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Hyper-Fibonacci and Hyper-Lucas Polynomials

Efruz Özlem Mersin

Department of Mathematics, Faculty of Science and Arts, Aksaray University, 68100, Aksaray, Turkey.

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Abstract. In this paper, hyper-Fibonacci and hyper-Lucas polynomials are defined and some of their algebraic and combinatorial properties such as the recurrence relations, summation formulas, and generating functions are presented. In addition, some relationships between the hyper-Fibonacci and hyper-Lucas polynomials are given.

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1. Introduction

The Fibonacci and Lucas number sequences, which are the most famous integer sequences, are defined by the formulas

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_0 = 0, \quad F_1 = 1,$$

and

$$L_{n+1} = L_n + L_{n-1} \quad \text{with} \quad L_0 = 2, \quad L_1 = 1$$

for $n \geq 1$ [12].

The Fibonacci and Lucas polynomials have attracted the attention of researchers as a generalization of the Fibonacci and Lucas number sequences [2–5, 9, 11, 13, 14, 16–19]. The Fibonacci and Lucas polynomials were defined by Catalan and Bicknell with initial conditions $F_0(x) = 0$, $F_1(x) = 1$, $L_0(x) = 2$ and $L_1(x) = x$, and for $n \geq 2$

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

and

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \tag{1.1}$$

respectively, where x is any variable quantity. The Fibonacci and Lucas polynomials give the Fibonacci and Lucas number sequences for $x = 1$, respectively. Moreover, they can also be extended to negative subscripts as

$$F_{-n}(x) = (-1)^{n-1} F_n(x) \quad \text{and} \quad L_{-n}(x) = (-1)^n L_n(x),$$

such as the Fibonacci and Lucas number sequences [10, 15]. Using the induction method, the following relationships can be obtained between the Fibonacci and Lucas polynomials:

$$xF_n(x) + L_n(x) = 2F_{n+1}(x) \tag{1.2}$$

Email address: efruzmersin@aksaray.edu.tr (E.Ö. Mersin)

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and

$$xF_n(x) - L_n(x) = -2F_{n-1}(x). \tag{1.3}$$

The generating functions for the Fibonacci and Lucas polynomials are as follows, respectively [11, 13]:

$$\sum_{n=0}^{\infty} F_n(x)\, t^n = \frac{t}{1 - xt - t^2}$$

and

$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{2 - xt}{1 - xt - t^2}.$$

As another generalization of the Fibonacci and Lucas number sequences, we encounter hyper-Fibonacci and hyper-Lucas numbers. These numbers were defined by Dil and Mező [6], by the formulas

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)} \quad \text{with} \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1$$

and

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)} \quad \text{with} \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r + 1,$$

where r is positive integer, F_n and L_n are the ordinary Fibonacci and Lucas numbers, respectively. The authors presented the generating functions

$$\sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1 - t - t^2)(1 - t)^r},$$

$$\sum_{n=0}^{\infty} L_n^{(r)} t^n = \frac{2 - t}{(1 - t - t^2)(1 - t)^r},$$

and the following recurrence relations for $n \geq 1$ and $r \geq 1$

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)},$$

$$L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}$$

for the hyper-Fibonacci and hyper-Lucas numbers [6]. The relationship between hyper-Fibonacci and Fibonacci numbers

$$F_n^{(r)} = \sum_{s=0}^{n} \binom{n + r - s - 1}{r - 1} F_s,$$

and similarly the relationship between hyper-Lucas and Lucas numbers

$$L_n^{(r)} = \sum_{s=0}^{n} \binom{n + r - s - 1}{r - 1} L_s$$

were obtained by Bahsi et al. [1].

The aim of this paper is to introduce hyper-Fibonacci and hyper-Lucas polynomials and to examine some algebraic and combinatoric properties of newly defined polynomials such as the recurrence relations, summation formulas and generating functions.

2. Main Results

Definition 2.1. Hyper-Fibonacci and hyper-Lucas polynomials are defined as

$$F_n^{(r)}(x) = \sum_{s=0}^{n} F_s^{(r-1)}(x) \quad \text{with} \quad F_n^{(0)}(x) = F_n(x), \quad F_0^{(r)}(x) = 0, \quad F_1^{(r)}(x) = 1$$

and

$$L_n^{(r)}(x) = \sum_{s=0}^{n} L_s^{(r-1)}(x) \quad \text{with} \quad L_n^{(0)}(x) = L_n(x), \quad L_0^{(r)}(x) = 2, \quad L_1^{(r)}(x) = x + 2r,$$

where r is positive integer, $F_n(x)$ and $L_n(x)$ are the ordinary Fibonacci and Lucas polynomials, respectively.

