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Research Article

Actions of generalized derivations on prime ideals in $*$-rings with applications

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Abstract

In this paper, we make use of generalized derivations to scrutinize the deportment of prime ideal satisfying certain algebraic $*$-identities in rings with involution. In specific cases, the structure of the quotient ring $\mathscr{R}/\mathscr{P}$ will be resolved, where $\mathscr{R}$ is an arbitrary ring and $\mathscr{P}$ is a prime ideal of $\mathscr{R}$ and we also find the behaviour of derivations associated with generalized derivations satisfying algebraic $*$-identities involving prime ideals. Finally, we conclude our paper with applications of the previous section's results.

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1. Introduction

The study of additive maps on rings possessing involution was initiated by Brešar et al. [10] and they characterized the additive centralizing mappings on the skew-symmetric elements of prime rings possessing involution. The algebra of derivations and generalized derivations play a crucial role in the study of $*$-functional identities and their applications. In, 2022, some work have been done by researcher on the structure of a quotient ring $\mathscr{R}/\mathscr{P}$ with the help of different additive mappings (See [7, 8, 15]). In this paper, we are interested in the study of rings with involution given as a quotient ring $\mathscr{R}/\mathscr{P}$, where $\mathscr{R}$ is an arbitrary ring and $\mathscr{P}$ is a prime ideal of $\mathscr{R}$ involving certain $*$-differential/functional identities on prime ideals. The originality in this work is that we use a derivation on $\mathscr{R}$ (and not on $\mathscr{R}/\mathscr{P}$) which satisfies an algebraic property on $\mathscr{R}$ with respect to $\mathscr{P}$.

If not otherwise stated, $\mathscr{R}$ in that manuscript always represents an associative ring with centre $Z(\mathscr{R})$. Retrieve that a proper ideal $\mathscr{P}$ is called prime if $a\mathscr{R}b \subseteq \mathscr{P}$ implies $a \in \mathscr{P}$ or $b \in \mathscr{P}$. In case $\mathscr{P} = (0)$, the ring is called prime. An additive mapping $f : \mathscr{R} \to \mathscr{R}$ is said to be left centralizer if $f(xy) = f(x)y$ holds for all $x, y \in \mathscr{R}$. By a derivation of $\mathscr{R}$, we mean an additive map $\phi : \mathscr{R} \to \mathscr{R}$ satisfying $\phi(xy) = \phi(x)y + x\phi(y)$ for all

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$x, y \in \mathscr{R}$. A first generalization of derivation on $\mathscr{R}$ is known in literature as generalized derivation which was introduced by Brešar [9], and defined as $\mathscr{F}(xy) = \mathscr{F}(x)y + x\phi(y)$ for all $x, y \in \mathscr{R}$, where ϕ is an associated derivation on $\mathscr{R}$. The analyses of generalized derivations have primarily been studied on operator algebras. Therefore, any investigation from the algebraic point of view might be fascinating (see [14, 21] for details). We refer to an involution $*$ on a ring $\mathscr{R}$ as a conjugate linear anti-automorphism of period two i.e., an additive map $x \mapsto x^*$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in \mathscr{R}$. A ring equipped with involution is referred to as ring with involution or $*$-ring. An element x in a ring with involution is said to be symmetric if $x^* = x$ and skew-symmetric if $x^* = -x$. The sets of all symmetric and skew-symmetric elements of $\mathscr{R}$ will be denoted by $H(\mathscr{R})$ and $S(\mathscr{R})$, respectively. Generally, involutions are considered to be the first kind if $Z(\mathscr{R}) \subseteq H(\mathscr{R})$, otherwise, they are considered of the second kind. $S(\mathscr{R}) \cap Z(\mathscr{R}) \neq \{0\}$ in the last-mentioned case. We refer the reader to [4, 13] for justification and amplification for the notations discussed above and key definitions.

2. Results on prime ideals

In the early nineties, after a memorable work on the structure theory of rings, a tremendous work has been established by Brešar considering centralizing mappings, commuting mappings, commutativity preserving (CP) mappings and strong commutativity preserving (SCP) mappings on some appropriate subset of rings. Since then it became a tempting research idea in the matrix theory/operator theory/ring theory for algebraists. Commutativity preserving (CP) maps were introduced and studied by Watkins [24] and further extended to SCP by Bell and Mason [6]. By mean SCP, is an additive map $\psi : \mathscr{R} \to \mathscr{R}$ satisfying $[\psi(x), \psi(y)] = [x, y]$ for all $x, y \in \mathscr{R}$. A great account of work has been done on these maps possessing a variety of derivations on many algebraic structures. See [12, 18, 19] and the references therein. In 2014, Ali et al. [1] studied the SCP maps in different way on rings possessing involution. They established the commutativity of prime ring of characteristic not two possessing second kind of involution satisfying $[\phi(x), \phi(x^*)] - [x, x^*] = 0$ for every $x \in \mathscr{R}$, where ϕ is a nonzero derivation of $\mathscr{R}$. Later, Dar and Khan [11] improved this result by studying the case of generalized derivations. Further, Khan and Ali [16] studied SCP maps as endomorphisms on rings with involution. Recently, Raza et al. [22] established the same result for b-generalized derivations. Very recently Khan et al. [17] proved the above result for prime ideals in rings with involution [17, Theorem 2.14]. Motivated by the above research, we deal the following result with a pair of generalized derivations in the case of prime ideals, as follows:

