}{2}(\underline{x}+\overline{x})$. By an n-dimensional interval vectors, we mean an ordered n-tuble of intervals

$$x=(x_1,x_2,...x_n).=\left(\left[\underline{x_1},\overline{x_1}\right],...,\left[\underline{x_n},\overline{x_n}\right]\right).$$

For example, a two-dimensional interval vector

$$x=(x_1,x_2)=\left(\left[\underline{x_1},\overline{x_1}\right],\left[\underline{x_2},\overline{x_2}\right]\right)$$

can be represented as a rectangle in the plane. Addition of interval vectors is defined by coordinate-wise addition of intervals and the scalar real multiplication by an interval vectors is also similar. For example; if two-dimensional interval vectors $x=([-1,2],[3,6])$ and $y=([-1,2],[3,6])$ are given, then

$$\begin{aligned}2x-3y&=(2[-1,2],2[3,6])+((-3)[-1,2],(-3)[3,6])\\&=([-2,4],[6,12])+([-6,3],[-18,-9])\\&=([-8,7],[-12,3]).\end{aligned}$$

Note that the set of all n-dimensional interval vectors is not a vector space. $x\preceq y$ iff $x_k\subseteq y_k$ for each $k=1,2,...,n$ is a partial order relation on the set of all n-dimensional interval vectors. The set of all n-dimensional interval vectors is denoted by $\mathbb{I}_{\mathbb{R}}^n$.

The product of two intervals $x=[\underline{x},\overline{x}]$ and $y=[\underline{y},\overline{y}]$ is given by $xy=[\underline{x},\overline{x}][\underline{y},\overline{y}]=[\min S,\max S]$ where $S=\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\}$.

Although we use the term n-dimensional, the algebraic meaning of this term should be questioned, since the set A is not a vector space. However, the set A has an algebraic structure, which we call quasilinear space, which is a generalization of classical vector spaces, first given by Aseev [8]. First, let's give the definition of quasilinear space.

A set X is called a quasilinear space, [8], on the field $\mathbb{K}$ of real or complex numbers, if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for all elements $x,y,z,v\in X$ and all $\alpha,\beta\in\mathbb{K}$:

$$x\preceq x, \tag{2.1}$$

$$x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \tag{2.2}$$

$$x = y \text{ if } x \preceq y \text{ and } y \preceq x, \tag{2.3}$$

$$x + y = y + x, \tag{2.4}$$

$$x + (y + z) = (x + y) + z, \tag{2.5}$$

$$\text{there exists an element (zero) } \theta \in X \text{ such that } x + \theta = x, \tag{2.6}$$

$$\alpha(\beta x) = (\alpha\beta)x, \tag{2.7}$$

$$\alpha(x + y) = \alpha x + \alpha y, \tag{2.8}$$

$$1x = x, \tag{2.9}$$

$$0x = \theta, \tag{2.10}$$

$$(\alpha + \beta)x \preceq \alpha x + \beta x, \tag{2.11}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{2.12}$$

$$\alpha x \preceq \alpha y \text{ if } x \preceq y. \tag{2.13}$$

Any linear space is a QLS with the partial order relation " $=$ ". Perhaps the most popular example of a nonlinear QLS on the field real numbers is $\mathbb{I}^1_{\mathbb{R}}$ with the inclusion relation "$\subseteq$".

Let us record some basic results from [8].

In a QLS X, the element θ is minimal, i.e., $x = \theta$ if $x \preceq \theta$. An element x' is called *inverse* of $x \in X$ if $x + x' = \theta$. The inverse is unique whenever it exists. An element x possessing inverse is called *regular*, otherwise is called *singular*.

Lemma 2.1. *[8] Suppose that each element x in QLS X has inverse element $x' \in X$. Then the partial order in X is determined by equality, the distributivity conditions hold, and consequently X is a linear space.*

In a real linear space, the equality is the only way to define a partial order such that the conditions (1)-(13) hold.

Let us give some assumption in quasilinear spaces. It will be assumed in what follows that $-x = (-1)x$. Note that the additive inverse x' may not be exists but if it exists then $x' = -x$. For example, the interval $[1, 2]$ is a singular element in $\mathbb{I}^1_{\mathbb{R}}$ since the inverse of the element $[1, 2]$ does not exists. However, $-[1, 2] = (-1)[1, 2] = [-2, -1] \in \mathbb{I}^1_{\mathbb{R}}$. Let us give an easy characterization of regular elements. An element x is regular in a QLS if and only if $x' = -x$, or equivalently, $x - x = \theta$. We should note that in a linear QLS, briefly in a linear space, each element is regular. Hence, the notions of regular and singular elements in linear spaces are redundant. Regular elements in $\mathbb{I}^1_{\mathbb{R}}$ is known as degenerate intervals and they are just the real numbers.

Definition 2.1. [10] Suppose that X is a QLS and $Y \subseteq X$. Then Y is called a *subspace of* X whenever Y is a QLS with the same partial order and the restriction to Y of the operations on X.

In [8] the concept of a subspace for a QLS was not defined. After detailed investigations we saw that the characterization of the definition is just the same as in linear subspaces.

Theorem 2.1. *[10] Y is a subspace of a QLS X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in Y$.*

Let Y be a subspace of a QLS X and suppose that each element x in Y has an inverse in Y. Then by Lemma 2.1 the partial order on Y is determined by the equality. In this case Y is a linear subspace of X.

An element x in X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. In a linear QLS, equivalently, in a linear space zero is the only symmetric element. X_r and X_s stand for the set of all regular and singular elements with zero in X, respectively. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular, symmetric* and *singular subspaces* of X, respectively. Regular subspace of X is a linear space while the singular subspace is a nonlinear QLS. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element.

