Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753 /msaen.1125031
10 (4) 208-216 (2022) - Research Article
ISSN: 2147-6268

Some Results on $\mathcal{W}_8$-Curvature Tensor in α-Cosymplectic Manifolds

Selahattin Beyendi

Abstract
The object of this paper is to study $\mathcal{W}_8$ curvature tensors in α-cosymplectic manifolds.

Keywords: *$\mathcal{W}_8$-curvature tensor; α-cosymplectic manifold; η-Ricci soliton.*

AMS Subject Classification (2020): *Primary: 53C25; Secondary: 53D15.*

1. Introduction

The geometry of contact Riemannian manifolds and related issues have received great attention in recent years. One of the most important of these is the almost cosymplectic manifolds presented by Goldberg and Yano [11] in 1969. A special variant of almost contact manifolds was presented by Kenmotsu [15] in 1972. Afterwards Kim and Pak in [16] described a new class of manifolds known as almost α-cosymplectic manifolds by combining almost cosymplectic and almost α-Kenmotsu manifolds, where α is a real number. Almost cosymplectic manifolds have been studied by many mathematicians in literature ([1], [2], [3], [7], [10], [16], [17], [20], [21], [28]) and many others. On the other hand, many different kinds of almost contact structures are defined in the literature. Pokhariyal and Mishra [23] have presented new tensor fields. In 1982, $\mathcal{W}$-curvature tensor have been studied by Pokhariyal [22]. Pokhariyal, described the curvature tensor $\mathcal{W}_8$ in this work. Many authors have worked on $\mathcal{W}$ curvature tensors ([4], [19], [25], [27], [29]). Ingalahalli et al. [13] have been studied the $\mathcal{W}_8$-curvature tensor on Kenmotsu manifolds. Also, Ruganzu et al. [26] have been studied the $\mathcal{W}_8$ curvature tensor on para Kenmotsu manifolds. By the motivations of all these studies, we, authors, in the present manuscript, are going to study the $\mathcal{W}_8$ curvature tensor on α-cosymplectic manifolds.

This manuscript has been structured as follows: After a brief presentation of α-cosymplectic manifolds we examine the cases $\xi-\mathcal{W}_8$ flat, $\varphi-\mathcal{W}_8$ semisymmetric, $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$, $\mathcal{W}_8.\mathcal{R} = 0$, $\mathcal{W}_8.\mathcal{W}_8 = 0$, $\mathcal{W}_8$-Ricci pseudosymmetric, $\mathcal{W}_8.\mathcal{Q} = 0$. Also, we examine η-Ricci solitons on α-cosymplectic manifolds satisfying $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$ and $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$.

Received : *02-06-2022,* ***Accepted :*** *19-07-2022*

2. Preliminaries

Let $(M^n, \varphi, \xi, \eta, g)$ be an n-dimensional $(n = 2m + 1)$ almost contact metric manifold, in which ξ is the structure vector field, φ is a $(1, 1)$-tensor field, g is the Riemannian metric and η is a 1-form. The (φ, ξ, η, g) structure satisfies the following conditions [6].

$$\varphi\xi = 0, \quad \eta(\varphi\xi) = 0, \quad \eta(\xi) = 1,$$

$$\varphi^2 X_1 = -X_1 + \eta(X_1)\xi, \quad g(X_1, \xi) = \eta(X_1),$$

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2),$$

for any $X_1, X_2 \in \chi(M)$; in which $\chi(M)$ represents the collection of all smooth vector fields of M.

If moreover

$$\nabla_{X_1}\xi = -\alpha\varphi^2 X_1,$$

$$(\nabla_{X_1}\eta)X_2 = \alpha[g(X_1, X_2) - \eta(X_1)\eta(X_2)],$$

in which ∇ indicates the Riemannian connection and α is a real number, in that case $(M^n, \varphi, \xi, \eta, g)$ is known a α-cosymplectic manifold [16].

