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Research Article

On T_1-reflection of topological spaces

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Abstract

This paper deals with some universal spaces. For every topological space X, the universal T_1 space is viewed as the bottom element of the lattice $\mathcal{L}_X$. The class of morphisms in **Top** orthogonal to all T_1 spaces is characterized. Also, we introduce some new separation axioms and characterize them. Moreover, we characterize topological spaces X for which the universal T_1 space associated with X is a spectral space. Finally, we give some characterizations of topological spaces such that their T_1-reflection are compact spaces.

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1. Introduction

It is well known that the subcategory $\mathbf{Top_1}$, whose objects are T_1-spaces, is a reflective subcategory in the category **Top** of all topological spaces. Let us recall the T_1-reflection of a given topological space. Starting from a topological space (X,τ), we take R_X the smallest *closed equivalence relation* defined on X which is the intersection of all closed equivalence relations on X (an equivalence relation on X is said to be closed if its equivalence classes are all closed in (X,τ)). The quotient space X/R_X with the quotient topology is the T_1-reflection of the topological space (X,τ) [10, 14, 20, 21].

If we denote the canonical surjection from X to X/R_X by μ_X , then for every continuous map f from (X,τ) to a given topological space (Y,γ) there exists a unique continuous map denoted by $T_1(f)$ from $T_1(X)$ (the T_1-reflection of (X,τ)) to $T_1(Y)$ making commutative the following diagram

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$$\begin{array}{ccc} (X,\tau) & \xrightarrow{\quad f \quad} & (Y,\gamma) \\ \mu_X \downarrow & \triangledown & \downarrow \mu_Y \\ (X/R_X, \tau/R_X) & \xrightarrow{T_1(f)} & (Y/R_Y, \gamma/R_Y) \end{array}$$

From the above properties, it is clear that T_1 is a covariant functor from the category of topological spaces **Top** into the full subcategory **Top₁** of **Top** whose objects are T_1-spaces (for more information see [13, 16, 21]).

The *lattice* structure of topologies on a set X has been studied for over five decades. Recall that a lattice is a partially ordered set in which every two elements have a unique supremum (also known as a least upper bound or join) and a unique infimum (also known a greatest lower bound or meet). Lattice theory is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra [15, 19, 22].

A *distributive lattice* is a lattice in which the operations of join and meet distribute over each other. A *complete lattice* is a partially ordered set in which all subsets have both a supremum (join) and an infimum (meet) [15].

This paper consists of four investigations. In the first one, we introduce the concept of $T_1(X)$ as a bottom element of the lattice $\mathcal{L}_X$ and we show that $\mathcal{L}_X$ is complete and non distributive. In the second investigation, some new separation axioms as $T_{(1,2)}$, $T_{(1,\rho)}$ and $T_{(1,S)}$ are introduced and characterized. The third investigation focuses on some categorical properties of the category T_1; more precisely, a characterization of the class of morphisms in **Top** rendered invertible by the functor T_1 is given. Finally, the fourth investigation focuses on topological spaces X in which $T_1(X)$ is a spectral space.

2. T_1-reflection and lattice

Let (X,τ) be a topological space. We denote by $\mathcal{L}_X$ the collection of all quotient topologies on X by closed equivalence relations. That is $\mathcal{L}_X = \{(X/r, \tau/r) : r$ is a closed equivalence relation on $X\}$. Since, given an equivalence relation on X means exactly the data of a partition of X, then we can define the binary relation $\leq$ on $\mathcal{L}_X$ by: If we set $\Sigma = (\Sigma^x)_{x\in X}$, then

$$\Sigma \leq \Sigma^* \text{ if and only if each element of } \Sigma^* \text{ is a union of elements of } \Sigma.$$

Equivalently,

$$\Sigma \leq \Sigma^* \text{ if and only if } \Sigma^x \subseteq (\Sigma^*)^x \text{ for any } x \in X$$

Proposition 2.1. *Let (X,τ) be a topological space and $\mathcal{L}_X$ its related lattice. Then, the following properties hold:*

(1) *$(\mathcal{L}_X, \leq)$ is a complete lattice.*
(2) *$T_1(X) = \mathbf{0}$ is the infimum (bottom) of $(\mathcal{L}_X, \leq)$.*
($\mathbf{0}$ is the quotient space $(X/R_X, \tau/R_X)$ of (X,τ), where R_X is the intersection of all closed equivalence relations).

Proof. An immediate consequence of the fact that:

(1) $\bigwedge(\Sigma_i)_{i\in I} = (X/r, \tau/r)$, where r is the intersection of all $r_i, i \in I$.
(2) $\bigvee(\Sigma_i)_{i\in I}$ is the intersection of all elements of $(\mathcal{L}_X, \leq)$ greater or equal to all Σ_i.

□

Remark 2.2. X with the indiscrete topology denoted by $\mathbf{1}$ is the supremum (top) of $(\mathcal{L}_X, \leq)$.

In the rest of this paper, given a topological space X, we denote the bottom element of $\mathcal{L}_X$ by $\mathbf{0} = (\mathbf{0}^x, x \in X)$.