Theorem 2.1. *Let $\mathscr{R}$ be a ring with involution $*$ of the second kind, $\mathscr{P}$ a prime ideal of $\mathscr{R}$ such that $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ and char$(\mathscr{R}/\mathscr{P}) \neq 2$. If $\mathscr{R}$ admits two generalized derivations $\mathscr{F}$ and $\mathscr{G}$ associated with two derivations ϕ and ψ respectively, such that $\mathscr{F} \neq I_{\mathscr{R}}$ or $\mathscr{G} \neq I_{\mathscr{R}}$ and satisfying the condition $[\mathscr{F}(x), \mathscr{G}(x^*)] \pm [x, x^*] \in \mathscr{P}$ for all $x \in \mathscr{R}$, then either $\mathscr{R}/\mathscr{P}$ is a commutative integral domain or $(\phi(\mathscr{R}) \subseteq \mathscr{P}$, $\psi(\mathscr{R}) \subseteq \mathscr{P})$.*

Proof. We have

$$[\mathscr{F}(x), \mathscr{G}(x^*)] \pm [x, x^*] \in \mathscr{P} \tag{2.1}$$

for all $x \in \mathscr{R}$. Consider either $\mathscr{F}$ or $\mathscr{G}$ or both are zero then we have $\pm[x, x^*] \in \mathscr{P}$ for all $x \in \mathscr{R}$. Then $\mathscr{R}/\mathscr{P}$ is commutative integral domain in view of [17, Lemma 2.2]. Now consider neither $\mathscr{F}$ nor $\mathscr{G}$ be zero. Linearization of (2.1), yields that

$$[\mathscr{F}(x), \mathscr{G}(y^*)] + [\mathscr{F}(y), \mathscr{G}(x^*)] \pm [x, y^*] \pm [y, x^*] \in \mathscr{P} \tag{2.2}$$

for all $x, y \in \mathscr{R}$. Now substitute yh_0 for y in (2.2), where $h_0 \in H(\mathscr{R}) \cap Z(\mathscr{R})$, we find that

$$\begin{aligned}[][\mathscr{F}(x), \mathscr{G}(y^*)]h_0 + [\mathscr{F}(x), y^*]\psi(h_0) + [\mathscr{F}(y), \mathscr{G}(x^*)]h_0 + [y, \mathscr{G}(x^*)]\phi(h_0) \\ \pm [x, y^*]h_0 \pm [y, x^*]h_0 \in \mathscr{P}\end{aligned} \tag{2.3}$$

for all $x, y \in \mathscr{R}$. Using (2.2) in (2.3), we obtain

$$[\mathscr{F}(x), y^*]\psi(h_0) + [y, \mathscr{G}(x^*)]\phi(h_0) \in \mathscr{P} \tag{2.4}$$

for all $x, y \in \mathscr{R}$. Taking k_0^2 for h_0 in (2.4) where $k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we get

$$2([\mathscr{F}(x), y^*]\psi(k_0) + [y, \mathscr{G}(x^*)]\phi(k_0))k_0 \in \mathscr{P} \tag{2.5}$$

for all $x, y \in \mathscr{R}$. Using the hypotheses $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$ and $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$, we arrive at

$$[\mathscr{F}(x), y^*]\psi(k_0) + [y, \mathscr{G}(x^*)]\phi(k_0) \in \mathscr{P} \tag{2.6}$$

for all $x, y \in \mathscr{R}$. Substituting yk_0 in place of y in (2.6) and using the fact that $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq P$, we obtain

$$-[\mathscr{F}(x), y^*]\psi(k_0) + [y, \mathscr{G}(x^*)]\phi(k_0) \in \mathscr{P} \tag{2.7}$$

for all $x, y \in \mathscr{R}$. Now replacing y by yk_0 in (2.2), where $k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we get

