Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 131-136, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1118409

Stability of Solutions for a Krichhoff-Type Plate Equation with Degenerate Damping

Fatma Ekinci[1]*, Erhan Pişkin[2]

Abstract
We investigate a Kirchhoff type plate equation with degenerate damping term. By potential well theory, we show the asymptotic stability of energy in the presence of a degenerate damping.

Keywords: Degenerate damping, Kirchhoff-type equation, Stability
2010 AMS: 35B40, 35B44, 35G61, 35L75

[1] *Department of Mathematics, Dicle University, 21280 Diyarbakır, Turkey, ORCID: 0000-0002-9409-3054*
[2] *Department of Mathematics, Dicle University, 21280 Diyarbakır, Turkey, ORCID: 0000-0001-6587-4479*
***Corresponding author**: ekincifatma2017@gmail.com
Received: 18 May 2022, **Accepted:** 2 August 2022, **Available online:** 30 September 2022

1. Introduction and Preliminaries

In this paper, we focus on the stability of solutions under the sufficent conditions for the following problem

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u-\left(\int\limits_\Omega |\nabla u|^2 dx\right)^\gamma \Delta u+|u|^\rho j'(u_t)=|u|^{q-1}u \ \text{ in } \ \Omega\times(0,+\infty), \\ u(x,0)=u_0(x), \ \ u_t(x,0)=u_1(x) \qquad \text{on } \ x\in\Omega, \\ u(x,t)=\frac{\partial}{\partial n}u(x,t)=0 \qquad \text{on } \ x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\gamma>0$, j' denotes the derivative of $j(\alpha)$ [1], n is the outer normal and Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. Also, here

$$\Delta u-\left(\int\limits_\Omega |\nabla u|^2 dx\right)^\gamma \Delta u \ \text{ and } \ |u|^\rho j'(u_t)$$

represent Kirchhoff-type term and degenerate damping term, respectively.

1.1 Kirchhoff-type plate problems

To motivation for this problem comes from the following equation so called Beam equation model

$$u_{tt}+\Delta^2 u-\left(\alpha+\beta\int\limits_\Omega |\nabla u|^2 dx\right)\Delta u=|u|^{q-2}u, \tag{1.2}$$

without source term $\left(|u|^{q-2}u\right)$ was firstly introduced by Woinowsky-Krieger [2] to describe the dynamic bucking of a hinged extensible beam under an axial force. It was extensively studied by several researcers in different contexs. In [3, 4], the authors showed the global attractor, convergence and unboundedness of solutions with $|u_t|^{p-2}u_t$ nonlinear damping term. Then, the model also was investigated in [5, 6] and the authors obtained the existence, decay estimates of solutions and blow up of solutions with both negative and positive initial energy with $|u_t|^{p-2}u_t$ nonlinear damping term.

Recently, Pereira et al. [7] and Pişkin and Yüksekkaya [8] studied the model (1.2) with u_t. Pereira et al. studied existence of the global solutions through the Faedo-Galerkin approximations and obtained the asymptotic behavior by using the Nakao method. Pişkin and Yüksekkaya proved the blow up of solutions with positive and negative initial energy.

1.2 Problems with degenerate damping

This kind of degenerate damping effects was firstly investigated by Levine and Serrin [9] and considered the following equation

$$\left(|u_t|^{l-2}u_t\right)_t - a\nabla.\left(|\nabla u|^{q-2}\nabla u\right) + b|u|^{\rho}|u_t|^{m-2}u_t = c|u|^{p-2}u.$$

The authors considered the blow up of solutions with negative initial energy for the case $\rho+m<p$ under several other restrictions imposed on the paremeters m,ρ,p,q. But Levine and Serrin obtain only blow up solution with negative initial energy without any guarantees that the solution has a local solution. Then, Pitts and Rammaha [10] proved global and local existence for $\rho+m\geq p$ and for the case $\rho<1$ established uniqueness. Also, the authors obtained blow up solutions for negative initial energy and $\rho+m<p$.

On the other hand, the hyperbolic models with degenerate damping are of much interest in material science and physics. It particularly appears in physics when the friction is modulated by the strains. There is a wide literature has degenerate damping terms, namely $\delta(u)h(u_t)$ where $\delta(u)$ is a positive function and h is nonlinear, (see [11]-[27]).

The remaining part of this paper is organized as follows: In the next section, we study the stability result.

Now, we present some preliminary material which will be helpful for the proof of our result. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\|=\|.\|_{L^2(\Omega)}$ and $L^q(\Omega)$ norm $\|.\|_q=\|.\|_{L^q(\Omega)}$.