Example 2.3. Let $X = \{a, b, c, d\}$ with the topology such that its family of closed subsets is:

$$\tau^c = \{\emptyset, X, \{a, b, c\}, \{a, b, d\}, \{a, b\}, \{c, d\}, \{c\}, \{d\}\}.$$

We can see that $\mathcal{L}_X = \{\mathbf{1}, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$ such that:

- $\mathbf{1} = X$
- $\Sigma_1 = \{\{a, b, c\}, \{d\}\}$
- $\Sigma_2 = \{\{a, b\}, \{c, d\}\}$
- $\Sigma_3 = \{\{a, b, d\}, \{c\}\}$
- $\Sigma_4 = \{\{a, b\}, \{c\}, \{d\}\}$

We have $\Sigma_4 \leq \Sigma_1$, $\Sigma_4 \leq \Sigma_2$ and $\Sigma_4 \leq \Sigma_3$. So that, Σ_4 is the bottom of $\mathcal{L}_X$ and then $\mathbf{0} = \Sigma_4$

Proposition 2.4. *Let X be a non empty set.*

(1) *If $\mid X \mid \leq 2$, then $(\mathcal{L}_X, \leq)$ is a distributive lattice.*
(2) *If $\mid X \mid \geq 3$, then there exists at least a topology on X such that $(\mathcal{L}_X, \leq)$ is not a distributive lattice.*

Proof. (1) If $\mid X \mid \leq 2$, then $(\mathcal{L}_X, \leq)$ is either $\{0 = 1\}$ or $\{0 \neq 1\}$ and thus a distributive lattice.

(2) Suppose $\mid X \mid \geq 3$. Let $X = \{a, b, c\} \cup Y$ such that a, b, c are three element not contained in Y. We equip X by the topology with nonempty closed sets which are union of elements of the following family:

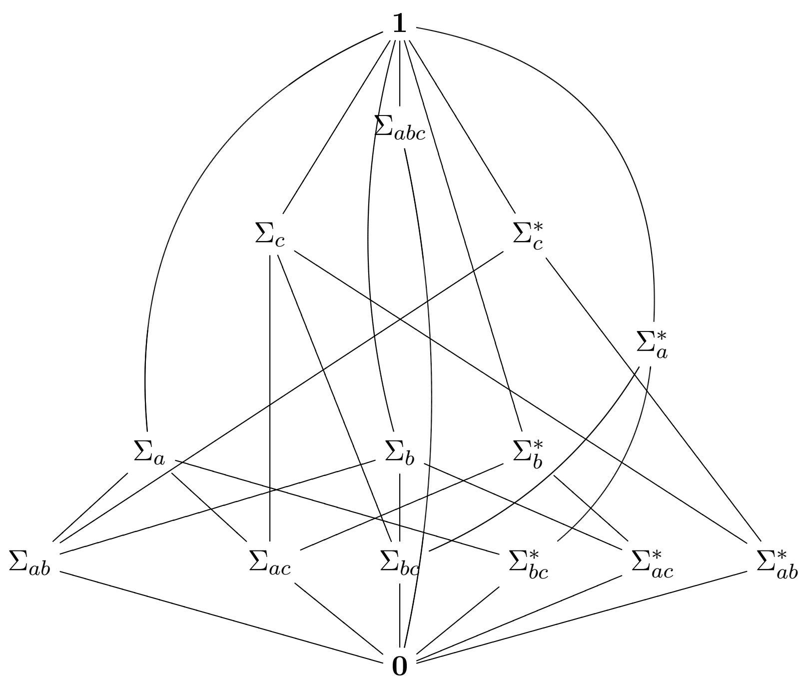
$$\{\{a\}, \{b\}, \{c\}, Y\}.$$

It is obviously seen that

$$(\mathcal{L}_X, \leq) = \{\mathbf{0}, \Sigma_{ij}, \Sigma^*_{ij}, \Sigma_i, \Sigma^*_i, \Sigma_{ijk}, \mathbf{1} : i, j, k \in \{a, b, c\}\}.$$

Where $\Sigma_{ij} = \{\{i\}, \{j\}, \{i, j\}^c\}$, $\Sigma^*_{ij} = \{\{i, j\}, \{k\}, Y\}$, $\Sigma_i = \{\{i\}, \{i\}^c\}$, $\Sigma^*_i = \{\{j, k\}, \{j, k\}^c\}$ and finally $\Sigma_{ijk} = \{\{a, b, c\}, Y\}$.

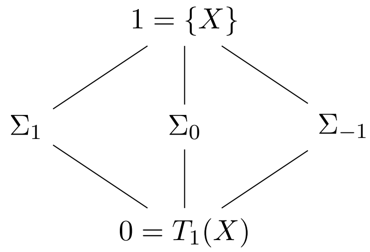
The lattice $(\mathcal{L}_X, \leq)$ can be described as follow:



We have $(\Sigma_a \wedge \Sigma_c^*) \vee \Sigma_b = \Sigma_b$ and $(\Sigma_a \vee \Sigma_b) \wedge (\Sigma_c^* \vee \Sigma_b) = \mathbf{1}$.