3. Main results

Theorem 3.1. *$\mathbb{I}_{\mathbb{R}}^{n}$ is a quasilinear space by the partial order relation $x \preceq y$ iff $\left[\underline{x_k}, \overline{x_k}\right] \subseteq \left[\underline{y_k}, \overline{y_k}\right]$ for each $k = 1, 2, ..., n$.*

Proof. Most of the proof comes from the known result in interval analysis, see [22]. Let us only verify two axioms. The zero is $\theta = (0, 0, ...0). = ([0, 0], ..., [0, 0])$ in $\mathbb{I}_{\mathbb{R}}^{n}$ and if $x, y \in \mathbb{I}_{\mathbb{R}}^{n}$ and $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned} (\alpha+\beta)x &= \left((\alpha+\beta)\left[\underline{x_1}, \overline{x_1}\right], ..., (\alpha+\beta)\left[\underline{x_n}, \overline{x_n}\right]\right) \\ &\preceq \left(\alpha\left[\underline{x_1}, \overline{x_1}\right], ..., \alpha\left[\underline{x_n}, \overline{x_n}\right]\right) + \left(\beta\left[\underline{x_1}, \overline{x_1}\right], ..., \beta\left[\underline{x_n}, \overline{x_n}\right]\right) \\ &= \alpha x + \beta x. \end{aligned}$$

Further, $x \preceq y$ means $\left[\underline{x_k}, \overline{x_k}\right] \subseteq \left[\underline{y_k}, \overline{y_k}\right]$ for each k and hence for every (positive or negative) $\alpha \in \mathbb{R}$, $\alpha\left[\underline{x_k}, \overline{x_k}\right] \subseteq \alpha\left[\underline{y_k}, \overline{y_k}\right]$. This implies $\alpha x \preceq \alpha y$. □

Example 3.1. The symmetric subspace of $\mathbb{I}_{\mathbb{R}}^{2}$ is $\left(\mathbb{I}_{\mathbb{R}}^{2}\right)_{sym} = \{([-a, a], [-b, b]) : a, b \in \mathbb{R}\}$. Further, the singular subspace of $\mathbb{I}_{\mathbb{R}}^{2}$ is just

$$\left(\mathbb{I}_{\mathbb{R}}^{2}\right)_{s} = \left\{\left(\left[\underline{x_1}, \overline{x_1}\right], \left[\underline{x_2}, \overline{x_2}\right]\right) : \underline{x_1} \neq \overline{x_1} \text{ or } \underline{x_2} \neq \overline{x_2}\right\} \cup \{([0, 0], [0, 0])\}$$

and the regular subspace is

$$\left(\mathbb{I}_{\mathbb{R}}^{2}\right)_{r} = \{([a, a], [b, b]) : a, b \in \mathbb{R}\} \equiv \{(\{a\}, \{b\}) : a, b \in \mathbb{R}\} \equiv \mathbb{R}^2.$$

Thus, we can see $\mathbb{R}^2$ as a regular subspace of $\mathbb{I}_{\mathbb{R}}^{2}$. The equivalence mentioned here means that there is a linear bijection and even an isometry when the normed space structure is introduced between these spaces. In general, $\mathbb{I}_{\mathbb{R}}^{n}$ has these special subspaces and we can see $\mathbb{R}^n \equiv \left(\mathbb{I}_{\mathbb{R}}^{n}\right)_r$ is a linear part of $\mathbb{I}_{\mathbb{R}}^{n}$.

Definition 3.1. [8] In a QLS X, a real function $\|.\|_X : X \longrightarrow \mathbb{R}$ is called a *norm* if the following conditions hold:

$$\|x\|_X > 0 \text{ if } x \neq 0, \tag{3.1}$$

$$\|x + y\|_X \leq \|x\|_X + \|y\|_X, \tag{3.2}$$

$$\|\alpha x\|_X = |\alpha| \, \|x\|_X, \tag{3.3}$$

$$\text{if } x \preceq y, \text{ then } \|x\|_X \leq \|y\|_X, \tag{3.4}$$

$$\begin{gathered} \text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \\ x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\|_X \leq \varepsilon \text{ then } x \preceq y. \end{gathered} \tag{3.5}$$

A quasilinear space X with a norm defined on it, is called *normed quasilinear* space (briefly, *normed QLS*). It follows from Lemma 2 that if any $x \in X$ has an inverse element $x' \in X$ then the concept of normed QLS coincides with the concept of real normed linear space. Hausdorff metric or norm metric on X is defined by the equality

$$h_X(x, y) = \inf\left\{r \geq 0 : x \preceq y + a_1^r, y \preceq x + a_2^r \text{ and } \|a_i^r\| \leq r, i = 1, 2\right\}.$$

Since $x \preceq y + (x - y)$ and $y \preceq x + (y - x)$, the quantity $h_X(x, y)$ is well-defined for any elements $x, y \in X$, and the function h_X satisfies all axioms of the metric. Further, $h_X(x, y)$ may not equal to $\|x - y\|_X$ if X is not a linear space, but always $h_X(x, y) \leq \|x - y\|_X$ for every $x, y \in X$ [8].

Example 3.2. A norm on $\mathbb{I}_{\mathbb{R}}^{n}$ is defined by

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k| = \max_k \left\{\max\left\{\left|\underline{x_k}\right|, \left|\overline{x_k}\right|\right\}\right\}$$

where $k \in \{1, 2, ..., n\}$ and $|x_k|$ is the absolute value of the interval x_k. Another important norm on $\mathbb{I}_{\mathbb{R}}^{n}$ is

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} = \left(\sum_{k=1}^{n} \left\{\max\left\{\left|\underline{x_k}\right|, \left|\overline{x_k}\right|\right\}\right\}^2\right)^{1/2}$$

which is perhaps the most important one. This norm is the classical norm of $\mathbb{I}_{\mathbb{R}}^n$. To prove $\|.\|_2$ is a norm on $\mathbb{I}_{\mathbb{R}}^n$ let us only verify the last condition. Let $\varepsilon > 0$ be given and let $x = (x_1, x_2, ...x_n). = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right)$ and $y = (y_1, y_2, ...y_n). = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right) \in \mathbb{I}_{\mathbb{R}}^n$. Assume that there exists an element $x_\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, ..., x_n^\varepsilon). = \left(\left[\underline{x_1^\varepsilon}, \overline{x_1^\varepsilon}\right], ..., \left[\underline{x_n^\varepsilon}, \overline{x_n^\varepsilon}\right]\right) \in \mathbb{I}_{\mathbb{R}}^n$ such that $x \preceq y + x_\varepsilon$ and $\|x_\varepsilon\|_2 = \left(\sum_{k=1}^n |x_k^\varepsilon|^2\right)^{1/2} \leq \varepsilon$. This implies, for each $k \in \{1, 2, ..., n\}$, $\left[\underline{x_k}, \overline{x_k}\right] \subseteq \left[\underline{y_k}, \overline{y_k}\right] + \left[\underline{x_k^\varepsilon}, \overline{x_k^\varepsilon}\right]$ and $|x_k^\varepsilon| = \max\left\{\left|\underline{x_k^\varepsilon}\right|, \left|\overline{x_k^\varepsilon}\right|\right\} \leq \varepsilon$. Now for $\varepsilon \to 0$ we get $|x_k^\varepsilon| \to 0$ for each $k \in \{1, 2, ..., n\}$ and this means $\|x_\varepsilon\|_2 \to 0$ and hence $x_\varepsilon \to 0$ in $\mathbb{I}_{\mathbb{R}}^n$. Eventually, we get $x \preceq y$.