The first few hyper-Fibonacci polynomials are

$$\begin{aligned} F_2^{(1)}(x) &= x + 1, \\ F_3^{(1)}(x) &= x^2 + x + 2, \\ F_4^{(1)}(x) &= x^3 + x^2 + 3x + 2, \\ F_5^{(1)}(x) &= x^4 + x^3 + 4x^2 + 3x + 3 \end{aligned}$$

and

$$\begin{aligned} F_2^{(2)}(x) &= x + 2, \\ F_3^{(2)}(x) &= x^2 + 2x + 4, \\ F_4^{(2)}(x) &= x^3 + 2x^2 + 5x + 6, \\ F_5^{(2)}(x) &= x^4 + 2x^3 + 6x^2 + 8x + 9. \end{aligned}$$

The first few hyper-Lucas polynomials are

$$\begin{aligned} L_2^{(1)}(x) &= x^2 + x + 4, \\ L_3^{(1)}(x) &= x^3 + x^2 + 4x + 4, \\ L_4^{(1)}(x) &= x^4 + x^3 + 5x^2 + 4x + 6, \\ L_5^{(1)}(x) &= x^5 + x^4 + 6x^3 + 5x^2 + 9x + 6 \end{aligned}$$

and

$$\begin{aligned} L_2^{(2)}(x) &= x^2 + 2x + 8, \\ L_3^{(2)}(x) &= x^3 + 2x^2 + 6x + 12, \\ L_4^{(2)}(x) &= x^4 + 2x^3 + 7x^2 + 10x + 18, \\ L_5^{(2)}(x) &= x^5 + 2x^4 + 8x^3 + 12x^2 + 19x + 24. \end{aligned}$$

We note that the hyper-Fibonacci and hyper-Lucas polynomials give the hyper-Fibonacci and hyper-Lucas numbers for $x = 1$, respectively.
Definition 2.1 yields the following recurrence relations for the hyper-Fibonacci and hyper-Lucas polynomials for $n \geq 1$ and $r \geq 1$:

$$F_n^{(r)}(x) = F_{n-1}^{(r)}(x) + F_n^{(r-1)}(x),$$

and

$$L_n^{(r)}(x) = L_{n-1}^{(r)}(x) + L_n^{(r-1)}(x).$$

Theorem 2.2. *The generating function for the hyper-Fibonacci polynomials is*

$$g(r) = \sum_{n=0}^{\infty} F_n^{(r)}(x)\, t^n = \frac{F_0(x) + t\,(F_1(x) - F_0(x)\,x)}{(1 - xt - t^2)(1 - t)^r} = \frac{t}{(1 - xt - t^2)(1 - t)^r}.$$

Proof. We use the mathematical induction method on r. Since

$$g(0) = \sum_{n=0}^{\infty} F_n^{(0)}(x)\, t^n = \sum_{n=0}^{\infty} F_n(x)\, t^n = \frac{t}{1 - xt - t^2},$$

the result is true for $r = 0$. Suppose that the result is true for r. Then, we have

$$g(r) = \sum_{n=0}^{\infty} F_n^{(r)}(x)\, t^n = \frac{t}{(1 - xt - t^2)(1 - t)^r}.$$

Now, we must show that the result is true for $r+1$. Considering the Cauchy product, we have

$$\begin{aligned} g(r+1) &= \sum_{n=0}^{\infty} F_n^{(r+1)}(x)\,t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n} F_s^{(r)}(x)\right)t^n \\ &= \left(\sum_{i=0}^{\infty} F_i^{(r)}(x)\,t^i\right)\left(\sum_{j=0}^{\infty} t^j\right) \\ &= \frac{t}{(1-xt-t^2)(1-t)^{r+1}}. \end{aligned}$$

□

Theorem 2.3. *The generating function for the hyper-Lucas polynomials is*

$$G(r) = \sum_{n=0}^{\infty} L_n^{(r)}(x)\,t^n = \frac{L_0(x) + t(L_1(x) - L_0(x)\,x)}{(1-xt-t^2)(1-t)^r} = \frac{2-xt}{(1-xt-t^2)(1-t)^r}.$$

Proof. We use the induction method on r. If $r=0$, then we have

$$G(0) = \sum_{n=0}^{\infty} L_n^{(0)}(x)\,t^n = \sum_{n=0}^{\infty} L_n(x)\,t^n = \frac{2-xt}{1-xt-t^2}.$$

Thus, the result is true for $r=0$. Assume that the result is true for $r=k$. Then,

$$G(k) = \sum_{n=0}^{\infty} L_n^{(k)}(x)\,t^n = L_0^{(k)}(x) + L_1^{(k)}(x)\,t + L_2^{(k)}(x)\,t^2 + L_3^{(k)}(x)\,t^3 + \cdots.$$