□

Example 2.5. Consider the topology in Proposition 2.4 when $X = \{-1, 0, 1\}$ ($|X| = 3$). So $X = \{-1, 0, 1\}$ is equipped by the discrete topology. Clearly $(\mathcal{L}_X, \leq) = \{\mathbf{0}, \Sigma_{-1}, \Sigma_0, \Sigma_1, \mathbf{1}\}$, where $\Sigma_i = \{\{i\}, \{j, k\}\}$ such that $\{i, j, k\} = \{-1, 0, 1\}$. Then $(\Sigma_{-1} \wedge \Sigma_0) \vee \Sigma_1 = \mathbf{0} \vee \Sigma_1 = \Sigma_1$, However, $(\Sigma_{-1} \vee \Sigma_1) \wedge (\Sigma_0 \vee \Sigma_1) = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}$. Such situation can be illustrated as follow:



3. Separation axioms

In [4], the authors have introduced the following separation axioms.

Definition 3.1. Let i, j be two integers such that $0 \leq i < j \leq 2$. A topological space X is said to be a $T_{(i,j)}$-space if its T_i-refection is a T_j-space. Thus, there are three new types of separation axioms; namely, $T_{(0,1)}, T_{(0,2)}$ and $T_{(1,2)}$.

More generally, those authors in [4] introduced the following definition.

Definition 3.2. Let F be a (covariant) functor from **Top** to itself, and P be a topological property. An object X of **Top** is said to be a $T_{(F,P)}$-object if $F(X)$ satisfies the property P.

Our goal, here, is to characterize topological spaces (X, τ) which is $T_{(1,i)}$ such that $i \in \{2, \rho, S\}$, where ρ is the Tychonoff functor and S is the Sobrification functor. We start with some definitions and properties which will be used along this paper.

Definition 3.3. Let (X, τ) be a topological space and C be a closed subset of X. C is called a z-closed set in X if and only if for any $x \in X$, $C \cap 0^x \neq \emptyset \Longrightarrow 0^x \subseteq C$.

Definition 3.4. Let (X, τ) be a topological space and O be an open subset of X. Then O is said to be a $z-$open set if its complementary is z-closed in X.

The collection of all z-open subsets of a topological space (X, τ) represents the open sets of a new topology on X denoted by τ_z. The topological space (X, τ_z) will be called the *z-space* of (X, τ).

Proposition 3.5. *Let X be a topological space and A a closed (resp. open) subset of X. Then*
A is z-closed (resp. z-open) if and only if $\mu_X^{-1}(\mu_X(A)) = A$.

Proof. Suppose that A is a z-closed. It is clear that $A \subseteq \mu_X^{-1}(\mu_X(A)$. Now, let $x \in \mu_X^{-1}(\mu_X(A)$. Then there exists $y \in A$ such that $\mu_X(x) = \mu_X(y)$. So that, $\Sigma^x = \Sigma^y \forall\ \Sigma \in \mathcal{L}_X$ and thus $x \in \Sigma^y \forall\ \Sigma\ \in\ \mathcal{L}_X$. Therefore $x \in \mathbf{0}^y \subseteq A$ which implies the equality.
The converse is Straightforward.

□

Example 3.6. Let $X = \{0, 1\}$ the Sierpinski space with the topology $\tau_X = \{\emptyset, \{0\}, X\}$. Then $T_1(X)$ is a one point set and thus $\mu_X^{-1}(\mu_X(\{1\})) = X$. So that $\{1\}$ is a closed set but it is not z-closed.

Corollary 3.7. *Let (X, τ) be a topological space. If A is a z-open (resp. z-closed) subset of X, then $\mu_X(O)$ is open (resp. closed) in $T_1(X)$.*

Theorem 3.8. *Let (X, τ) be a topological space. Then the following properties are equivalent:*

(1) *X is a $T_{(1,2)}$-space;*
(2) *If $x, y \in X$ such that $\mathbf{0}^x \neq \mathbf{0}^y$, then there exists two disjoint z-open sets U and V such that $x \in U$ and $y \in V$.*

Proof.
(1) $\Rightarrow$ (2) Suppose that $T_1(X)$ is a T_2-space. If $x, y \in X$ such that $\mathbf{0}^x \neq \mathbf{0}^y$, then $\overline{x} \neq \overline{y}$ in $T_1(X)$. So that there exist $O_1, O_2 \in \tau_{/R}$ which are disjoint and $\overline{x} \in O_1$, $\overline{y} \in O_2$. Thus $x \in \mu_X^{-1}(O_1)$, $y \in \mu_X^{-1}(O_2)$ and $\mu_X^{-1}(O_1), \mu_X^{-1}(O_2)$ are two disjoint z-open sets.
(2) $\Rightarrow$ (1) Let $\overline{x} \neq \overline{y}$ in $T_1(X)$ which means that $\mathbf{0}^x \neq \mathbf{0}^y$. Then there exist two disjoint z-open sets U and V such that $x \in U$ and $y \in V$ and then $\mu_X(U), \mu_X(V)$ are two disjoint open sets in $T_1(X)$ containing respectively x and y. This fact completes the proof.

□

Theorem 3.9. *Let (X, τ) be a topological space. Then the following properties are equivalent:*

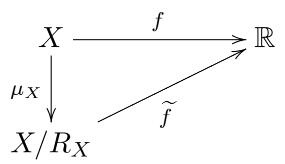
(1) *X is a $T_{(1,\rho)}$-space;*
(2) *For every z-closed A and every $x \in X \backslash A$ there exists $f \in \mathcal{C}(X)$ satisfying $f(A) = \{0\}$ and $f(x) = 1$.*

Proof.