$$\begin{aligned} -[\mathscr{F}(x), \mathscr{G}(y^*)]k_0 - [\mathscr{F}(x), y^*]\psi(k_0) + [\mathscr{F}(y), \mathscr{G}(x^*)]k_0 + [y, \mathscr{G}(x^*)]\phi(k_0) \\ \mp [x, y^*]k_0 \pm [y, x^*]k_0 \in \mathscr{P} \end{aligned} \tag{2.8}$$

for all $x, y \in \mathscr{R}$. Application of (2.7) in (2.8) yields

$$-[\mathscr{F}(x), \mathscr{G}(y^*)]k_0 + [\mathscr{F}(y), \mathscr{G}(x^*)]k_0 \mp [x, y^*]k_0 \pm [y, x^*]k_0 \in \mathscr{P} \tag{2.9}$$

for all $x, y \in \mathscr{R}$. Since $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq P$, it follows that

$$-[\mathscr{F}(x), \mathscr{G}(y^*)] + [\mathscr{F}(y), \mathscr{G}(x^*)] \mp [x, y^*] \pm [y, x^*] \in \mathscr{P} \tag{2.10}$$

for all $x, y \in \mathscr{R}$. Now combining (2.2) and (2.10), we get

$$2([\mathscr{F}(y), \mathscr{G}(x^*)] \pm [y, x^*]) \in \mathscr{P}$$

for all $x, y \in \mathscr{R}$. This implies that

$$[\mathscr{F}(y), \mathscr{G}(x^*)] \pm [y, x^*] \in \mathscr{P} \tag{2.11}$$

for all $x, y \in \mathscr{R}$. Replacing x by x^* in (2.11), we get

$$[\mathscr{F}(y), \mathscr{G}(x)] \pm [y, x] \in \mathscr{P} \text{ for all } x, y \in \mathscr{R}.$$

Thus in view of [23, Theorem 1.4], we get $\mathscr{R}/\mathscr{P}$ is commutative integral domain or $(\phi(\mathscr{R}) \subseteq \mathscr{P}, \psi(\mathscr{R}) \subseteq \mathscr{P})$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 2.1:

Corollary 2.2. *Let $\mathscr{R}$ be a ring with involution $*$ of the second kind, $\mathscr{P}$ a prime ideal of $\mathscr{R}$ such that $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ and $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with a derivation ϕ such that $\mathscr{F} \neq I_{\mathscr{R}}$ and satisfying the condition $[\mathscr{F}(x), \mathscr{F}(x^*)] \pm [x, x^*] \in \mathscr{P}$ for all $x \in \mathscr{R}$, then either $\mathscr{R}/\mathscr{P}$ is a commutative integral domain or $\phi(\mathscr{R}) \subseteq \mathscr{P}$.*

Corollary 2.3 ([17, Theorem 2.14])**.** *Let $\mathscr{R}$ be a ring with involution $*$ of the second kind, $\mathscr{P}$ a prime ideal of R such that $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ and $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$. If d_1 and d_2 are derivations of $\mathscr{R}$ satisfying the condition $[d_1(x), d_2(x^*)] - [x, x^*] \in \mathscr{P}$ for all $x \in \mathscr{R}$, then $\mathscr{R}/\mathscr{P}$ is a commutative integral domain.*

Now we establish the following results in the light of above theorem, which may help to develop the interests of readers:

Theorem 2.4. *Let $\mathscr{R}$ be a ring with involution $*$ of the second kind, $\mathscr{P}$ a prime ideal of $\mathscr{R}$ such that $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ and $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with a derivation ϕ, such that $[\mathscr{F}(x \circ x^*), x] \in \mathscr{P}$ for all $x \in \mathscr{R}$, then either $\mathscr{R}/\mathscr{P}$ is a commutative integral domain or $\phi(\mathscr{R}) \subseteq \mathscr{P}$.*