Then, it is also well known that [21]

$$\mathcal{R}(X_1, X_2)\xi = \alpha^2[\eta(X_1)X_2 - \eta(X_2)X_1], \tag{2.1}$$

$$\mathcal{R}ic(X_1, \xi) = -\alpha^2(n-1)\eta(X_1), \tag{2.2}$$

$$\mathcal{R}ic(\xi, \xi) = -\alpha^2(n-1), \tag{2.3}$$

$$\mathcal{Q}\xi = -\alpha^2(n-1)\xi \tag{2.4}$$

for all $X_1, X_2 \in \chi(M)$, in which $\mathcal{R}$, $\mathcal{R}ic$, $\mathcal{Q}$ indicates the curvature tensor, Ricci tensor and Ricci operator $g(\mathcal{Q}X_1, X_2) = \mathcal{R}ic(X_1, X_2)$ on M. Using (2.1), one can easily conclude that

$$\mathcal{R}(\xi, X_1)X_2 = \alpha^2[\eta(X_2)X_1 - g(X_1, X_2)\xi] \tag{2.5}$$

$$\mathcal{R}(X_1, \xi)\xi = \alpha^2[\eta(X_1)\xi - X_1]. \tag{2.6}$$

An α-cosymplectic manifold is known to be an η-Einstein manifold if Ricci tensor $\mathcal{R}ic$ satisfies condition

$$\mathcal{R}ic(X_1, X_2) = \lambda_1 g(X_1, X_2) + \lambda_2\eta(X_1)\eta(X_2) \tag{2.7}$$

in which λ_1, λ_2 are certain scalars. The manifold is known as Einstein when $\lambda_2 = 0$ in eq. (2.7).

On the other hand, η-Ricci solitons on α-cosymplectic manifolds have the following properties [30]:

$$\mathcal{R}ic(X_1, X_2) = -(\alpha + \lambda)g(X_1, X_2) + (\alpha - \mu)\eta(X_1)\eta(X_2), \tag{2.8}$$

$$\mathcal{R}ic(X_1, \xi) = -(\lambda + \mu)\eta(X_1) \tag{2.9}$$

for all $X_1, X_2 \in \chi(M)$.

3. $\xi - \mathcal{W}_8$-flat α-Cosymplectic Manifolds

In this part, we consider $\xi - \mathcal{W}_8$-flat in α-cosymplectic manifolds.

Definition 3.1. An α-cosymplectic manifold is known to be $\xi - \mathcal{W}_8$-flat if

$$\mathcal{W}_8(X_1, X_2)\xi = 0 \tag{3.1}$$

for all $X_1, X_2 \in \chi(M)$. $\mathcal{W}_8$-curvature tensor [22] is defined as

$$\mathcal{W}_8(X_1, X_2)X_3 = \mathcal{R}(X_1, X_2)X_3 + \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)X_3 - \mathcal{R}ic(X_2, X_3)X_1] \tag{3.2}$$

in which $\mathcal{R}ic$ and $\mathcal{R}$ are Ricci tensor and the curvature tensor of the manifold, respectively. By using of (3.1) in (3.2), we get

$$\mathcal{R}(X_1, X_2)\xi + \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)\xi - \mathcal{R}ic(X_2, \xi)X_1] = 0. \tag{3.3}$$

By virtue of (2.1), (2.2) in (3.3) and on simplification, we have

$$\alpha^2[\eta(X_1)X_2 - \eta(X_2)X_1] + \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)\xi + \alpha^2(n-1)\eta(X_2)X_1] = 0. \tag{3.4}$$

When the inner product is taken with ξ in eq. (3.4) and on simplification, one has

$$\mathcal{R}ic(X_1, X_2) = -\alpha^2(n-1)\eta(X_1)\eta(X_2).$$

In conclusion, one has the theorem given below:

Theorem 3.1. *Let M be an α-cosymplectic manifold satisfying $\xi - \mathcal{W}_8$-flat condition, then the manifold is a special kind of η-Einstein manifold.*

4. $\varphi - \mathcal{W}_8$-semisymmetric condition in α-cosymplectic manifolds

At this part, we examine $\varphi - \mathcal{W}_8$-semisymmetric condition in α cosymplectic manifolds.