(A1) $\rho,p,q\geq 0$; $\rho\leq\frac{n}{n-2}$, $q+1\leq\frac{2n}{n-2}$ if $n\geq 3$. There exist positive constants c, c_0, c_1 such that for all $\alpha,\beta\in R$ $j(\alpha):R\to R$ be a C^1 convex real function satisfies

- $j(\alpha)\geq c|\alpha|^{p+1}$,
- $j'(\alpha)$ is single valued and $|j'(\alpha)|\leq c_0|\alpha|^p$,
- $(j'(\alpha)-j'(\beta))(\alpha-\beta)\geq c_1|\alpha-\beta|^{p+1}$.

(A2) $u_0(x)\in H_0^2(\Omega)$, $u_1(x)\in L^2(\Omega)$, $|u(\tau)|^{\rho}j(u_t)\in L^2(\Omega\times(0,T))$.

The said solution of (1.1) satisfies the energy identity

$$E'(t)=-\int_{\Omega}|u(\tau)|^{\rho}j(u_t)(\tau)\,dxd\tau\leq 0 \tag{1.3}$$

where

$$E(t)=\frac{1}{2}\left[\|u_t\|^2+\|\Delta u\|^2+\|\nabla u\|^2+\frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right]-\frac{1}{q+1}\|u\|_{q+1}^{q+1} \tag{1.4}$$

and

$$E(0)=\frac{1}{2}\left[\|u_1\|^2+\|\Delta u_0\|^2+\|\nabla u_0\|^2+\frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right]-\frac{1}{q+1}\|u_0\|_{q+1}^{q+1}. \tag{1.5}$$

Moreover, by computation, we get $E(t)$ is a non-increasing function, then

$$E(t)\leq E(0). \tag{1.6}$$

Now, we define

$$\alpha_1=\lambda_1^{-\frac{2}{q-1}},\quad E_1=\left(\frac{1}{2(\gamma+1)}-\frac{1}{q+1}\right)\alpha_1^{q+1},$$

$$\alpha_2=\left(\frac{1}{(q+1)\lambda_1^2}\right)^{\frac{1}{q-1}},\quad E_2=\frac{q+1}{2}\left(\frac{1}{2}-\frac{1}{q+1}\right)\alpha_2^{q+1},$$

$$W_0=\left\{(\alpha,E)\in R^2, 0\leq\alpha<\alpha_2, 0<E<E_2\right\},$$

$$V=\left\{(\alpha,E)\in R^2, \alpha>\alpha_1, 0<E<E_1\right\}$$

where λ_1 is the embedding constant (where $H_0^2(\Omega)$ is embedded into $L^{q+1}(\Omega)$).

2. Stability

This section is devoted to prove the stability of solutions for problem (1.1).

Lemma 2.1. *Assume that* $(A1)$ *and* $(A2)$ *hold and* $\left(\|u_0\|_{q+1}, E(0)\right) \in W_0$, *then*

$$\left(\|u(t)\|_{q+1}, E(t)\right) \in W_0,\ \ t \geq 0, \tag{2.1}$$

and

$$E(t) \geq \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right] + \frac{1}{4}\|\nabla u(t)\|^2,\ \ t \geq 0. \tag{2.2}$$

Proof. By using the embedding theorem and (1.6), we get

$$\begin{aligned} E_2 &> E(0) \geq E(t) \\ &\geq \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right] \\ &\quad + \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}\lambda_1^{-2}\|u(t)\|_{q+1}^2 - \frac{1}{2}\|u(t)\|_{q+1}^{q+1} \\ &\geq \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right] \\ &\quad + \frac{1}{4}\|\nabla u(t)\|^2 + h\left(\|u(t)\|_{q+1}\right), \end{aligned} \tag{2.3}$$

where $h(\alpha) = \frac{1}{4}\lambda_1^{-2}\alpha^2 - \frac{1}{2}\alpha^{q+1}$, for $\alpha \geq 0$. It is not difficult to verify that $h(\alpha)$ reaches its maximum E_2 for $\alpha = \alpha_2$, $h(\alpha)$ is strictly decreasing for $\alpha \geq \alpha_2$ and $h(\alpha) \to -\infty$ as $\alpha \to \infty$. By the continuity of $\|u(t)\|_{q+1}$ and $\alpha(0) = \|u_0\|_{q+1} < \alpha_2$, $\alpha(t) < \alpha_2$ for all $t \geq 0$. Further, $E(t) < E_2$ by (2.3). Then, (2.1) holds.