Proof. We have
$$[\mathscr{F}(x \circ x^*), x] \in \mathscr{P} \quad \text{for all} \quad x \in \mathscr{R}. \tag{2.12}$$
Substituting h for x in (2.12), where $h \in H(\mathscr{R})$, we obtain
$$2[\mathscr{F}(h^2), h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}).$$
Since $\operatorname{char}(\mathscr{R}/\mathscr{P}) \neq 2$, the above expression reduces to
$$[\mathscr{F}(h^2), h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}). \tag{2.13}$$
Writing $h + h_0$ instead of h in (2.13), where $h_0 \in H(\mathscr{R}) \cap Z(\mathscr{R})$ and using (2.13), we get
$$[\mathscr{F}(h_0^2), h] + 2[\mathscr{F}(hh_0), h] \in \mathscr{P} \tag{2.14}$$
for all $h \in H(\mathscr{R})$. This gives
$$[\mathscr{F}(h_0), h]h_0 + 2[\mathscr{F}(h), h]h_0 \in \mathscr{P} \tag{2.15}$$
for all $h \in H(\mathscr{R})$. Taking $h_0 = k^2$, where $k \in S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$, we find that
$$[\mathscr{F}(k), h]k + 2[\mathscr{F}(h), h] \in \mathscr{P} \tag{2.16}$$
for all $h \in H(\mathscr{R})$. Putting $-h$ for h in (2.16), we obtain
$$-[\mathscr{F}(k), h]k + 2[\mathscr{F}(h), h] \in \mathscr{P} \tag{2.17}$$
for all $h \in H(\mathscr{R})$. Combining (2.16) and (2.17), we get
$$4[\mathscr{F}(h), h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}).$$
Since $\operatorname{char}(\mathscr{R}/\mathscr{P}) \neq 2$, this implies that
$$[\mathscr{F}(h), h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}). \tag{2.18}$$
Linearization of (2.18) gives that
$$[\mathscr{F}(h), h'] + [\mathscr{F}(h'), h] \in \mathscr{P} \quad \text{for all} \quad h, h' \in H(\mathscr{R}). \tag{2.19}$$
Replacing h' by $k'k_0$ in (2.19), where $k' \in S(\mathscr{R})$ and $k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we get
$$[\mathscr{F}(h), k'k_0] + [\mathscr{F}(k'k_0), h] \in \mathscr{P} \tag{2.20}$$
for all $h \in H(\mathscr{R})$ and $k' \in S(\mathscr{R})$, which further gives that
$$[\mathscr{F}(h), k']k_0 + [\mathscr{F}(k'), h]k_0 + [k', h]\phi(k_0) \in \mathscr{P} \tag{2.21}$$
for all $h \in H(\mathscr{R})$ and $k' \in S(\mathscr{R})$. Now consider (2.18) and replacing h by $k'k_0$ in (2.18), we get
$$[\mathscr{F}(k'k_0), k'k_0] \in \mathscr{P} \quad \text{for all} \quad k' \in S(\mathscr{R}) \quad \text{and} \quad k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R}). \tag{2.22}$$
This implies that
$$[\mathscr{F}(k'), k'] \in \mathscr{P} \quad \text{for all} \quad k' \in S(\mathscr{R}). \tag{2.23}$$
Linearization of (2.23) yields that
$$[\mathscr{F}(k), k'] + [\mathscr{F}(k'), k] \in \mathscr{P} \quad \text{for all} \quad k, k' \in S(\mathscr{R}). \tag{2.24}$$
Replacing k' by $h'k_0$ in (2.24), where $k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we obtain
$$[\mathscr{F}(k), h'k_0] + [\mathscr{F}(h'k_0), k] \in \mathscr{P} \quad \text{for all} \quad h' \in H(\mathscr{R}). \tag{2.25}$$
This can be further written as
$$[\mathscr{F}(k), h']k_0 + [\mathscr{F}(h'), k]k_0 + [h', k]\phi(k_0) \in \mathscr{P} \tag{2.26}$$
for all $h' \in H(\mathscr{R})$ and $k \in S(\mathscr{R})$. Substituting h for h' and k for k' in (2.26), we get
$$[\mathscr{F}(k'), h]k_0 + [\mathscr{F}(h), k']k_0 + [h, k']\phi(k_0) \in \mathscr{P} \tag{2.27}$$
for all $h \in H(\mathscr{R})$ and $k' \in S(\mathscr{R})$. Combining (2.21) and (2.27), we obtain
$$2([\mathscr{F}(h), k'] + [\mathscr{F}(k'), h])k_0 \in \mathscr{P} \tag{2.28}$$

for all $h \in H(\mathscr{R})$ and $k' \in S(\mathscr{R})$. This implies that

$$[\mathscr{F}(h), k'] + [\mathscr{F}(k'), h] \in \mathscr{P} \tag{2.29}$$

for all $h \in H(\mathscr{R})$ and $k' \in S(\mathscr{R})$. Now consider

$$\begin{aligned} 4[\mathscr{F}(x), x] &= [2\mathscr{F}(x), 2x] \\ &= [\mathscr{F}(2x), 2x] \\ &= [\mathscr{F}(h+k'), h+k'] \\ &= [\mathscr{F}(h), h] + [\mathscr{F}(h), k'] + [\mathscr{F}(k'), h] + [\mathscr{F}(k'), k']. \end{aligned}$$

Thus in view of (2.18), (2.23) and (2.29), we obtain $4[\mathscr{F}(x), x] \in \mathscr{P}$ for all $x \in \mathscr{R}$. Since $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$, it follows that $[\mathscr{F}(x), x] \in \mathscr{P}$ for all $x \in \mathscr{R}$. Hence [23, Lemma 1.1], we get the required results. This completes the proof of the theorem. □