1) $\Rightarrow$ 2) :

Let A be z-closed and $x \in X \backslash A$. Then $\mu_X(A)$ is a closed set in $T_1(X)$ and $\overline{x} \notin \mu_X(A)$. Since $T_1(X)$ is a Tychonoff space then there exists a continuous map $f : X/R_X \to \mathbb{R}$ such that $f(\mu_X(A)) = \{0\}$ and $f(\overline{x}) = 1$. So that the map $f \circ \mu_X$ satisfies the condition in 2).
2) $\Rightarrow$ 1) :

Let A be closed in $T_1(X)$ and $\overline{x} \notin A$. Then $\mu_X^{-1}(A)$ is z-closed and $x \notin \mu_X^{-1}(A)$. Using the second item, there is $f \in \mathcal{C}(X)$ such that $f\left(\mu_X^{-1}(A)\right) = \{0\}$ and $f(x) = 1$. By the reflection of $\mathbf{Top_1}$ in $\mathbf{Top}$ there exists a unique continuous function $\widetilde{f} \in \mathcal{C}\left(X/R_X\right)$ such that $\widetilde{f} \circ \mu_X = f$.



Thus $\widetilde{f}(A) = \widetilde{f}\left(\mu_X\left(\mu_X^{-1}(A)\right)\right) = f\left(\mu_X^{-1}(A)\right) = \{0\}$ and $\widetilde{f}\left(\overline{x}\right) = \widetilde{f}(\mu_X(x)) = f\left(x\right) = 1$. This fact complete the proof.

□

The next goal is to characterize topological spaces such that their T_1 reflection give a sober space. Recall that a topological space is said to be sober if every irreducible closed set is a closure of a singleton (a subset C of a topological space X is said to be irreducible if and only if, for any two open sets O_1, O_2 of X, if $C \cap O_1 \neq \emptyset$ and $C \cap O_2 \neq \emptyset$ then $C \cap O_1 \cap O_2 \neq \emptyset$.)

Proposition 3.10. *Let (X, τ) be a topological space and A be a subset of $T_1(X)$. Then the following properties are equivalent:*

(1) *A is irreducible.*
(2) *$\mu_X^{-1}(A)$ is irreducible in the z-space.*

Proof. Suppose A is irreducible in $T_1(X)$.

Let O_1, O_2 be z-open sets such that $\mu_X^{-1}(A) \cap O_1 \neq \emptyset$ and $\mu_X^{-1}(A) \cap O_2 \neq \emptyset$. Since $\mu_X\left(\mu_X^{-1}(A) \cap O_i\right) = A \cap \mu_X(O_i)$ and $\mu_X(O_i)$ is open in $T_1(X)$, then $A \cap \mu_X(O_1) \cap \mu_X(O_2) \neq \emptyset$. Thus $\mu_X^{-1}(A \cap \mu_X(O_1) \cap \mu_X(O_2)) = \mu_X^{-1}(A) \cap O_1 \cap O_2 \neq \emptyset$. So that, $\mu_X^{-1}(A)$ is irreducible in the z-space.

Conversely, suppose $\mu_X^{-1}(A)$ is irreducible in the z-space.

Let A_1, A_2 be open in $T_1(X)$ such that $A \cup A_1 \neq \emptyset$ and $A \cup A_2 \neq \emptyset$. Since $\mu_X^{-1}(A \cup A_i) = \mu_X^{-1}(A) \cup \mu_X^{-1}(A_i)$ and $\mu_X^{-1}(A_i)$ is open in the z-space, then $\mu_X^{-1}(A) \cup \mu_X^{-1}(A_1) \cup \mu_X^{-1}(A_2) \neq \emptyset$. Thus,
$\mu_X\left(\mu_X^{-1}(A) \cup \mu_X^{-1}(A_1) \cup \mu_X^{-1}(A_2)\right) = A \cap A_1 \cap A_2 \neq \emptyset$. So that, A is irreducible in $T_1(X)$. □

The following result gives the required characterization. First, we can note that $T_1(X)$ is sober if and only if the unique irreducible closed sets are the singletons (every singleton set in a T_1-space is closed).

Theorem 3.11. *Let (X, τ) be a topological space. Then the following properties are equivalent:*

(1) *$T_1(X)$ is sober.*
(2) *If F is irreducible and closed in the z-space then $\mathbf{0}^x = \mathbf{0}^y$ for all $x, y \in F$.*

Proof. Suppose $T_1(X)$ is sober. Let F be an irreducible z-closed. Then $\mu_X(F)$ is an irreducible closed set in $T_1(X)$. Since $T_1(X)$ is sober, there exists $\overline{x} \in T_1(X)$ such that $\mu_X(F) = \overline{\{\overline{x}\}} = \{\overline{x}\}$. Finally, $F = \mu_X^{-1}(\{\overline{x}\})$, so $\mathbf{0}^x = \mathbf{0}^y$ for all $x, y \in F$.