To obtain (2.2), it remains to the note that $h(\alpha) \geq 0$ whenever $0 \leq \alpha < \alpha_2$. Then (2.2) comes after at once. □

Lemma 2.2. *Assume that* $(A1)$ *and* $(A2)$ *hold, then*

$$\|\nabla u(t)\|^2 \geq 2\|u(t)\|_{q+1}^{q+1}\ \ or\ \ \|\nabla u(t)\|^2 - \|u(t)\|_{q+1}^{q+1} \geq \frac{1}{2}\|\nabla u(t)\|^2. \tag{2.4}$$

Furthermore, we have for constant C

$$\begin{cases} \|u_t(t)\| \in L^2(\Omega), \\ \|\nabla u(t)\| \leq C,\ \ \|u(t)\|_{q+1} \leq C,\ \|u_t(t)\| \leq C,\ \ \|\Delta u(t)\| \leq C. \end{cases} \tag{2.5}$$

Proof. By using the embedding theorem, we get

$$\begin{aligned} \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\|u(t)\|_{q+1}^{q+1} &\geq \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}\lambda_1^{-2}\|u(t)\|_{q+1}^2 - \frac{1}{2}\|u(t)\|_{q+1}^{q+1} \\ &= \frac{1}{4}\|\nabla u(t)\|^2 + h\left(\|u(t)\|_{q+1}\right). \end{aligned}$$

Since $h(\alpha) \geq 0$, if $0 \leq \alpha < \alpha_2$ and $0 \leq \|u(t)\|_{q+1} < \alpha_2$ by Lemma 1, (2.4) is true.

The initial result in (2.5) comes from the assumption $(A2)$. The remainder of results in (2.5) follows (1.6), (2.2) and (2.4). □

Lemma 2.3. *Let* $\left(\|u_0\|_{q+1}, E(0)\right) \in W_0$ *and* $E(t) \geq \eta$, *where* $\eta > 0$ *is a constant, then there exists* $\delta = \delta(\eta) > 0$ *such that*

$$\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u(t)\|^{2(\gamma+1)} - \|u(t)\|_{q+1}^{q+1} \geq \delta,\ \ t \geq 0. \tag{2.6}$$

Proof. From the definition of $E(t)$ and $E(t) \geq \eta$, we get

$$\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u(t)\|^{2(\gamma+1)} \geq 2\eta, \ \ t \geq 0. \tag{2.7}$$

Now, we suppose by contradiction that (2.6) does not hold. By (2.4), there is a sequences $t_n \subset R^+$ as follows

$$\begin{aligned} & \|u_t(t_n)\|^2 + \|\Delta u(t_n)\|^2 + \|\nabla u(t_n)\|^2 + \|\nabla u(t_n)\|^{2(\gamma+1)} - \|u(t_n)\|_{q+1}^{q+1} \\ \geq \ & \|u_t(t_n)\|^2 + \|\Delta u(t_n)\|^2 + \|\nabla u(t_n)\|^{2(\gamma+1)} + \frac{1}{2}\|\nabla u(t_n)\|^2 \to 0, \ \ (n \to \infty). \end{aligned}$$

Then, we get

$$\|u_t(t_n)\|^2 \to 0, \|\Delta u(t_n)\|^2 \to 0, \|\nabla u(t_n)\|^{2(\gamma+1)} \to 0, \ \|\nabla u(t_n)\|^2 \to 0, \ \ n \to \infty.$$

This is imposible since (2.7) and yield the desired result. This completes the proof of lemma. □

Theorem 2.4. *Assume that* $(A1)$ *and* $(A2)$ *hold, we get*

$$\lim_{t\to\infty} E(t_n) = 0, \ \lim_{t\to\infty} \|\Delta u(t_n)\|^2 = 0. \tag{2.8}$$

Proof. Assume that (2.8) fails, then there exists $\eta > 0$ such that $E(t) \geq \eta$ for all $t \geq 0$ since (1.6) and $E(t) \geq 0$. Multiplying both sides of (1.1) by u, integrating them over $[T,t] \times \Omega$ $(0 < T \leq t \leq \infty)$ and integrating by parts, we have

$$\begin{aligned} & (u_t(s), u(s))|_{s=T}^{t} \\ = \ & \int_T^t \Big[2\|u_t(s)\|^2 - \Big(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\nabla u(s)\|^{2(\gamma+1)} - \|u(s)\|_{q+1}^{q+1} \Big) \\ & - \int_\Omega |u(s)|^\rho u(s) j'(u_t)(s)\, dx \Big] ds \\ = \ & \int_T^t (K_1 + K_2 + K_3)\, ds. \end{aligned} \tag{2.9}$$