For $r=k+1$, we have

$$\begin{aligned} G(k+1) &= \sum_{n=0}^{\infty} L_n^{(k+1)}(x)\,t^n = L_0^{(k+1)}(x) + L_1^{(k+1)}(x)\,t + L_2^{(k+1)}(x)\,t^2 + L_3^{(k+1)}(x)\,t^3 + \cdots \\ tG(k+1) &= L_0^{(k+1)}(x)\,t + L_1^{(k+1)}(x)\,t^2 + L_2^{(k+1)}(x)\,t^3 + \cdots. \end{aligned}$$

Then, by subtracting the above equalities, we get

$$\begin{aligned} (1-t)\,G(k+1) &= L_0^{(k+1)}(x) + \left(L_1^{(k+1)}(x) - L_0^{(k+1)}(x)\right)t + \left(L_2^{(k+1)}(x) - L_1^{(k+1)}(x)\right)t^2 \\ &\qquad + \left(L_3^{(k+1)}(x) - L_2^{(k+1)}(x)\right)t^3 \cdots \\ &= L_0^{(k)}(x) + L_1^{(k)}(x)\,t + L_2^{(k)}(x)\,t^2 + L_3^{(k)}(x)\,t^3 + \cdots \\ &= G(k). \end{aligned}$$

This completes the proof. □

Theorem 2.4. *If $n \geq 1$ and $r \geq 1$, then there are the following relationships between hyper-Fibonacci and Fibonacci polynomials, and similarly between hyper-Lucas and Lucas polynomials:*

(i) $F_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s(x)$,

(ii) $L_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} L_s(x)$.

Proof. (i) According to the Euler-Seidel algorithm in [7], the symmetric infinite matrix with entries a_n^r has the following recurrence relation:

$$a_n^0 = a_n, \quad a_0^n = a^n \quad (n \geq 0),$$

$$a_n^r = a_n^{r-1} + a_{n-1}^r \quad (n \geq 1, r \geq 1),$$

where (a_n) and (a^n) are two real initial sequences. Then, the entries a_n^r has the following symmetric relation [6]:

$$a_n^r = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} a_0^i + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} a_s^0. \tag{2.1}$$

For the values $a_n^0 = F_{n+1}^{(0)}(x) = F_{n+1}(x)$ and $a_0^n = F_1^{(r)}(x) = 1$, equation (2.1) turns into the following form:

$$a_{n+1}^{r+1} = \sum_{i=1}^{r+1} \binom{n+r-i+1}{n} + \sum_{s=1}^{n+1} \binom{n+r-s+1}{r} F_{s+1}(x).$$

Then, we have

$$\begin{aligned} a_{n+1}^{r+1} &= \sum_{i=0}^{r} \binom{n+r-i}{n} + \sum_{s=0}^{n} \binom{n+r-s}{r} F_{s+2}(x) \\ &= \sum_{k=0}^{r} \binom{n+k}{n} + \sum_{b=0}^{n} \binom{r+b}{r} F_{n-b+2}(x), \end{aligned}$$

where $k = r - i$ and $b = n - s$. From the following property of the combinations [8]

$$\sum_{t=a}^{c} \binom{t}{a} = \binom{c+1}{a+1},$$

we have

$$\begin{aligned} a_{n+1}^{r+1} &= \binom{n+r+1}{n+1} + \sum_{b=0}^{n} \binom{r+b}{r} F_{n-b+2}(x) \\ &= \sum_{b=0}^{n+1} \binom{r+b}{r} F_{n-b+2}(x). \end{aligned}$$

Then,

$$a_{n-1}^r = F_n^{(r)}(x) = \sum_{b=0}^{n-1} \binom{n+b-1}{r-1} F_{n-b}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s(x).$$

(ii) The proof is similar to the proof of (i).

□

Theorem 2.5. *If $r \geq 1$ and $n \geq 2$, then there are the following recurrence relations for the hyper-Fibonacci and hyper-Lucas polynomials*

(i) $F_n^{(r)}(x) = xF_{n-1}^{(r)}(x) + F_{n-2}^{(r)}(x) + \binom{n+r-2}{r-1}$,

(ii) $L_n^{(r)}(x) = xL_{n-1}^{(r)}(x) + L_{n-2}^{(r)}(x) - x\binom{n+r-2}{r-1} + 2\binom{n+r-1}{r-1}$.

Proof. (i) The proof is similar to the proof of (ii).