Conversely, if A is irreducible and closed in $T_1(X)$ then by Proposition 3.10 $\mu_X^{-1}(A)$ is irreducible. Since $\mu_X^{-1}(A)$ is z-closed then using the second item in the theorem we have $\mathbf{0}^x = \mathbf{0}^y$ for all $x, y \in \mu_X^{-1}(A)$. This fact implies that A is a singleton which complete the proof. □

It is known that Tychonoff spaces are Hausdorff and Hausdorff spaces are all sober. Then the following result is immediate.

Proposition 3.12.

$$T_{(1,\rho)} \Longrightarrow T_{(1,2)} \Longrightarrow T_{(1,S)}$$

Remark 3.13. The converse of the implications in Proposition 3.12 does not hold.

Examples 3.14.

- Let $X = \mathbb{R} \cup \{\alpha\}$ such that $\alpha \notin \mathbb{R}$. We equip X by the topology containing all open sets in the standard topology of the real line and also all cofinite sets containing α. X with this topology is a T_1-space which is sober but not T_2. We deduce that $T_{(1,S)}$ does not implies $T_{(1,2)}$.
- Let $X = \{(a, b) \in \mathbb{R} \ : \ b \geq 0\}$ and τ the topology on X defined by it's basis containing:
  - all open discs contained in X.
  - all sets of the form $U_{a,r} = B((a, 0), r) \cap X \setminus \{(x, 0) \ : \ x \in [a - r, a + r] \setminus \{a\}\}$ such that $r \in \mathbb{R}_+^*$

  (X, τ) is a T_2-space which is not Tychonoff. We deduce that $T_{(1,2)}$ does not implies $T_{(1,\rho)}$.

4. The class of continuous maps orthogonal to all T_1 spaces

It is worth noting that reflective subcategories arise throughout mathematics, via several examples such as the free group and free ring functors in algebra, various compactification functors in topology, and completion functors in analysis: cf. [20, p.90]. Recall from [20, p.89] that a subcategory D of a category C is called reflective (in C) if the inclusion functor $I : D \longrightarrow C$ has a left adjoint functor $F : C \longrightarrow D$; i.e., if, for each object A of C, there exist an object $F(A)$ of D and a morphism $\mu_A : A \longrightarrow F(A)$ in C such that, for each object X in D and each morphism $f : A \longrightarrow X$ in C, there exists a unique morphisme $f : F(A) \longrightarrow X$ in D such that $\widetilde{f} \circ \mu_A = f$.

The concept of reflections in categories has been investigated by several authors (see for example [8, 9, 11, 12, 16, 17] and [21]). This concept serves the purpose of unifying various constructions in mathematics. Historically, the concept of reflections in categories seems to have its origin in the universal extension property of the Stone-Cech compactification of a Tychonoff space.

A morphism $f : A \longrightarrow B$ and an object X in a category C are called orthogonal [13], if the mapping $hom_C(f; X) : hom_C(B; X) \longrightarrow hom_C(A; X)$ that takes g to gf is bijective. For a class of morphisms σ (resp., a class of objects D), we denote by $\sigma^{\perp}$ the class of objects orthogonal to every f in σ (resp., by $D^{\perp}$ the class of morphisms orthogonal to all X in D) [13]. The orthogonality class of morphisms $D^{\perp}$ associated with a reflective subcategory D of a category C satisfies the following identity $D^{\perp\perp} = D$ [1, Proposition 2.6]. Thus, it is of interest to give explicitly the class $D^{\perp}$. Note also that, if $I : D \longrightarrow C$ is the inclusion functor and $F : C \longrightarrow D$ is a left adjoint functor of I, then the class $D^{\perp}$ is the collection of all morphisms of C rendered invertible by the functor F [1, Proposition 2.3]. This section is devoted to the study of the orthogonal class $T_1^{\perp}$; hence we will provide a characterization of morphisms rendered invertible by the functor T_1. The following concepts are needed.

Definition 4.1. Let $f : X \to Y$ be a continuous map between two topological spaces.

(1) f is said to be $l-injective$ (or $l-one-to-one$) if for each $x, y \in X$ and each $\Sigma = (\Sigma^x, x \in X) \in \mathcal{L}_X$, we have

$$\Sigma^x \neq \Sigma^y \Longrightarrow \exists\, \Sigma^* \in \mathcal{L}_Y \; satisfying \; (\Sigma^*)^{f(x)} \neq (\Sigma^*)^{f(y)}.$$

(2) f is said to be $l-surjective$ (or $l-onto$) if for each $y \in Y$

$$\exists\, x \in X \; such\, that \; \forall\, \Sigma^* \in \mathcal{L}_Y \; we\ have \; (\Sigma^*)^{f(x)} = (\Sigma^*)^{y}$$

(3) f is said to be $l-bijective$ if it is both $l-injective$ and $l-surjective$.

Examples 4.2.

(1) Every onto continuous map is $l-$onto.

(2) A $l-$onto continuous map may not be onto.
Let $X = \{0, 1\}$ with the discreet topology and $Y = \{a, b, c\}$ with the topology $\tau = \{\emptyset, Y, \{a\}, \{b, c\}\}$. Let $f : X \to Y$ be defined by $f(0) = a$ and $f(1) = b$.
Clearly, f is a continuous map which is not onto.
Since $\mathcal{L}_Y = \{\{X\}, \{\{a\}, \{b, c\}\}\}$ then f is $l-$onto.