4. Quasilinear independence and basis

In this section, we will give some algebraic definitions [9, 11]. Let X be a QLS and $\{x_k\}_{k=1}^n$ be a subset of X where n is a positive integer. *A (linear) combination of the set* $\{x_k\}_{k=1}^n$ is an element z of X in the form

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = z$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ are real scalars. On the other hand, *a* quasilinear *combination of the set* $\{x_k\}_{k=1}^n$ is an element $z \in X$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \preceq z$$

for some real scalars $\alpha_1, \alpha_2, ..., \alpha_n$. Hence, the quasilinear combination, briefly ql-combination, is defined by the partial order relation on X. In fact, the definition of linear combination in a QLS is also depend on the partial order relation and it can be defined as in the following form; a linear combination of the set $\{x_k\}_{k=1}^n$ is an element z of X such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \preceq z \text{ and } z \preceq \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n,$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ are real scalars. In a linear QLS, this is the definition of the classical linear combination since the relation "$\preceq$" turns to the relation "=". Clearly, a linear combination of $\{x_k\}_{k=1}^n$, is a quasilinear combination of $\{x_k\}_{k=1}^n$, but not conversely. For any nonempty subset A of a QLS X, we know that the *span* of A is written by SpA and

$$SpA = \{\sum_{k=1}^n \alpha_k x_k : x_1, x_2, ..., x_n \in A,\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R},\ n \in \mathbb{N}\}.$$

However, $QspA$, the *quasispan (q-span*, for short) *of* A, is defined by the set of all possible quasilinear combinations of A, that is,

$$\begin{aligned} QspA &= \{x \in X : \sum_{k=1}^n \alpha_k x_k \preceq x, \\ \text{for some } x_1, x_2, ..., x_n &\in A \text{ and for some scalars } \alpha_1, \alpha_2, ..., \alpha_n\}. \end{aligned}$$

Obviously, $SpA \subseteq QspA$. Further, $SpA = QspA$ for some linear QLS (linear space), hence, the notion of $QspA$ is redundant in linear spaces. Moreover, we say A *quasi spans* X whenever $QspA = X$.

Let us give an example from the quasilinear space of compact intervals.

Example 4.1. Let $X = \mathbb{I}_{\mathbb{R}}^1$ and take $A = \{[1, 3]\}$, a singleton in X. The q-span of A is

$$QspA = \{x \in \mathbb{I}_{\mathbb{R}}^1 : \lambda [1, 3] \subseteq x,\ \lambda \in \mathbb{R}\}.$$

For example, $[2, 7] \in QspA$ since $2 [1, 3] \subseteq [2, 7]$ whereas $[2, 7] \notin SpA$ since there is no $\lambda \in \mathbb{R}$ satisfying $\lambda [1, 3] = [2, 7]$. Further, $[2, 3] \notin QspA$ since we cannot find any $\lambda \in \mathbb{R}$ satisfying the condition $\lambda [1, 3] \subseteq [2, 3]$. Clearly, $QspA \neq \mathbb{I}_{\mathbb{R}}^1$ Let $B = \{\{1\}\}$, another singleton in X. It consist of a regular element or degenerate interval. For any $x \in X$, clearly, we can write $\lambda.\{1\} \subseteq x$ for some $\lambda \in \mathbb{R}$. This means $QspB = X$. It can be easily shown that a singleton arising from nonzero regular element can quasispans X. A singular element cannot quasi spans X.

Theorem 4.1. *Let $A = \{x_1, x_2, ..., x_n\}$ be a subset of the QLS X. Then $QspA$ is a subspace of X.*

Definition 4.1. (*Quasilinear independence* and *dependence)* A set $A = \{x_1, x_2, ..., x_n\}$ in a QLS X is called *quasilinear independent (briefly ql-independent*) whenever the inequality

$$\theta \preceq \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \tag{4.1}$$

holds if and only if $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. Otherwise, A is called *quasilinear dependent (briefly ql-dependent).*

If we recall again that every linear space is a QLS with the relation "=", it can be seen that the notions of quasilinear independence and dependence coincide with linear independence and dependence.

Example 4.2. Consider the singleton $A = \{[1,2]\}$ in $\mathbb{I}^1_{\mathbb{R}}$. It is obvious that $\{0\} = [0,0] \subseteq \alpha[1,2]$ if and only if $\alpha = 0$ where $\{0\}$ is the zero element of $\mathbb{I}^1_{\mathbb{R}}$. Therefore, A is ql-independent. However, the singleton $B = \{[-1,2]\}$ is ql-dependent since $[0,0] \subseteq \beta[-1,2]$ for $\beta = 2 \neq 0$. This is a unusual case since a non-zero singleton is obviously linear independent in linear spaces. On the other hand, the set $\{[1,2], [-1,2]\}$ is ql-dependent. In general, we can see from the definition that any subset including an element related to zero must be ql-dependent in a QLS. This is a generalization of the well-known fact that a subset including zero must be linear independent in linear spaces.