Theorem 2.5. *Let $\mathscr{R}$ be a ring with involution $*$ of the second kind, $\mathscr{P}$ a prime ideal of $\mathscr{R}$ such that $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ and $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with a derivation ϕ, such that $[\mathscr{F}(x) \circ x^*, x] \in \mathscr{P}$ for all $x \in \mathscr{R}$, then either $\mathscr{R}/\mathscr{P}$ is a commutative integral domain or $\phi(\mathscr{R}) \subseteq \mathscr{P}$.*

Proof. By the given hypothesis, we have

$$[\mathscr{F}(x) \circ x^*, x] \in \mathscr{P} \quad \text{for all} \quad x \in \mathscr{R}. \tag{2.30}$$

Substituting h for x in (2.30) where $h \in H(\mathscr{R})$, we obtain

$$[\mathscr{F}(h) \circ h, h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}). \tag{2.31}$$

This can be further written as

$$[\mathscr{F}(h)h + h\mathscr{F}(h), h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}). \tag{2.32}$$

Replacing h by $h + h_0$ where $h_0 \in H(\mathscr{R}) \cap Z(\mathscr{R})$ in (2.32), we get

$$2[\mathscr{F}(h), h]h_0 + [\mathscr{F}(h_0), h]h + h[\mathscr{F}(h_0), h] + 2[\mathscr{F}(h_0), h]h_0 \in \mathscr{P} \tag{2.33}$$

for all $h \in H(\mathscr{R})$. Substituting $-h$ for h in (2.33), then we have

$$2[\mathscr{F}(h), h]h_0 + [\mathscr{F}(h_0), h]h + h[\mathscr{F}(h_0), h] - 2[\mathscr{F}(h_0), h]h_0 \in \mathscr{P} \tag{2.34}$$

for all $h \in H(\mathscr{R})$. Combination of (2.33) and (2.34) gives that

$$2[\mathscr{F}(h), h]h_0 + [\mathscr{F}(h_0), h]h + h[\mathscr{F}(h_0), h] \in \mathscr{P} \tag{2.35}$$

for all $h \in H(\mathscr{R})$. Again substituting $h + h_0$ for h in (2.35), we get

$$4[\mathscr{F}(h_0), h]h_0 \in \mathscr{P} \tag{2.36}$$

for all $h \in H(\mathscr{R})$. Using the primeness of $\mathscr{P}$ and the fact that *char*$(\mathscr{R}/\mathscr{P}) \neq 2$, we obtain

$$[F(h_0), h] \in \mathscr{P} \quad \text{or} \quad h_0 \in \mathscr{P}.$$

Consider, if $h_0 \in \mathscr{P}$ for all $h_0 \in H(\mathscr{R}) \cap Z(\mathscr{R})$, then $k^2 \in \mathscr{P}$ for all $k \in S(\mathscr{R}) \cap Z(\mathscr{R})$. This further gives that $k \in \mathscr{P}$, which contradicts the fact $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$. Thus, we have

$$[\mathscr{F}(h_0), h] \in \mathscr{P} \tag{2.37}$$

for all $h \in H(\mathscr{R})$. In view of (2.37), the relation (2.33) reduces to

$$2[\mathscr{F}(h), h]h_0 \in \mathscr{P} \tag{2.38}$$

for all $h \in H(\mathscr{R})$. The primeness of $\mathscr{P}$ and the conditions $\mathrm{char}(\mathscr{R}/\mathscr{P}) \neq 2$, $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ forces that

$$[\mathscr{F}(h), h] \in \mathscr{P} \quad \text{for all} \quad h \in H(\mathscr{R}). \tag{2.39}$$

Now following the same lines of proof as we did after (2.18), we get either $\mathscr{R}/\mathscr{P}$ is a commutative integral domain or $\phi(\mathscr{R}) \subseteq \mathscr{P}$. This completes the proof of the theorem. □

Theorem 2.6. *Let $\mathscr{R}$ be a ring with involution $*$ of the second kind, $\mathscr{P}$ a prime ideal of $\mathscr{R}$ such that $S(\mathscr{R})\cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$ and char$(\mathscr{R}/\mathscr{P}) \neq 2$. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with a derivation ϕ, such that $[\mathscr{F}(x), x\circ x^*] \in \mathscr{P}$ for all $x \in \mathscr{R}$, then either $\mathscr{R}/\mathscr{P}$ is a commutative integral domain or $\phi(\mathscr{R}) \subseteq \mathscr{P}$.*

Proof. Given that

$$[\mathscr{F}(x), x\circ x^*] \in \mathscr{P} \ \text{ for all } \ x \in \mathscr{R}. \tag{2.40}$$