(ii) Considering Theorem 2.4 and equation (1.1), we get

$$\begin{aligned}
L_n^{(r)}(x) &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_s(x)\\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(xL_{s-1}(x)+L_{s-2}(x)\right)\\
&= x\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_{s-1}(x)+\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_{s-2}(x)\\
&= x\sum_{s=-1}^{n-1}\binom{n+r-(s+1)-1}{r-1}L_s(x)+\sum_{s=-2}^{n-2}\binom{n+r-(s+2)-1}{r-1}L_s(x)\\
&= x\left[\sum_{s=0}^{n-1}\binom{(n-1)+r-s-1}{r-1}L_s(x)+\binom{n+r-1}{r-1}L_{-1}(x)\right]\\
&+\sum_{s=0}^{n-2}\binom{(n-2)+r-s-1}{r-1}L_s(x)+\binom{n+r-2}{r-1}L_{-1}(x)+\binom{n+r-1}{r-1}L_{-2}(x)\\
&= x\left[L_{n-1}^{(r)}(x)+\binom{n+r-1}{r-1}(-x)\right]+\left[L_{n-2}^{(r)}(x)+\binom{n+r-2}{r-1}(-x)+\binom{n+r-1}{r-1}\left(x^2+2\right)\right]\\
&= xL_{n-1}^{(r)}(x)+L_{n-2}^{(r)}(x)-x\binom{n+r-2}{r-1}+2\binom{n+r-1}{r-1}.
\end{aligned}$$

□

Theorem 2.6. *If $n \geq 1$ and $r \geq 1$, then there are the following relationships between the hyper-Fibonacci and hyper-Lucas polynomials:*

(i) $xF_n^{(r)}(x)+L_n^{(r)}(x)=2F_{n+1}^{(r)}(x)$,

(ii) $xF_n^{(r)}(x)-L_n^{(r)}(x)=-2\left(F_{n-1}^{(r)}(x)+\binom{n+r-1}{r-1}\right)$.

Proof. By using Theorem 2.4, equations (1.2) and (1.3), we have

(i)

$$\begin{aligned}
xF_n^{(r)}(x)+L_n^{(r)}(x) &= x\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}F_s(x)+\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_s(x)\\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(xF_s(x)+L_s(x)\right)\\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}2F_{s+1}(x)\\
&= 2\sum_{s=1}^{n+1}\binom{n+r-(s-1)-1}{r-1}F_s(x)\\
&= 2\left(\sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}F_s(x)-\binom{n+r}{r-1}F_0(x)\right)\\
&= 2F_{n+1}^{(r)}(x),
\end{aligned}$$

(ii)

$$\begin{aligned} xF_n^{(r)}(x) - L_n^{(r)}(x) &= x\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}F_s(x) - \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_s(x) \\ &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}(xF_s(x) - L_s(x)) \\ &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}(-2F_{s-1}(x)) \\ &= -2\sum_{s=-1}^{n-1}\binom{n+r-(s+1)-1}{r-1}F_s(x) \\ &= -2\left(\sum_{s=0}^{n-1}\binom{(n-1)+r-s-1}{r-1}F_s(x) + \binom{n+r-1}{r-1}F_{-1}(x)\right) \\ &= -2\left(F_{n-1}^{(r)}(x) + \binom{n+r-1}{r-1}\right). \end{aligned}$$

□

Theorem 2.7. *If $n \geq 1$ and $r \geq 1$, then the summation formulas*

(i) $\sum_{s=0}^{r} F_n^{(s)}(x) = F_{n+1}^{(r)}(x) + (1-x)F_n(x) - F_{n-1}(x)$,

(ii) $\sum_{s=0}^{r} L_n^{(s)}(x) = L_{n+1}^{(r)}(x) + (1-x)L_n(x) - L_{n-1}(x)$

are valid.

Proof. (i) By using Theorem 2.4, we have

$$\begin{aligned} \sum_{s=1}^{r} F_n^{(s)}(x) &= \sum_{s=1}^{r}\left(\sum_{t=0}^{n}\binom{n+s-t-1}{s-1}F_t(x)\right) \\ &= \sum_{t=0}^{n}\left(F_t(x)\sum_{s=1}^{r}\binom{n+s-t-1}{s-1}\right) \\ &= \sum_{t=0}^{n}\binom{n+r-t}{r-1}F_t(x) \\ &= \sum_{t=0}^{n+1}\binom{(n+1)+r-t-1}{r-1}F_t(x) - F_{n+1}(x) \\ &= F_{n+1}^{(r)}(x) - F_{n+1}(x). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{s=0}^{r} F_n^{(s)}(x) &= F_n^{(0)}(x) + \sum_{s=1}^{r} F_n^{(s)}(x) \\ &= F_n^{(0)}(x) + F_{n+1}^{(r)}(x) - F_{n+1}(x) \\ &= F_n(x) + F_{n+1}^{(r)}(x) - xF_n(x) - F_{n-1}(x) \\ &= F_{n+1}^{(r)}(x) + (1-x)F_n(x) - F_{n-1}(x). \end{aligned}$$

(ii) The proof is similar to the proof of (i).

□

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Author have read and agreed to the published version of the manuscript.

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