By (1.6), (2.2) and (2.5), we have

$$\int_T^t K_1 ds = \int_T^t 2\|u_t(s)\|^2 ds \leq 4E^{\frac{1}{2}}(0) \left(\int_T^t \|u_t(s)\|^2 ds \right)^{\frac{1}{2}} \left(\int_T^t ds \right)^{\frac{1}{2}} \leq C_1 \left(\int_T^t ds \right)^{\frac{1}{2}}. \tag{2.10}$$

Here and in the next positive constant C_i not depend on t and T. From Lemma 3, we have

$$\begin{aligned} \int_T^t K_2 ds &= -\int_T^t \Big(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\nabla u(s)\|^{2(\gamma+1)} - \|u(s)\|_{q+1}^{q+1} \Big) ds \\ &\leq -\delta \int_T^t ds. \end{aligned} \tag{2.11}$$

Set

$$H(t) = E_1 - E(t).$$

From (1.3), we have

$$H'(t) = -E'(t) = \int_\Omega |u(t)|^\rho j(u_t)(t)\, dx \geq 0. \tag{2.12}$$

Form (2.12) and since $E(t) \geq 0$ for $t \geq 0$ and $H(t) \in C(0,\infty)$ we reach at $\int_\Omega |u(t)|^\rho j'(u_t)(t)\,dx \in L^1(0,\infty)$, using Holder inequality, (2.4) and embedding theorem $H_0^2(\Omega) \hookrightarrow L^{\rho+p}(\Omega)$, we have

$$\begin{aligned}
\int_T^t K_3 ds &= -\int_T^t \int_\Omega |u(s)|^\rho u(s)\, j'(u_t)(s)\,dxds \\
&\leq \int_T^t \int_\Omega |u(s)|^{\rho+1-\frac{\rho+p+1}{p+1}} |u(s)|^{\frac{\rho+p+1}{p+1}} |u_t(s)|^p\,dxds \\
&\leq \left(\int_T^t \int_\Omega |u|^\rho\, j(u_t)(s)\,dxds\right)^{\frac{p}{p+1}} \left(\int_T^t \int_\Omega |u(s)|^{\rho+p+1}\,dxds\right)^{\frac{1}{p+1}} \\
&\leq C_2 \left(\int_T^t H'(s)\,ds\right)^{\frac{p}{p+1}} \left(\int_T^t \|u(s)\|_{\rho+p+1}^{\rho+p+1}\,ds\right)^{\frac{1}{p+1}} \\
&\leq C_3 \left(\int_T^t \|\nabla u(s)\|^{\rho+p+1}\,ds\right)^{\frac{1}{p+1}} \leq C_4 \left(\int_T^t ds\right)^{\frac{1}{p+1}}.
\end{aligned} \tag{2.13}$$

Then from (2.9)-(2.13), as $p+1 \leq 2$, we know

$$(u_t(s), u(s))|_{s=T}^t \leq C_1 \left(\int_T^t ds\right)^{\frac{1}{2}} + C_4 \left(\int_T^t ds\right)^{\frac{1}{p+1}} - \delta \int_T^t ds \leq C_5 \left(\int_T^t ds\right)^{\frac{1}{p+1}} - \delta \int_T^t ds. \tag{2.14}$$

Moreover, by applying Holder inequality and (2.5),

$$|(u_t(s), u(s))| \leq C_6 \left(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\nabla u(s)\|^{2(\gamma+1)}\right) < \infty.$$

In turn, we arrive a result that is in contradiction with (2.14) for fixing T when $t \to \infty$. Therefore, we derive $\lim_{t\to\infty} E(t) = 0$ and $\lim_{t\to\infty} \|\Delta u(t)\|^2 = 0$ by (2.2). This completes the proof. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] V. Barbu, I. Lasiecka, M. A. Rammaha, *Existence and uniqueness of solutions to wave equations with nonlinear degenerate damping and source terms*, Control Cybernet., **34**(3) (2005), 665-687.

[2] S. Woinowsky-Krieger, *The effect of axial force on the vibration of hinged bars*, J Appl Mech., **17** (1950), 35-36.

[3] J. A. Esquivel-Avila, *Dynamic analysis of a nonlinear Timoshenko equation*, Abstr Appl Anal., **2010** (2011), 1-36.

[4] J. A. Esquivel-Avila, *Global attractor for a nonlinear Timoshenko equation with source terms*, Math Sci., **7**(32) (2013), 1-8.

[5] E. Pişkin, *Existence, decay and blow up of solutions for the extensible beam equation with nonlinear damping and source terms*, Open Math., **13** (2005), 408-420.