Linearizing (2.40), we obtain

$$[\mathscr{F}(x), x\circ y^*] + [\mathscr{F}(x), y\circ x^*] + [\mathscr{F}(y), x\circ x^*]$$
$$+[\mathscr{F}(x), y\circ y^*] + [\mathscr{F}(y), y\circ x^*] + [\mathscr{F}(y), x\circ y^*] \in \mathscr{P} \tag{2.41}$$

for all $x, y \in \mathscr{R}$. Substituting $-y$ for y in (2.41) and combining it with (2.41), we get

$$2([\mathscr{F}(x), y\circ y^*] + [\mathscr{F}(y), y\circ x^*] + [\mathscr{F}(y), x\circ y^*]) \in \mathscr{P} \tag{2.42}$$

for all $x, y \in \mathscr{R}$. Since char$(\mathscr{R}) \neq 2$, this implies that

$$[\mathscr{F}(x), y\circ y^*] + [\mathscr{F}(y), y\circ x^*] + [\mathscr{F}(y), x\circ y^*] \in \mathscr{P} \tag{2.43}$$

for all $x, y \in \mathscr{R}$. Replacing x by xh_0 in (2.43) where $h_0 \in H(\mathscr{R}) \cap Z(\mathscr{R})$, we obtain

$$[\mathscr{F}(x), y\circ y^*]h_0 + [x, y\circ y^*]\phi(h_0) + [\mathscr{F}(y), y\circ x^*]h_0 + [\mathscr{F}(y), x\circ y^*]h_0 \in \mathscr{P} \tag{2.44}$$

for all $x, y \in \mathscr{R}$. Application of (2.43) into (2.44), yields that

$$[x, y\circ y^*]\phi(h_0) \in \mathscr{P} \ \text{ for all } \ x, y \in \mathscr{R}. \tag{2.45}$$

Taking $h_0 = k_0^2$ in (2.45), we get

$$2[x, y\circ y^*]\phi(k_0)k_0 \in \mathscr{P} \ \text{ for all } \ x, y \in \mathscr{R}.$$

for all $x, y \in \mathscr{R}$. This further gives that

$$[x, y\circ y^*]\phi(k_0) \in \mathscr{P} \ \text{ for all } \ x, y \in \mathscr{R}. \tag{2.46}$$

Replacing x by xk_0 in (2.43), where $k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, find that

$$[\mathscr{F}(x), y\circ y^*]k_0 + [x, y\circ y^*]\phi(k_0) - [\mathscr{F}(y), y\circ x^*]k_0 + [\mathscr{F}(y), x\circ y^*]k_0 \in \mathscr{P} \tag{2.47}$$

for all $x, y \in \mathscr{R}$. Using (2.46) in (2.47), we obtain

$$[\mathscr{F}(x), y\circ y^*]k_0 - [\mathscr{F}(y), y\circ x^*]k_0 + [\mathscr{F}(y), x\circ y^*]k_0 \in \mathscr{P} \tag{2.48}$$

for all $x, y \in \mathscr{R}$. Since $\mathscr{P}$ is a prime ideal of $\mathscr{R}$ and $S(\mathscr{R}) \cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$, we conclude that

$$[\mathscr{F}(x), y\circ y^*] - [\mathscr{F}(y), y\circ x^*] + [\mathscr{F}(y), x\circ y^*] \in \mathscr{P} \tag{2.49}$$

for all $x, y \in \mathscr{R}$. Subtracting (2.49) from (2.43), we obtain

$$2[\mathscr{F}(y), y\circ x^*] \in \mathscr{P}$$

for all $x, y \in \mathscr{R}$. This implies that

$$[\mathscr{F}(y), y\circ x^*] \in \mathscr{P} \ \text{ for all } \ x, y \in \mathscr{R}. \tag{2.50}$$

Substituting x by h_0 in (2.50) where $h_0 \in H(\mathscr{R}) \cap Z(\mathscr{R})$ and using the hypothesis of $S(\mathscr{R})\cap Z(\mathscr{R}) \nsubseteq \mathscr{P}$, implies that $[\mathscr{F}(y), y] \in \mathscr{P}$ for all $y \in \mathscr{R}$. Thus in view of [23, Lemma 1.1], we conclude our result. This completes the proof of the theorem. □