Example 4.3. In $\mathbb{I}^2_{\mathbb{R}}$, let $v_1 = ([-2,1], [0,0])$ and $v_2 = ([0,0], [-2,3])$. Then the set $\{v_1, v_2\}$ is ql-dependent since

$$([0,0], [0,0]) \subseteq \lambda_1 v_1 + \lambda_2 v_2 = ([-2,1], [-2,3])$$

for $\lambda_1 = \lambda_2 = 1$ where $([0,0], [0,0])$ is the zeros of $\mathbb{I}^2_{\mathbb{R}}$. However, $\{u_1, u_2\}$ is ql-independent where $u_1 = ([-2,-1], [0,0])$ and $u_2 = ([0,0], [2,3])$. On the other hand, let $u = ([-2,2], [-3,3])$ then the singleton $\{u\}$ is ql-dependent in $\mathbb{I}^2_{\mathbb{R}}$ since $([0,0], [0,0]) \subseteq u$.

Definition 4.2. A ql-independent subset A of a QLS X which quasi spans X is called *a basis (or Hamel basis)* for X.

Remark 4.1. For any $a \in \mathbb{R}$, the singleton $\{\{a\}\}$ is a basis for $\mathbb{I}^1_{\mathbb{R}}$. Further, $B = \{([1,1], [0,0]), ([0,0], [1,1])\}$ is a basis for $\mathbb{I}^2_{\mathbb{R}}$. In general, $B = \{([1,1], [0,0], ..., [0,0]), ..., ([0,0], [0,0], ..., [1,1])\}$ is a basis for $\mathbb{I}^n_{\mathbb{R}}$. As can be seen, a basis of $\mathbb{I}^n_{\mathbb{R}}$ is a set of degenerate intervals of $\mathbb{I}^n_{\mathbb{R}}$.

Following example is extraordinary since it presents an example of QLS which has no basis. This is an unusual case since all linear spaces have a (Hamel) basis.

Example 4.4. Let us consider singular subspace

$$\{\{0\}\} \cup \{[a,b] : a < b \text{ and } a, b \in \mathbb{R}\} = (\mathbb{I}^1_{\mathbb{R}})_s$$

of $\mathbb{I}^1_{\mathbb{R}}$. This quasilinear space has no basis. Any singleton $\{[a,b]\}$ in $(\mathbb{I}^1_{\mathbb{R}}))_s$ cannot quasi spans $(\mathbb{I}^1_{\mathbb{R}})_s$ where $a < b$.

Now let us introduce the notion of dimension of a QLS. Our investigation shows that it is necessary to split it into two different notion as regular and singular dimension. Previously, let us give analog of a classical definition.

Definition 4.3. Let S be a ql-independent subset of the QLS X. S is called *maximal ql-independent subset* of X whenever S is ql-independent, but any superset of S is ql-dependent.

Definition 4.4. *Regular (Singular) dimension* of any QLS X is the cardinality of any maximal ql-independent subsets of $X_r(X_s)$. If this number is finite then X is said to be finite regular (singular)-dimensional, otherwise; is said to be infinite regular (singular)-dimensional. Regular dimension is denoted by $r\text{-}\dim X$ and singular dimension is denoted by $s\text{-}\dim X$. If $r\text{-}\dim X = a$ and $s\text{-}\dim X = b$ then we say that X *is an* (a_r, b_s)*-dimensional QLS* where a and b are natural numbers or ∞.

Remark 4.2. The above definition means that $r\text{-}\dim X$ is classical definition of dimension of the linear space X_r. So, $r\text{-}\dim X = \dim X_r$. Notice that a non-trivial singular subspace of a QLS cannot be a linear space. Further, we can easily see that any QLS is $(n_r, 0_s)$-dimensional if and only if it is n-dimensional linear space. In this respect, the trivial linear space $\{0\}$ is a $(0_r, 0_s)$-dimensional QLS. Later, we will give an example of a $(0_r, 0_s)$-dimensional QLS other than $\{0\}$.

Let us determine dimensions of some nonlinear QLSs.

Example 4.5. It isn't hard to prove that $\mathbb{I}^n_{\mathbb{R}}$ is (n_r, n_s)-dimensional QLSs, that is, n-dimensional nonlinear QLS. Consider again the singular subspace $(\mathbb{I}^1_{\mathbb{R}})_s$ of $\mathbb{I}^1_{\mathbb{R}}$. $r - \dim (\mathbb{I}^1_{\mathbb{R}})_s = 0$ since $((\mathbb{I}^1_{\mathbb{R}})_s)_r = \{0\}$. Further, $\{[1,2]\}$ is ql-independent in $((\mathbb{I}^1_{\mathbb{R}})_s)_s$ and so $s - \dim (\mathbb{I}^1_{\mathbb{R}})_s = 1$. Hence, $(\mathbb{I}^1_{\mathbb{R}})_s$ is $(0_r, 1_s)$ $-$dimensional. Obviously, $\left(\mathbb{I}^1_{\mathbb{R}}\right)_r$ is $(1_r, 0_s)$ $-$dimensional. In this respect, $\mathbb{R}$ is also $(1_r, 0_s)$ $-$dimensional

If $X = (\mathbb{I}^2_{\mathbb{R}})_s \cup \{([t,t], [0,0]) : t \in \mathbb{R}\}$ then X is a subspace of $\mathbb{I}^2_{\mathbb{R}}$ and $r - \dim X = 1$ since $X_r = \{([t,t], [0,0]) : t \in \mathbb{R}\}$. Further, the set $\{u_1, u_2\}$ in Example 4.3 is ql-independent. This proves $s - \dim X = 2$. Hence X is a $(1_r, 2_s)$ $-$dimensional QLS.

Consider the QLS $X=\Omega_C(c_0)$, the set of all closed bounded subsets of the Banach space c_0. Regular subspace X_r is equivalent to c_0, the linear space of all sequences convergent to zero, and so $r-\dim X=\infty$. Let us define the set

$$\begin{aligned}\{\{(t,0,0,...) &: 1\le t\le 4\},\{(0,t,0,...):1\le t\le 4\},...\}\\ &= \{[1,4]\odot e_1,[1,4]\odot e_2,...\}\end{aligned}$$

where

$$[1,4]\odot e_k=\{(0,...,0,\overset{k.\ term}{s},0...):s\in[1,4]\}$$

is ql-independent in X_s, where e_k's are coordinate vectors of c_0, $k=1,2,...$.Therefore, $s-\dim X=\infty$ and so $X=\Omega_C(c_0)$ is an $(\infty_r:\infty_s)$-dimensional QLS. In general, an infinite-dimensional linear space E is a $(\infty_r,0_s)$-dimensional QLS while $\Omega_C(E)$ is (∞_r,∞_s)-dimensional QLS.