[6] E. Pişkin, N. Irkıl, *Blow up positive initial-energy solutions for the extensible beam equation with nonlinear damping and source terms*, Ser. Math. Inform., **31**(3) (2016), 645-654.

[7] D. C. Pereira, H. Nguyen, C. A. Raposo, C. H. M. Maranhao, *On the solutions for an extensible beam equation with internal damping and source terms*, Differential Equations & Applications, **11**(3) (2019), 367-377.

[8] E. Pişkin, H. Yüksekkaya, *Non-existence of solutions for a Timoshenko equations with weak dissipation*, Math Morav., **22**(2) (2018), 1-9.

[9] H. A. Levine, J. Serrin, *Global nonexistence theorems for quasilinear evolution with dissipation*, Arch. Rational Mech. Anal, **137** (1997), 341-361.

[10] D. R. Pitts, M. A. Rammaha, *Global existence and nonexistence theorems for nonlinear wave equations*, Indiana Uni. Math. J., **51**(6) (2002), 1479-1509.

[11] V. Barbu, I. Lasiecka, M. A. Rammaha, *Blow-up of generalized solutions to wave equations with nonlinear degenerate damping and source terms*, Indiana Uni. Math. J., **56**(3) (2007), 995-1022.

[12] V. Barbu, I. Lasiecka, M.A. Rammaha, *On nonlinear wave equations with degenerate damping and source terms*, Trans. Amer. Math. Soc., **357**(7) (2005), 2571-2611.

[13] Q. Hu, H. Zhang, *Blow up and asymptotic stability of weak solutions to wave equations with nonlinear degenerate damping and source terms*, Electron J. Differ. Eq., **2007** (76) (2007), 1-10.

[14] S. Xiao, W. Shubin, *A blow-up result with arbitrary positive initial energy for nonlinear wave equations with degenerate damping terms*, J. Part. Diff. Eq., **32** (2019), 181-190.

[15] F. Ekinci, E. Pişkin, *Nonexistence of global solutions for the Timoshenko equation with degenerate damping*, Menemui Mat., **43**(1) (2021), 1-8.

[16] E. Pişkin, F. Ekinci, *General decay and blowup of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms*, Math. Meth. App. Sci., **42**(16) (2019), 1-21.

[17] E. Pişkin, F. Ekinci, *Local existence and blow up of solutions for a coupled viscoelastic Kirchhoff-type equations with degenerate damping*, Miskolc Math. Notes, **22**(2) (2021), 861-874.

[18] E. Pişkin, F. Ekinci, *Blow up of solutions for a coupled Kirchhoff-type equations with degenerate damping terms*, Applications & Applied Mathematics, **14**(2) (2019), 942-956.

[19] E. Pişkin, F. Ekinci, K. Zennir, *Local existence and blow-up of solutions for coupled viscoelastic wave equations with degenerate damping terms*, Theor. Appl. Mech., **47**(1) (2020), 123-154.

[20] E. Pişkin, F. Ekinci, *Global existence of solutions for a coupled viscoelastic plate equation with degenerate damping terms*, Tbilisi Math. J., **14**(2021), 195-206.

[21] E. Pişkin, F. Ekinci, H. Zhang, *Blow up, lower bounds and exponential growth to a coupled quasilinear wave equations with degenerate damping terms*, Dynamics of Continuous, Discrete and Impulsive Systems, In press.

[22] F. Ekinci, E. Pişkin, S. M. Boulaaras, I. Mekawy, *Global existence and general decay of solutions for a quasilinear system with degenerate damping terms*, J. Funct. Spaces, **2021** (2021), 4316238.

[23] F. Ekinci, E. Pişkin, *Blow up and exponential growth to a Petrovsky equation with degenerate damping*, Univers. J. Math. Appl., **4**(2) (2021), 82-87.

[24] F. Ekinci, E. Pişkin, *Global existence and growth of solutions to coupled degeneratly damped Klein-Gordon equations*, Al-Qadisiyah Journal of Pure Science, **27**(1) (2022), 29-40.

[25] F. Ekinci, E. Pişkin, *Growth of solutions for fourth order viscoelastic system*, Sigma Journal of Engineering and Natural Sciences, (2021), 1-7.

[26] F. Ekinci, E. Pişkin, K. Zennir, *Existence, blow up and growth of solutions for a coupled quasi-linear viscoelastic Petrovsky equations with degenerate damping terms*, Journal of Information and Optimization Sciences, (2021), 1-29.

[27] E. Pişkin, F. Ekinci, *Blow up, exponential growth of solution for a reaction-diffusion equation with multiple nonlinearities*, Tbilisi Math. J., **12**(4) (2019), 61-70.