3. Applications

This section presents applications of the results proved in Section 2. Primarily, we explore the structure of prime rings with involution and find the specific forms of generalized derivations. Throughout the section, Q will denote the ring of quotients of $\mathscr{R}$ with center C. The center C of Q is known as the extended centroid of $\mathscr{R}$. It is well-known that if $\mathscr{R}$ is a prime ring, then Q is also a prime ring (see [4] for details). In [5], Bell and Daif investigated a certain kind of commutativity preserving maps as follows: Let S be a subset of R. A map $f : S \to \mathscr{R}$ is called strong commutativity preserving (SCP) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. Precisely, they proved that if a semiprime ring $\mathscr{R}$ admits a derivation which is SCP on a right ideal ρ, then $\rho \subseteq Z(\mathscr{R})$. In particular, $\mathscr{R}$ is commutative if $\rho = \mathscr{R}$. In [12], Deng and Ashraf proved that if there exist a derivation d of a semiprime ring $\mathscr{R}$ and a map $f : I \to \mathscr{R}$ defined on a nonzero ideal I of $\mathscr{R}$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then $\mathscr{R}$ contains a nonzero central ideal. In particular, they showed that R is commutative if $I = \mathscr{R}$. In [18–20], authors have studied SCP conditions for generalized derivations on prime and semiprime rings. On the other hand, Ali et al. [2] established a relationship between the commutativity of a prime ring R with involution $*$ involving strong commutativity preserving mappings. The following result is a natural generalization of the classical theorem proved in [2, Theorem 1].

Theorem 3.1. *Let $\mathscr{R}$ be a prime ring of char$(\mathscr{R}) \neq 2$ with involution $*$ of the second kind. If $\mathscr{R}$ admits two generalized derivations $\mathscr{F}$ and $\mathscr{G}$ associated with two derivations ϕ and ψ respectively, such that such that $\mathscr{F} \neq 1_{\mathscr{R}}$ or $\mathscr{G} \neq 1_{\mathscr{R}}$ satisfying the condition $[\mathscr{F}(x), \mathscr{G}(x^*)] \pm [x, x^*] = 0$ for all $x \in \mathscr{R}$, then either $\mathscr{R}$ is a commutative integral domain or there exist $a, b \in C$ such that $\mathscr{F}(x) = ax$ and $\mathscr{G}(x) = bx$ for all $x \in \mathscr{R}$ with $ab = \mp 1$.*

Proof. First, we consider the case

$$[\mathscr{F}(x), \mathscr{G}(x^*)] + [x, x^*] = 0 \tag{3.1}$$

for all $x \in \mathscr{R}$. Application of Theorem 2.1 yields that $\mathscr{R}$ is a commutative integral domain or $\phi = \psi = 0$. The latter case gives us $\mathscr{F}$ and $\mathscr{G}$ are left centralizers of $\mathscr{R}$. In view of [3, Lemma 2.3], there exist $a, b \in Q$ such that $\mathscr{F}(x) = ax$ and $\mathscr{G}(x) = bx$ for all $x \in \mathscr{R}$. Therefore, (3.1) becomes

$$[ax, bx^*] + [x, x^*] = 0 \tag{3.2}$$

for all $x \in \mathscr{R}$. A direct linearization of (3.2) yields that

$$[ax, by^*] + [ay, bx^*] + [x, y^*] + [y, x^*] = 0 \tag{3.3}$$

for all $x, y \in \mathscr{R}$. Replacing x by xk_0, where $0 \neq k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we obtain

$$([ax, by^*] - [ay, bx^*] + [x, y^*] - [y, x^*])k_0 = 0$$

for all $x, y \in \mathscr{R}$, which gives that

$$[ax, by^*] - [ay, bx^*] + [x, y^*] - [y, x^*] = 0 \tag{3.4}$$

for all $x, y \in \mathscr{R}$. Combining (3.3) and (3.4), we get

$$2([ax, by^*] + [x, y^*]) = 0$$

for all $x, y \in \mathscr{R}$. This implies that

$$[ax, by] + [x, y] = 0 \tag{3.5}$$

for all $x, y \in \mathscr{R}$. If $a = 0$ or $b = 0$, then $\mathscr{R}$ is commutative. Thus we may now assume that $a \neq 0$ and $b \neq 0$. Substituting yr for y in (3.5), we get

$$by[ax, r] + y[x, r] = 0$$

for all $x, y, r \in \mathscr{R}$. In particular, for $x = k_0$, where $0 \neq k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we have

$$by[a, r] = 0$$

for all $y, r \in \mathscr{R}$. Since $b \neq 0$ and $\mathscr{R}$ is a prime ring, we conclude that $a \in C$. Similarly, we can also prove that $b \in C$. From (3.5), we have

$$ab[x, y] + [x, y] = 0$$

for all $x, y \in \mathscr{R}$ and hence for all $x, y \in Q$ (see [4, Theorem 6.4.4]). So, the above expression can be written as

$$(ab + 1)[x, y] = 0$$

for all $x, y \in Q$. The primeness of Q forces that Q is commutative or $ab + 1 = 0$, which is the required result.
By using the similar arguments, we can prove the result for the case $[\mathscr{F}(x), \mathscr{G}(x^*)] - [x, x^*] = 0$ for all $x \in \mathscr{R}$. Thereby the proof of the theorem is now completed. □