(3) A one-to-one continuous map need not be $l-$one-to-one.
Let $X = \{0, 1\}$ be equipped with the discrete topology and $Y = \{0, 1\}$ be equipped with the trivial topology. Let $f = 1_X : X \to Y$. Then f is one-to-one and continuous.
However, there exists $\Sigma = \{\{0\}, \{1\}\} \in \mathcal{L}_X$, $\Sigma^0 \neq \Sigma^1$ but $(\Sigma^*)^{f(0)=0} = (\Sigma^*)^{f(1)=1} = \{0, 1\}$ for all $\Sigma^* \in \mathcal{L}_Y = \{\{X\}\}$.

(4) A $l-$one-to-one continuous map need not be one-to-one.
Let X be a topological space which is not a T_1-space. Of course, μ_X is a $l-$injective continuous map, but it is not injective.

Lemma 4.3. *Let $f : X \to Y$ be a continuous map. Then the following properties hold.*

(1) *f is $l-$injective if and only if $T_1(f)$ is injective.*
(2) *f is $l-$surjective if and only if $T_1(f)$ is surjective.*
(3) *f is $l-$bijective if and only if $T_1(f)$ is bijective.*
(4) *Let $g : Y \to Z$ be a continuous map. If two among f, g and $g \circ f$ are $l-$bijective, then it is the third one.*

Proof.

(1) Suppose that $T_1(f)$ is injective. Let $x, y \in X$ such that there exists $\Sigma \in \mathcal{L}_X$ satisfying $\Sigma^x \neq \Sigma^y$. It follows that $\mu_X(x) \neq \mu_X(y)$. Consequently, $T_1(f)(\mu_X(x)) \neq T_1(f)(\mu_X(y))$. Hence, $\mu_Y(f(x)) \neq \mu_Y(f(y))$. Thus, there exists $\Sigma^* \in \mathcal{L}_Y$ such that $(\Sigma^*)^{f(x)} \neq (\Sigma^*)^{f(y)}$. Therefore, f is $l-$injective.
Conversely, suppose that f is $l-$injective. Let $x, y \in X$ such that $T_1(f)(\mu_X(x)) = T_1(f)(\mu_X(y))$. Then $\mu_Y(f(x)) = \mu_Y(f(y))$. Hence, for all $\Sigma^* \in \mathcal{L}_Y$ we have $(\Sigma^*)^{f(x)} = (\Sigma^*)^{f(y)}$. Since f is $l-$injective, we conclude that, for all $\Sigma \in \mathcal{L}_X$ we have $\Sigma^x = \Sigma^y$, thus $\mu_X(x) = \mu_X(y)$. Therefore, $T_1(f)$ is one-to-one.
(2) Suppose that $T_1(f)$ is surjective. Let $y \in Y$. Then, there exists $x \in X$ such that $T_1(f)(\mu_X(x)) = \mu_Y(y)$. Hence, $\mu_Y(f(x)) = \mu_Y(y)$; consequently, for each $\Sigma^* \in \mathcal{L}_Y$, $(\Sigma^*)^{f(x)} = (\Sigma^*)^y$. Therefore, f is $l-$surjective.
Conversely, suppose that f is $l-$surjective. Let $y \in Y$. Then, there exists $x \in X$ such that for each $\Sigma^* \in \mathcal{L}_Y$, $(\Sigma^*)^{f(x)} = (\Sigma^*)^y$. Hence, $\mu_Y(f(x)) = \mu_Y(y) = T_1(f)(\mu_X(x))$. Thus, $T_1(f)$ is surjective.
(3) Direct consequence of (1) and (2).
(4) This is a direct consequence of (3) and the fact that T_1 is a functor.
Indeed, If f, g and $g \circ f$ are l-bijective then $T_1(f), T_1(g)$ and $T_1(g \circ f)$ are bijective. We have $T_1(g \circ f) = T_1(f) \circ T_1(g)$. Thus, if two among $T_1(f), T_1(g)$ and $T_1(g \circ f)$ are bijective then it is the third one.

□

Definition 4.4. Let X and Y be two topological spaces and f a map from X to Y. f is said to be z-closed if and only if $\mu_Y^{-1}(\mu_Y(f(C)))$ is z-closed for every z-closed $C \subseteq X$.

Remark 4.5. Let X and Y be two topological spaces and f a map from X to Y. It is clear that f is z-closed if and only if $T_1(f)$ is closed.

Now, we are in a position to present the main result of this paragraph. The following theorem characterizes all morphisms in **Top** rendered invertible by the functor T_1.

Theorem 4.6. *Let $f : X \longrightarrow Y$ be a continuous map. Then the following statements are equivalent:*

(1) *$T_1(f)$ is a homeomorphism;*
(2) *f is a l-bijective, z-closed map.*

5. T_1-spectral spaces

The main result of this section is the characterization of topological spaces X such that $T_1(X)$ is a spectral space.

Let us first recall that a topological space X is said to be spectral if the following axioms hold [18]:

(i) X is a sober space;
(ii) X is compact and has a basis of compact open sets;
(iii) the family of compact open sets of X is closed under finite intersections.