Definition 4.1. An α-cosymplectic manifold is known to be $\varphi - \mathcal{W}_8$-semisymmetric if

$$\mathcal{W}_8(X_1, X_2).\varphi = 0 \tag{4.1}$$

for all $X_1, X_2 \in \chi(M)$.

At this time, eq. (4.1) becomes

$$(\mathcal{W}_8(X_1, X_2).\varphi)X_3 = \mathcal{W}_8(X_1, X_2)\varphi X_3 - \varphi\mathcal{W}_8(X_1, X_2)X_3 = 0. \tag{4.2}$$

Making use of (3.2) in (4.2), we obtain

$$\mathcal{R}(X_1, X_2)\varphi X_3 - \varphi\mathcal{R}(X_1, X_2)X_3 + \frac{1}{n-1}[\mathcal{R}ic(X_2, X_3)\varphi X_1 - \mathcal{R}ic(X_2, \varphi X_3)X_1] = 0. \tag{4.3}$$

By using $X_1 = \xi$ in (4.3) and with the help of (2.2), (2.5) equations and on simplification, we get

$$\alpha^2 g(X_2, \varphi X_3)\xi + \alpha\eta(X_3)\varphi X_2 + \frac{1}{n-1}\mathcal{R}ic(X_2, \varphi X_3)\xi = 0. \tag{4.4}$$

When X_3 by φX_3 is replaced in eq. (4.4), one has

$$\alpha^2 g(X_2, X_3)\xi = -\frac{1}{n-1}\mathcal{R}ic(X_2, X_3)\xi. \tag{4.5}$$

By taking inner product with ξ in (4.5), we obtain

$$\mathcal{R}ic(X_2, X_3) = -\alpha^2(n-1)g(X_2, X_3).$$

In conclusion, one has the theorem given below:

Theorem 4.1. *Let M be an α-cosymplectic manifold satisfying $\varphi - \mathcal{W}_8$-semisymmetric condition, then the manifold is an Einstein manifold.*

5. α-cosymplectic manifolds satisfying $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$ condition

At this part, we examine α-cosymplectic manifold satisfying $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$. Then, we get

$$\begin{aligned} &\mathcal{R}(\xi, X_1)\mathcal{W}_8(X_2, X_3)X_4 - \mathcal{W}_8(\mathcal{R}(\xi, X_1)X_2, X_3)X_4 \\ &-\mathcal{W}_8(X_2, \mathcal{R}(\xi, X_1)X_3)X_4 - \mathcal{W}_8(X_2, X_3)\mathcal{R}(\xi, X_1)X_4 = 0. \end{aligned} \tag{5.1}$$

By using (2.5) in (5.1), we have

$$\begin{aligned} &\alpha^2\eta(\mathcal{W}_8(X_2, X_3)X_4)X_1 - \alpha^2 g(X_1, \mathcal{W}_8(X_2, X_3)X_4)\xi \\ &-\alpha^2\eta(X_2)\mathcal{W}_8(X_1, X_3)X_4 + \alpha^2 g(X_1, X_2)\mathcal{W}_8(\xi, X_3)X_4 \\ &-\alpha^2\eta(X_3)\mathcal{W}_8(X_2, X_1)X_4 + \alpha^2 g(X_1, X_3)\mathcal{W}_8(X_2, \xi)X_4 \\ &-\alpha^2\eta(X_4)\mathcal{W}_8(X_2, X_3)X_1 + \alpha^2 g(X_1, X_4)\mathcal{W}_8(X_2, X_3)\xi = 0. \end{aligned} \tag{5.2}$$