In a finite dimensional linear space X let us recall that each $x\in X$ has a unique representation

$$x=\sum_{k=1}^{n}a_kb_k$$

where n is the dimension of X, $B=\{b_1,b_2,...,b_n\}$ is a basis of X and $a_1,a_2,...,a_n$ are corresponding scalars. Since a consolidate QLS has a basis we can give a similar representation. Let X be a $(n_r:n_s)-$dimensional (finite-dimensional) QLS where n_r and n_s are positive integers, and $n_r=n_s$. Let us try to give a representation in X. If y is any element of X then the *floor* $F_y=\{x\in X_r:x\preceq y\}$ of y have many regular elements. From linear algebra any $x\in F_y$ has a unique representation

$$x=\sum_{k=1}^{n}\alpha_k^x b_k$$

where each α_k^x, $k=1,2,...,n$, is a real scalar depending on x. Now let us consider the supremum with respect to the partial order relation "$\preceq$" on the QLS X. Thus, by the definition of consolidate space, we get the representation

$$y=\sup\{x\in X_r:x\preceq y\}=\sup\{\sum_{k=1}^{n}\alpha_k^x b_k:x\preceq y,\ x\in X_r\}$$

of each element y in X. That is, any element of a (nonlinear) consolidate QLS can be represented by the basis elements and by the supremum with respect to "$\preceq$". More practically, we can write

$$y=\sup_{\substack{x\preceq y\\ x\in X_r}}\sum_{k=1}^{n}\alpha_k^x b_k. \tag{4.2}$$

Theorem 4.2. *Any $y\in\mathbb{I}_{\mathbb{R}}^n$ has a unique representation*

$$y=\sup_{\substack{x\preceq y\\ x\in X_r}}\sum_{k=1}^{n}\alpha_k^x b_k$$

where $B=\{b_k\}_{k=1}^n=\{([1,1],[0,0],...,[0,0]),...,([0,0],[0,0],...,[1,1])\}$ is the standard basis of $\mathbb{I}_{\mathbb{R}}^n$ and the supremum is calculated by the partial order "$\preceq$" on $\mathbb{I}_{\mathbb{R}}^n$.

Proof. Let us first write y explicitly;

$$y=(y_1,y_2,...,y_n)=\left(\left[\underline{y_1},\overline{y_1}\right],...,\left[\underline{y_n},\overline{y_n}\right]\right).$$

Now take an arbitrary $t_k\in y_k$ and constitute the degenerate interval $[t_k,t_k]$. Obviously, $[t_k,t_k]\subseteq y_k$ for each k and hence, $([t_k,t_k])\preceq(y_k)=y$. Since $t=(t_k)\in\mathbb{R}^n$ has a unique representation $t=\sum\limits_{k=1}^{n}t_ke_k$, we can say

$([t_k, t_k]) := x \in (\mathbb{I}_{\mathbb{R}}^n)_r$ has the unique representation $x = ([t_k, t_k]) = \sum_{k=1}^{n} t_k b_k$. On the other hand, for any y_k there may be a lot of $t_k \in y_k$. In other words, there may be a lot of $[t_k, t_k] \subseteq y_k$. You can easily see that, for each k,

$$\begin{aligned} y_k &= \sup_{\subseteq} \{[t_k, t_k] : [t_k, t_k] \subseteq y_k\} \\ &= \sup_{\subseteq} \left\{ \sum_{k=1}^{n} t_k b_k : [t_k, t_k] \subseteq y_k \right\}. \end{aligned}$$

This representation is still unique by the properties of the suprema by the partial order relation $\subseteq$. Thus, we get

$$\begin{aligned} y &= (y_k)_{k=1}^n = \left(\sup_{\subseteq} \left\{ \sum_{k=1}^{n} t_k b_k : [t_k, t_k] \subseteq y_k \right\} \right)_{k=1}^n \\ &= \sup_{\preceq} \left\{ \sum_{k=1}^{n} t_k^x b_k : ([t_k, t_k])_{k=1}^n = x \preceq y = (y_k)_{k=1}^n \right\}. \end{aligned}$$

The last supremum, of course, is taken over "$\preceq$" relation on $\mathbb{I}_{\mathbb{R}}^n$ and the representation is obviously unique. $\square$

The representation is also known as the *super position* of y in $\mathbb{I}_{\mathbb{R}}^n$.

Example 4.6. Let us give the super position of $y = ([-1, 3], [2, 2])$ in $\mathbb{I}_{\mathbb{R}}^2$ where $y_1 = [-1, 3]$ and $y_2 = [2, 2]$. By the discussion in the proof

$$y_1 = [-1, 3] = \sup_{\subseteq} \{[t, t] : [t, t] \subseteq y_1\} = \sup_{\subseteq} \{[t, t] : t \in [-1, 3]\}$$

and

$$\begin{aligned} y_2 &= [2, 2] = \sup_{\subseteq} \{[t, t] : [t, t] \subseteq y_2\} = \sup_{\subseteq} \{[t, t] : t \in [2, 2]\} \\ &= \sup_{\subseteq} \{[2, 2]\} = [2, 2] . \end{aligned}$$

Hence,

$$\begin{aligned} y &= ([-1, 3], [2, 2]) = \left(\sup_{\subseteq} \left\{ \sum_{k=1}^{2} t_k b_k : [t_k, t_k] \subseteq y_k \right\} \right)_{k=1}^2 \\ &= \sup_{\subseteq} \{t_1 ([1, 1], [0, 0]) + t_2 ([0, 0], [1, 1]) : [t_k, t_k] \subseteq y_k, k = 1, 2\} \\ &= \sup_{\preceq} \left\{ \begin{array}{c} [t_1, t_1] ([1, 1], [0, 0]) + [t_2, t_2] ([0, 0], [1, 1]) \\ : x = ([t_1, t_1], [t_2, t_2]) \preceq y \end{array} \right\}. \end{aligned}$$

Definition 4.5. A quasilinear space X is called *consolidate (solid-floored) QLS* whenever $y = \sup\{x \in X_r : x \preceq y\}$ for each $y \in X$. Otherwise, X is called *a non-consolidate QLS*, briefly, *nc-QLS*.