Let $Spec(R)$ denote the set of prime ideals of a commutative ring R with identity. Recall that, the Zariski topology also known as the hull-kernel topology for $Spec(R)$ is defined by letting $C \subseteq Spec(R)$ to be closed if and only if there exists an ideal $\mathcal{A}$ of R such that $C = \{\mathcal{P} \in Spec(R) : \mathcal{P} \supseteq \mathcal{A}\}$. Hochster [18] has proved that a topological space is homeomorphic to the prime spectrum of some ring equipped with the Zariski topology if and only if it is spectral. In lattice theory, a spectral space is characterized by the fact that it is homeomorphic to the prime spectrum of a bounded (with a 0 and a 1) distributive lattice.

Note that spectral spaces are of interest not only in (topological) ring and lattice theory, but also in computer science, in particular, in domain theory. In order to motivate the reader, we give some links between the previous axioms (i) and (ii) and functional analysis.

Bratteli and Elliott showed in [7] that a topological space X is homeomorphic to the primitive spectrum of an approximately finite-dimensional $C^\star$-algebra (called A:F $C^\star$-algebra) equipped with the Jacobson topology if and only if it has a countable basis and it satisfies the above axioms (i) and (ii). By the way, $C^\star$-algebra and foliation theory are strongly linked. Thus, there must be some link between spectral spaces and foliation theory; this was done by the authors of [5] and [6].

Recently, some authors (for example [2] and [3]) have been interested on particular type of spectral spaces constructed from some compactifications (namely, the one point-compactification for [3], and the Walman compactification and the T_0-compactification for [2]).

In order to pursue this type of investigations for spectral spaces, we are interested in topological spaces in which the T_1-reflection is a spectral space.

The following result represents a characterization of topological spaces such that their T_1-reflection are compact spaces.

Proposition 5.1. *Let X be a topological space. Then the following statements are equivalent.*

(i) *$T_1(X)$ is compact;*

(ii) *If $\{F_i : i \in I\}$ is a collection of z-closed subsets of X with $\bigcap_{i\in I} F_i = \emptyset$, then there is a finite subset $J \subset I$ such that $\bigcap_{i\in J} F_i = \emptyset$.*

(iii) *For all z-open cover of X there exists a finite z-open subcover.*

Proof.

$(i) \Rightarrow (ii)$

Let $\{F_i : i \in I\}$ be a collection of z-closed subsets of X such that $\cap\{F_i : i \in I\} = \emptyset$. Since $\mu_X(F_i)$ are closed in $T_1(X)$ which is compact, there exists a finite subset J of I such that $\cap\{\mu_X(F_i) : i \in J\} = \emptyset$. Thus, $\cap\{F_i : i \in J\} = \emptyset$.

$(ii) \Rightarrow (i)$

Let $\{F_i : i \in I\}$ be a collection of closed subsets of $T_1(X)$. Then, $\{\mu_X^{-1}(F_i) : i \in I\}$ is a collection of z-closed subsets of X and thus by (ii) there exists a finite subset J of I such that $\{\mu_X^{-1}(F_i) : i \in J\} = \emptyset$. Hence, $\{\mu_X(\mu_X^{-1}(F_i)) : i \in J\} = \emptyset$. Since μ_X is onto, we have $\{F_i : i \in J\} = \emptyset$. Therefore, $T_1(X)$ is a compact space.

$(ii) \Leftrightarrow (iii)$ It is sufficient to use the fact that the complement of a z-closed is a z-open. □

The previous result means that $T_1(X)$ is compact if and only if the z-space of (X, τ) is compact.

Remark 5.2. If X is compact then $T_1(X)$ is compact.

Example 5.3. The real line with the right topology is not compact but it's T_1-reflection is a singleton which is of course a compact space.

Proposition 5.4. *$\mu_X : (X, \tau_z) \to (T_1(X), \tau_{/R_X})$ is a continuous map.*

Proof. Straightforward. □

Proposition 5.5. *$\mu_X : (X, \tau_z) \to (T_1(X), \tau_{/R_X})$ is an open map.*

Proof. This is a direct deduction by Proposition 3.5. □

Recall that a continuous map $q : Y \to Z$ is said to be a *quasihomeomorphism* if $U \to q^{-1}(U)$ defines a bijection $\mathcal{O}(Z) \to \mathcal{O}(Y)$ [14], where $\mathcal{O}(Y)$ is the set of all open subsets of the space Y.

Proposition 5.6. *$\mu_X : (X, \tau_z) \to (T_1(X), \tau_{/R_X})$ is a quasi-homeomorphism.*

Proof. Let

$$\begin{array}{rcl} \Psi : \mathcal{C}(T_1(X)) & \to & \mathcal{C}(X) \\ C & \mapsto & \mu_X^{-1}(C) \end{array}$$

(1) **Ψ is Onto**: Let C be a $z-$closed subset of X, then $\mu_X^{-1}(\mu_X(C)) = C$ and thus, $\mu_X(C)$ is a closed subset of $T_1(X)$. Hence, there exists $V = \mu_X(C)$ such that $\Psi(V) = C$.