By using inner product with ξ in (5.2) and the aid of (3.2) and on simplification, we obtain

$$\begin{aligned} &\alpha^2\eta(\mathcal{W}_8(X_2, X_3)X_4)\eta(X_1) - \alpha^2 g(X_1, \mathcal{W}_8(X_2, X_3)X_4) \\ &-\alpha^2\eta(X_2)\eta(\mathcal{W}_8(X_1, X_3)X_4) + \alpha^2 g(X_1, X_2)\eta(\mathcal{W}_8(\xi, X_3)X_4) \\ &-\alpha^2\eta(X_3)\eta(\mathcal{W}_8(X_2, X_1)X_4) + \alpha^2 g(X_1, X_3)\eta(\mathcal{W}_8(X_2, \xi)X_4) \\ &-\alpha^2\eta(X_4)\eta(\mathcal{W}_8(X_2, X_3)X_1) + \alpha^2 g(X_1, X_4)\eta(\mathcal{W}_8(X_2, X_3)\xi) = 0. \end{aligned} \tag{5.3}$$

By using (2.1), (2.2) and (2.6) in (5.3), we get

$$\begin{aligned} &-\alpha^2 g(X_1, \mathcal{R}(X_2, X_3)X_4) - \alpha^4 g(X_1, X_2)g(X_3, X_4) + \alpha^4 g(X_1, X_3)g(X_2, X_4) \\ &-\alpha^4 g(X_2, X_1)\eta(X_4)\eta(X_3) + \alpha^4 g(X_1, X_4)\eta(X_2)\eta(X_3) \\ &-\tfrac{1}{n-1}\alpha^2[\mathcal{R}ic(X_2, X_1)\eta(X_3)\eta(X_4) - \mathcal{R}ic(X_1, X_4)\eta(X_3)\eta(X_2)] = 0. \end{aligned} \tag{5.4}$$

It is assumed that $\{e_i : i = 1, 2, ..., n\}$ is an orthonormal frame field at any point of the manifold. Then contracting $X_1 = X_2 = e_i$ in (5.4), we have

$$\mathcal{R}ic(X_3, X_4) = \alpha^2(1-n)g(X_3, X_4) - [\frac{scal}{n-1} + n\alpha^2]\eta(X_3)\eta(X_4).$$

In conclusion, one has the theorem given below:

Theorem 5.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$ condition, then the manifold is an η-Einstein manifold.*

6. α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{R} = 0$ condition

At this part, we examine α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{R} = 0$ condition. Then, we have

$$\begin{aligned} &\mathcal{W}_8(\xi, X_4)\mathcal{R}(X_1, X_2)X_3 - \mathcal{R}(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 \\ &-\mathcal{R}(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 = 0. \end{aligned} \tag{6.1}$$

If $X_3 = \xi$ is used in eq. (6.1), then one gets

$$\begin{aligned} &\mathcal{W}_8(\xi, X_4)\mathcal{R}(X_1, X_2)\xi - \mathcal{R}(\mathcal{W}_8(\xi, X_4)X_1, X_2)\xi \\ &-\mathcal{R}(X_1, \mathcal{W}_8(\xi, X_4)X_2)\xi - \mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)\xi = 0. \end{aligned} \tag{6.2}$$

By taking (2.1) in (6.2) and making the necessary simplifications, we get

$$-\alpha^2\eta(\mathcal{W}_8(\xi, X_4)X_1)X_2 + \alpha^2\eta(\mathcal{W}_8(\xi, X_4)X_2)X_1 - \mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)\xi = 0. \tag{6.3}$$

If eqs. (2.2), (2.5) and (3.2) is used in eq. (6.3), then one obtains

$$\begin{aligned} &\alpha^4[g(X_1, X_4)X_2 - g(X_4, X_2)X_1 + \eta(X_4)\eta(X_1)X_2 - \eta(X_4)\eta(X_2)X_1] \\ &-\alpha^2\mathcal{R}(X_1, X_2)X_4 + \tfrac{\alpha^2}{n-1}[\mathcal{R}ic(X_4, X_1)X_2 - \mathcal{R}ic(X_4, X_2)X_1] = 0. \end{aligned} \tag{6.4}$$