The supremum in this definition is taken on the order relation "$\preceq$" in the definition of a QLS. Above definition assumes $\sup\{x \in X_r : x \preceq y\}$ exists for each $y \in X$. Implicitly, we say that X is consolidate if and only if $y = \sup F_y$, for each $y \in X$.

We signify that any linear space is a consolidate QLS: Indeed, $X_r = X$ for any linear space X and so

$$y = \sup\{x \in X_r : x \preceq y\} = \sup\{x \in X_r : x = y\} = \sup\{y\} = y$$

for any element y in X.

Example 4.7. $\mathbb{I}_{\mathbb{R}}^n$ is a consolidate QLS. Singular subspace of $\mathbb{I}_{\mathbb{R}}^1$ is a nc-QLS since $F_y = \emptyset$ for the element $y = [1, 2]$ in $(\mathbb{I}_{\mathbb{R}}^1)_s$. Further,

$$\mathcal{B} = \{[a, b] : a \leq 0 \leq b, \;\; a, b, 0 \in \mathbb{R}\}$$

is another nc-subspace of $\mathbb{I}_{\mathbb{R}}^1$. $(\mathbb{I}_{\mathbb{R}}^1)_{sym}$ is also a nc-QLS subspace of $\mathcal{B}$ and of $\mathbb{I}_{\mathbb{R}}^1$.

Definition 4.6. For two quasilinear spaces $(X, \leq)$ and $(Y, \preceq)$, Y is called compatible contains X whenever $X \subseteq Y$ and the partial order relation $\leq$ on X is the restriction of the partial order relation $\preceq$ on Y. We briefly use the symbol $X \overline{\subseteq} Y$ in this case. We write $X \equiv Y$ whenever $X \overline{\subseteq} Y$ and $Y \overline{\subseteq} X$.

Remark 4.3. Hence $X \equiv Y$ means X and Y are the same sets with the same partial order relations which make one each quasilinear space. However, we may write $X = Y$ for $X \equiv Y$ whenever the relations are clear from context.

Definition 4.7. Let X be a QLS. *Consolidation of* X is the smallest consolidate QLS $\widehat{X}$ which compatible contains X, that is, if there exists another consolidate QLS Y which compatible contains X then $\widehat{X} \subseteq Y$.

Clearly, $\widehat{X} = X$ for some consolidate QLS X. Whether each QLS has a consolidation is not know yet. This notion is unnecessary for consolidate QLSs, hence is in linear spaces.

Theorem 4.3. *Consolidation of* $(\mathbb{I}_{\mathbb{R}}^1)_s$ *is* $\mathbb{I}_{\mathbb{R}}^1$.

Proof. Obviously, $\mathbb{I}_{\mathbb{R}}^1$ compatible contains $(\mathbb{I}_{\mathbb{R}}^1)_s$. Suppose that Z is another consolidate QLS containing $(\mathbb{I}_{\mathbb{R}}^1)_s$. For an arbitrary element x of $\mathbb{I}_{\mathbb{R}}^1$ we will show that $x \in Z$. If $x \in (\mathbb{I}_{\mathbb{R}}^1)_s$ then the proof is clear. If $x \notin (\mathbb{I}_{\mathbb{R}}^1)_s$ then x have to be a degenerate interval that is an element of $(\mathbb{I}_{\mathbb{R}}^1)_r$. Hence, $x = [a, a]$ for an $a \in \mathbb{R}$. Assume that $[a, a] \notin Z$. For any $\varepsilon > 0$ we have that $[a - \varepsilon, a + \varepsilon] \in (\mathbb{I}_{\mathbb{R}}^1)_s$ and so $[a - \varepsilon, a + \varepsilon] \in Z$. Since Z is consolidate,

$$[a - \varepsilon, a + \varepsilon] = \sup\{y \subseteq [a - \varepsilon, a + \varepsilon] : y \in Z_r\}$$

for any $\varepsilon > 0$. This means there exists an element $u_\varepsilon \in Z_r$ such that $u_\varepsilon \subseteq [a - \varepsilon, a + \varepsilon]$ in Z. Therefore, we have $[a, a] \in Z_r$, otherwise; the set $[a - \varepsilon, a + \varepsilon]$ cannot be a closed set in $\mathbb{R}$ and so this conflicts with the fact that $[a - \varepsilon, a + \varepsilon] \in (\mathbb{I}_{\mathbb{R}}^1)_s$. Thus, the assumption $[a, a] \notin Z$ is incorrect. □

For any element y of a QLS X, the set

$$F_y^{\widehat{X}} = \left\{z \in \left(\widehat{X}\right)_r : z \preceq y\right\}$$

denotes the floor of y in $\widehat{X}$ and sometimes $F_y^{\widehat{X}}$ is said to be the *floor of* y *in the consolidation.* For a consolidate QLS, this notion is unnecessary. But the concept is important in a nc-QLS, especially, in producing of an inner-product on a QLS.

Definition 4.8. Let X be a quasilinear space having a consolidation $\widehat{X}$. A mapping $\langle\ ,\ \rangle : X \times X \to \Omega(\mathbb{K})$ is called an inner product on X if for any $x, y, z \in X$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied :

$$\text{If } x, y \in X_r \text{ then } \langle x, y\rangle \in \Omega(\mathbb{K})_r \equiv \mathbb{K}, \tag{4.3}$$

$$\langle x + y, z\rangle \subseteq \langle x, z\rangle + \langle y, z\rangle, \tag{4.4}$$

$$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle, \tag{4.5}$$

$$\langle x, y\rangle = \langle y, x\rangle, \tag{4.6}$$

$$\langle x, x\rangle \geq 0 \text{ for } x \in X_r \text{ and } \langle x, x\rangle = 0 \Leftrightarrow x = 0, \tag{4.7}$$

$$\|\langle x, y\rangle\|_{\Omega(\mathbb{R})} = \sup\left\{\|\langle a, b\rangle\|_{\Omega(\mathbb{R})} : a \in F_x^{\widehat{X}}, b \in F_y^{\widehat{X}}\right\}, \tag{4.8}$$

$$\text{if } x \preceq y \text{ and } u \preceq v \text{ then } \langle x, u\rangle \subseteq \langle y, v\rangle, \tag{4.9}$$

$$\begin{gathered}\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \\ x \preceq y + x_\varepsilon \text{ and } \langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta) \text{ then } x \preceq y,\end{gathered} \tag{4.10}$$

where $\mathbb{K}$ is real or complex field and $\Omega(\mathbb{K})$ denotes the quasilinear space of the family of all compact subsets of $\mathbb{K}$. Further $S_\varepsilon(\theta)$ is the zero-centered ε-radius closed circle in $\mathbb{K}$. A quasilinear space with an inner product is called an inner product quasilinear space, briefly, IPQLS.