(2) **Ψ is One-to-one:** Let C_1 and C_2 be two closed subsets of $T_1(X)$ such that $\Psi(C_1) = \Psi(C_2)$ then $\mu_X^{-1}(C_1) = \mu_X^{-1}(C_2)$. Hence $C_1 = C_2$(since μ_X is onto).

□

Definition 5.7. Let (X, τ) be a topological space and A be a nonempty subset of X. We say that A is z-connexe if it is connexe in the z-space. Equivalently, if $A \subseteq O_1 \cup O_2$ then $A \subseteq O_1$ or $A \subseteq O_2$ for every disjoint z-open sets O_1, O_2.

Definition 5.8. Let (X, τ) be a topological space and A be a nonempty subset of X. We say that X is z-totally-disconnected if we have:

$$A \text{ is z-connexe} \Rightarrow \forall x, y \in A \text{ we have } \mathbf{0}^x = \mathbf{0}^y.$$

Theorem 5.9. *Let (X, τ) be a topological space. Then the following properties hold:*

(1) *If A is z-connexe then $\mu_X(A)$ is connexe in $T_1(X)$.*
(2) *If A is connexe in $T_1(X)$ then $\mu_X^{-1}(A)$ is z-connexe.*

Proof.

(1) Suppose A is z-connexe. Let $O_1, O_2 \in \tau/R_X$ be disjoint such that $\mu_X(A) \subseteq O_1 \cup O_2$. Then $A \subseteq \mu_X^{-1}(O_1) \cup \mu_X^{-1}(O_2)$. Since $\mu_X^{-1}(O_1), \mu_X^{-1}(O_2)$ are disjoint z-open and A is z-connexe then $A \subseteq \mu_X^{-1}(O_1)$ or $A \subseteq \mu_X^{-1}(O_2)$ and thus $\mu_X(A) \subseteq O_1$ or $\mu_X(A) \subseteq O_2$. So that $\mu_X(A)$ is connexe.
(2) Suppose A is connexe in $T_1(X)$. Let O_1, O_2 be two disjoint z-open sets which satisfy $\mu_X^{-1}(A) \subseteq O_1 \cup O_2$. So, $A \subseteq \mu_X(O_1) \cup \mu_X(O_2)$. Since A is connexe and $\mu_X(O_1), \mu_X(O_2)$ are disjoint and open in $T_1(X)$, then $A \subseteq \mu_X(O_1)$ or $A \subseteq \mu_X(O_2)$. Thus $\mu_X^{-1}(A) \subseteq O_1$ or $\mu_X^{-1}(A) \subseteq O_2$ and $\mu_X^{-1}(A)$ is z-connexe.

□

Lemma 5.10 ([18])**.** *Let X be a T_1-space. Then X is spectral if and only if X is compact and totally disconnected. (i.e., X is a Stone space)*

In the particular case of a T_1-space it will be sufficient to prove that (X, τ) is compact and totally disconnected.

Our goal now is to characterize topological space such that its T_1-reflection is a spectral space.

Theorem 5.11. *Let (X, τ) be a topological space. Then $(T_1(X), \tau_{/R_X})$ is a spectral space if and only if the following properties hold:*

(i) *For all z-open cover of X there exists a finite z-open subcover.*
(ii) *X is z-totally-disconnected.*

Proof. We will use the Lemma 5.10 for this proof.

" $\Longrightarrow$ " :

Suppose that $(T_1(X), \tau_{/R_X})$ is a spectral space. This is equivalent to say that $T_1(X)$ is compact and totally disconnected. By Proposition 5.1, we have $T_1(X)$ is compact, if and only if, for all z-open cover of X there exists a finite z-open subcover. It is now sufficient to prove that X is z-totally disconnected.

Let A be z-connexe and $x, y \in A$. Using the fact that $T_1(X)$ is totally disconnected and the Theorem 5.9 we have $\mu_X(A)$ is connexe in $T_1(X)$.

Now, since $T_1(X)$ is totally disconnected and $\mu_X(A)$ is connexe then $\overline{x} = \overline{y}$ which implies that $\mathbf{0}^x = \mathbf{0}^y$. Finally, X is z-totally disconnected.

" $\Longleftarrow$ " :

On one a hand, using the item **(i)** in the Theorem and the Proposition 5.1 we have $T_1(X)$ is compact. On an other hand we have to prove that $T_1(X)$ is totally disconnected.

Let A be connexe in $T_1(X)$ and $\overline{x}, \overline{y} \in A$. By Theorem 5.9 we have $\mu_X^{-1}(A)$ is z-connexe.

Now, since X is z-totally-disconnected and $x, y \in \mu_X^{-1}(A)$, then $\mathbf{0}^x = \mathbf{0}^y$. Thus $\overline{x} = \overline{y}$. This fact complete the proof.

□

6. Conclusion

Using Lattice's tools, we have succeeded to characterize the orthogonal of the subcategory of T_1-spaces in the category of all topological spaces. We have defined some new separation axioms related to the T_1-reflection of a topological space and studied their fundamental properties and characterizations. Also, we have characterized topological space such that its reflection in the subcategory of T_1-spaces gives a spectral space or a compact space.

In future work, we will research the possibility of similar work for the T_2-reflection of topological spaces.

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