Corollary 3.2. *Let $\mathscr{R}$ be a prime ring of char($\mathscr{R}$) $\neq 2$ with involution $*$ of the second kind. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with derivation ϕ such that $[\mathscr{F}(x), \mathscr{F}(x^*)] \pm [x, x^*] = 0$ for all $x \in \mathscr{R}$, then either $\mathscr{R}$ is a commutative integral domain or there exists $a \in C$ such that $\mathscr{F}(x) = ax$ for all $x \in \mathscr{R}$ with $a^2 = \mp 1$.*

Corollary 3.3 ([11, Theorem 2.3])**.** *Let $\mathscr{R}$ be a noncommutative prime ring of char($\mathscr{R}$) $\neq 2$ with involution $*$ of the second kind. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with derivation ϕ such that $[\mathscr{F}(x), \mathscr{F}(x^*)] - [x, x^*] = 0$ for all $x \in \mathscr{R}$, then $\mathscr{F}(x) = x$ for all $x \in \mathscr{R}$ or $\mathscr{F}(x) = -x$ for all $x \in \mathscr{R}$.*

Theorem 3.4. *Let $\mathscr{R}$ be a prime ring of char($\mathscr{R}$) $\neq 2$ with involution $*$ of the second kind. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with derivation ϕ such that $[\mathscr{F}(x \circ x^*), x] = 0$ for all $x \in \mathscr{R}$, then either $\mathscr{R}$ is a commutative integral domain or there exists $a \in C$ such that $\mathscr{F}(x) = ax$ for all $x \in \mathscr{R}$.*

Proof. By the assumption, we have

$$[\mathscr{F}(x \circ x^*), x] = 0 \tag{3.6}$$

for all $x \in \mathscr{R}$. In view of Theorem 2.4, $\mathscr{R}$ is commutative integral domain or $\phi = 0$. If $\phi = 0$, then $\mathscr{F}$ is a left centralizer of $\mathscr{R}$, and hence there exists $a \in Q$ such that $\mathscr{F}(x) = ax$ [3, Lemma 2.3]. Thus, the relation (3.6) can be written as

$$[a(x \circ x^*), x] = 0 \tag{3.7}$$

for all $x \in \mathscr{R}$. Writing $x + y$ in place of x in (3.7), we get

$$[a(x \circ y^*), x] + [a(x \circ x^*), y] + [a(y \circ x^*), x] + [a(x \circ y^*), y]$$

$$+[a(y \circ y^*), x] + [a(y \circ x^*), y] = 0 \tag{3.8}$$

for all $x, y \in \mathscr{R}$. Taking $x = -x$ in (3.8) and combining it with the obtained relation, we obtain

$$2([a(x \circ y^*), x] + [a(x \circ x^*), y] + [a(y \circ x^*), x]) = 0$$

for all $x, y \in \mathscr{R}$. Since char($\mathscr{R}$) $\neq 2$, it follows that

$$[a(x \circ y^*), x] + [a(x \circ x^*), y] + [a(y \circ x^*), x] = 0 \tag{3.9}$$

for all $x, y \in \mathscr{R}$. In particular, for $x = k_0$, where $0 \neq k_0 \in S(\mathscr{R}) \cap Z(\mathscr{R})$, we have

$$-2[a, y]k_0^2 = 0$$

for all $y \in \mathscr{R}$. Using the hypotheses of the theorem, we conclude that $a \in C$. This proves the theorem. □

The proof of the following theorems are almost identical to the proof of Theorem 3.4, so we write these results without proof.

Theorem 3.5. *Let $\mathscr{R}$ be a prime ring of char($\mathscr{R}$) $\neq 2$ with involution $*$ of the second kind. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with derivation ϕ such that $[\mathscr{F}(x) \circ x^*, x] = 0$ for all $x \in \mathscr{R}$, then either $\mathscr{R}$ is a commutative integral domain or there exists $a \in C$ such that $\mathscr{F}(x) = ax$ for all $x \in \mathscr{R}$.*

Theorem 3.6. *Let $\mathscr{R}$ be a prime ring of char($\mathscr{R}$) $\neq 2$ with involution $*$ of the second kind. If $\mathscr{R}$ admits a generalized derivation $\mathscr{F}$ associated with derivation ϕ such that $[\mathscr{F}(x), x \circ x^*] = 0$ for all $x \in \mathscr{R}$, then either $\mathscr{R}$ is a commutative integral domain or there exists $a \in C$ such that $\mathscr{F}(x) = ax$ for all $x \in \mathscr{R}$.*

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