Theorem 4.4. *For $x = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right)$ and $y = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right) \in \mathbb{I}_{\mathbb{R}}^n$, the equality*

$$\langle x, y\rangle = \sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right]$$

defines an inner-product and hence $\mathbb{I}_{\mathbb{R}}^n$ is an IPQLS on the field $\mathbb{R}$ by this inner product.

Proof. Let $x, y \in \left(\mathbb{I}_{\mathbb{R}}^n\right)_r$ then $x = ([x_1, x_1], ..., [x_n, x_n])$ and $y = ([y_1, y_1], ..., [y_n, y_n])$. So,

$$\begin{aligned} \langle x, y\rangle &= \sum_{k=1}^{n} [x_k, x_k] [y_k, y_k] \\ &= \sum_{k=1}^{n} \{x_k y_k\} \in \Omega(\mathbb{R})_r \equiv \mathbb{R}. \end{aligned}$$

Later three condition can be easily verified. Now for $x \in \left(\mathbb{I}_{\mathbb{R}}^n\right)_r$, $\langle x, x\rangle = \sum\limits_{k=1}^{n} \{x_k x_k\} = \sum\limits_{k=1}^{n} \left\{|x_k|^2\right\} \in \Omega(\mathbb{R})_r \equiv \mathbb{R}$ and so we can write $\langle x, x\rangle \geq 0$. Easily we can see that $\langle x, x\rangle = 0 \Leftrightarrow x = 0$. Let us now verify the equality

$$\|\langle x, y\rangle\|_{\Omega(\mathbb{R})} = \sup\left\{\|\langle a, b\rangle\|_{\Omega(\mathbb{R})} : a \in F_x^{\widehat{X}}, b \in F_y^{\widehat{X}}\right\}$$

where $X = \mathbb{I}_{\mathbb{R}}^n$. Since X is consolidate $\widehat{X} = X$ and

$$\begin{aligned} \|\langle x, y\rangle\|_{\Omega(\mathbb{R})} &= \left\|\sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right]\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sum_{k=1}^{n} \sup_{\subset} \left\{\langle [t_k, t_k], [s_k, s_k]\rangle : [t_k, t_k] \subset \left[\underline{x_k}, \overline{x_k}\right], [s_k, s_k] \subset \left[\underline{y_k}, \overline{y_k}\right]\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sum_{k=1}^{n} \sup_{\subset} \left\{\langle [t_k, t_k], [s_k, s_k]\rangle : [t_k, t_k] \in F_{\left[\underline{x_k}, \overline{x_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}, [s_k, s_k] \in F_{\left[\underline{y_k}, \overline{y_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sup\left\{\sum_{k=1}^{n} \langle [t_k, t_k], [s_k, s_k]\rangle : [t_k, t_k] \in F_{\left[\underline{x_k}, \overline{x_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}, [s_k, s_k] \in F_{\left[\underline{y_k}, \overline{y_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sup\left\{\langle a, b\rangle : a \in F_x^{\mathbb{I}_{\mathbb{R}}^n}, b \in F_y^{\mathbb{I}_{\mathbb{R}}^n}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \sup\left\|\left\{\langle a, b\rangle : a \in F_x^{\mathbb{I}_{\mathbb{R}}^n}, b \in F_y^{\mathbb{I}_{\mathbb{R}}^n}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \sup\left\{\|\langle a, b\rangle\| : a \in F_x^{\mathbb{I}_{\mathbb{R}}^n}, b \in F_y^{\mathbb{I}_{\mathbb{R}}^n}\right\} \end{aligned}$$

were $a = ([t_1, t_1], [t_2, t_2], ..., [t_n, t_n])$ and $b = ([s_1, s_1], [s_2, s_2], ..., [s_n, s_n])$ are degenerate interval vectors obeying the above equality chain. Now let us only verify the last axiom of the inner product. Let us assume that for any $\varepsilon > 0$ there exists an element $x_\varepsilon = \left(\left[\underline{x_{1_\varepsilon}}, \overline{x_{1_\varepsilon}}\right], ..., \left[\underline{x_{n_\varepsilon}}, \overline{x_{n_\varepsilon}}\right]\right) \in \mathbb{I}_{\mathbb{R}}^n$ such that

$$x = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right) \preceq y = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right) + x_\varepsilon$$

and $\langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta)$. This implies, for each $k \in \{1, 2, ..., n\}$, $[x_k, x_k] \subseteq [y_k, y_k] + \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right]$. Since

$$\langle x_\varepsilon, x_\varepsilon\rangle = \sum_{k=1}^{n} \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \subseteq S_\varepsilon(\theta),$$

we get

$$\left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] = \left\langle \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right], \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right]\right\rangle \subseteq S_\varepsilon(\theta)$$

for each k. Since $\varepsilon \to 0$ implies $\|S_\varepsilon(\theta)\|_{\Omega(\mathbb{R})} \to 0$, we obtain

$$\left\langle \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right], \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \right\rangle \to \{0\}$$

in $\Omega(\mathbb{R})$. This brings us $[x_k, x_k] \subseteq [y_k, y_k]$ for each k. Eventually we can say $x \preceq y$.

Verification of remaining axioms are easy. $\square$

Let us see verification of the condition (26) in the proof by an easy example in $\mathbb{I}^n_{\mathbb{R}}$ in order to well-understanding of the condition.

Example 4.8. Let us consider $x = ([-3,3], [2,5])$, $y = ([-1,3], [2,2])$ in $\mathbb{I}^2_{\mathbb{R}}$.

$$\begin{aligned}
\|\langle x,y\rangle\|_{\Omega(\mathbb{R})} &= \left\| \sum_{k=1}^{2} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right] \right\|_{\Omega(\mathbb{R})} = \|[-3,3][-1,3] + [2,5][2,2]\|_{\Omega(\mathbb{R})} \\
&= \left\| \begin{array}{c} \sup\{[t,t][s,s] : [t,t] \subset [-3,3], [s,s] \subset [-1,3]\} \\ +\sup\{[t,t][s,s] : [t,t] \subset [2,5], [s,s] \subset [2,2]\} \end{array} \right\|_{\Omega(\mathbb{R})} \\
&= \left\| \begin{array}{c} \sup\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-3,3]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-1,3]}\right\} \\ +\sup\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,5]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,2]}\right\} \end{array} \right\|_{\Omega(\mathbb{R})} \\
&= \left\| \begin{array}{c} \sup\{\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-3,3]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-1,3]}\right\} \\ +\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,5]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,2]}\right\}\} \end{array} \right\|_{\Omega(\mathbb{R})} \\
&= \left\| \sup\left\{\langle a,b\rangle : a \in F^{\mathbb{I}^2_{\mathbb{R}}}_{x}, b \in F^{\mathbb{I}^2_{\mathbb{R}}}_{y}\right\} \right\|_{\Omega(\mathbb{R})} \\
&= \sup \left\| \left\{\langle a,b\rangle : a \in F^{\mathbb{I}^2_{\mathbb{R}}}_{x}, b \in F^{\mathbb{I}^2_{\mathbb{R}}}_{y}\right\} \right\|_{\Omega(\mathbb{R})} \\
&= \sup \left\{ \|\langle a,b\rangle\| : a \in F^{\mathbb{I}^2_{\mathbb{R}}}_{x}, b \in F^{\mathbb{I}^2_{\mathbb{R}}}_{y}\right\}.
\end{aligned}$$

Remark 4.4. The norm derived from this inner product is obtained in a usual way for any

$$x = (x_1, x_2, ...x_n). = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right) \in \mathbb{I}^n_{\mathbb{R}} :$$

$$\begin{aligned}
\|x\| &= \sqrt{\|\langle x,x\rangle\|_{\Omega(\mathbb{R})}} = \left(\left\| \sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{x_k}, \overline{x_k}\right] \right\|_{\Omega(\mathbb{R})} \right)^{1/2} \\
&= \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2} = \left(\sum_{k=1}^{n} \left\{\max\left\{\left|\underline{x_k}\right|, \left|\overline{x_k}\right|\right\}\right\}^2 \right)^{1/2} = \|x\|_2 .
\end{aligned}$$

This shows that the inner-product norm is just the 2-norm on $\mathbb{I}^n_{\mathbb{R}}$. For $n = 1$ if $x = [\underline{x}, \overline{x}] \in \mathbb{I}^1_{\mathbb{R}}$ then

$$\begin{aligned}
\|x\| &= \sqrt{\|\langle x,x\rangle\|_{\Omega(\mathbb{R})}} = \left(\left\| \sum_{k=1}^{1} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{x_k}, \overline{x_k}\right] \right\|_{\Omega(\mathbb{R})} \right)^{1/2} = \left(\left|[\underline{x}, \overline{x}]\,[\underline{x}, \overline{x}]\right|\right)^{1/2} \\
&= \left(\left|[\min S, \max S]\right|\right)^{1/2}, \, where\ S = \{\underline{x}^2, \underline{x}\overline{x}, \overline{x}\underline{x}, \overline{x}^2\} \\
&= \left(\max\{\min S, \max S\}\right)^{1/2}, \, where\ S = \{\underline{x}^2, \underline{x}\overline{x}, \overline{x}\underline{x}, \overline{x}^2\} \\
&= \left(\max\left\{\left|a^2\right| : a \in [\min S, \max S]\right\}\right)^{1/2} \\
&= \left(\left|[\underline{x}, \overline{x}]\right|^2\right)^{1/2} = \left|[\underline{x}, \overline{x}]\right| .
\end{aligned}$$

Note in general that $[\underline{x}, \overline{x}]\,[\underline{x}, \overline{x}] \neq [\underline{x}, \overline{x}]^2$ where $[\underline{x}, \overline{x}]^2$ is defined as $[\underline{x}, \overline{x}]^2 = \left\{t^2 : t \in [\underline{x}, \overline{x}]\right\}$ in $\mathbb{I}^1_{\mathbb{R}}$. However, $\|x\|^2 = \left|[\underline{x}, \overline{x}]\,[\underline{x}, \overline{x}]\right| = \left|[\underline{x}, \overline{x}]\right|^2$.

Definition 4.9. Let $x = ([x_1, x_1], ..., [x_n, x_n])$ and $y = ([y_1, y_1], ..., [y_n, y_n])$ be two elements in $\mathbb{I}^n_{\mathbb{R}}$. x and y are called *orthogonal* if

$$\langle x, y \rangle = \sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right] = [0, 0] = \{0\} .$$

Any set A in $\mathbb{I}^n_{\mathbb{R}}$ is called *orthogonal* if each two elements in A are orthogonal. Moreover, if we know each elements of A has norm 1 then A is called *orthonormal.*

Example 4.9. Let us consider $x = ([-3, 3], [0, 0]), y = ([0, 0], [2, 5])$ in $\mathbb{I}^2_{\mathbb{R}}$. Obviously x and y are orthogonal. These two elements are singular elements which are orthogonal. $([-3, -3], [0, 0])$ and $y = ([0, 0], [2, 2])$ are regular (degenerate) orthogonal elements. The set

$$A = \{([1, 1], [0, 0], ..., [0, 0]), ([0, 0], [1, 2], ..., [0, 0]), ..., ([0, 0], [0, 0], ..., [1, n])$$

is an orthogonal set in $\mathbb{I}^n_{\mathbb{R}}$ which is not a basis. However,

$$B = \{([1, 1], [0, 0], ..., [0, 0]), ([0, 0], [1, 1], ..., [0, 0]), ..., ([0, 0], [0, 0], ..., [1, 1])$$

is an orthonormal basis in $\mathbb{I}^n_{\mathbb{R}}$.

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