Putting $X_2 = \xi$ in (6.4) and the aid of (2.2), (2.5), we have

$$\alpha^4 \eta(X_4)\eta(X_1)\xi + \frac{\alpha^2}{n-1}\mathcal{R}ic(X_4, X_1)\xi = 0. \tag{6.5}$$

When the inner product is taken with ξ in eq. (6.5), one has

$$\mathcal{R}ic(X_1, X_4) = -\alpha^2(n-1)\eta(X_1)\eta(X_4).$$

In conclusion, one has the theorem given below:

Theorem 6.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{R} = 0$ condition, then the manifold is a special kind of η-Einstein manifold.*

7. α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{W}_8 = 0$ condition

At this part, we examine α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{W}_8 = 0$ condition. Then, we have

$$\begin{array}{l}\mathcal{W}_8(\xi, X_4)\mathcal{W}_8(X_1, X_2)X_3 - \mathcal{W}_8(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 \\ -\mathcal{W}_8(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \mathcal{W}_8(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 = 0.\end{array} \tag{7.1}$$

By using (3.2) in (7.1), we obtain

$$\begin{array}{l}\mathcal{R}(\xi, X_4)\mathcal{W}_8(X_1, X_2)X_3 + \frac{1}{n-1}[\mathcal{R}ic(\xi, X_4)\mathcal{W}_8(X_1, X_2)X_3 - \mathcal{R}ic(X_4, \mathcal{W}_8(X_1, X_2)X_3)\xi] \\ -\mathcal{R}(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 - \frac{1}{n-1}[\mathcal{R}ic(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 - \mathcal{R}ic(X_2, X_3)\mathcal{W}_8(\xi, X_4)X_1] \\ -\mathcal{R}(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \frac{1}{n-1}[\mathcal{R}ic(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \mathcal{R}ic(\mathcal{W}_8(\xi, X_4)X_2, X_3)X_1] \\ -\mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 - \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 - \mathcal{R}ic(X_2, \mathcal{W}_8(\xi, X_4)X_3)X_1] = 0.\end{array} \tag{7.2}$$

Putting $X_2 = X_3 = \xi$ in (7.2) and with the help of (2.2), (2.3), (2.6), (3.2) equations and on simplification, we get

$$\mathcal{R}ic(X_1, X_4)\xi = -\alpha^2(n-1)g(X_1, X_4)\xi. \tag{7.3}$$

When the inner product is taken with ξ in eq. (7.3), one has

$$\mathcal{R}ic(X_1, X_4) = -\alpha^2(n-1)g(X_1, X_4).$$

In conclusion, one has the theorem given below:

Theorem 7.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{W}_8 = 0$ condition, then the manifold is an Einstein manifold.*

8. α-cosymplectic manifolds satisfying $\mathcal{W}_8$-Ricci pseudosymmetric condition

At this part, we examine $\mathcal{W}_8$-Ricci pseudosymmetric α-cosymplectic manifolds.
The notion of Ricci pseudosymmetric manifold has been presented by Deszcz ([8], [9]). In the case of Riemannian, the geometric comment of Ricci pseudosymmetric manifolds has been presented by [14]. A Riemannian manifold (M, g) is known Ricci pseudosymmetric ([8], [9], [12], [24]) if the tensor $\mathcal{R}.\mathcal{R}ic$ and the Tachibana tensor $\mathcal{Q}(g, \mathcal{R}ic)$ are linearly dependent, where

$$(\mathcal{R}(X_1, X_2).\mathcal{R}ic)(X_3, X_4) = -\mathcal{R}ic(\mathcal{R}(X_1, X_2)X_3, X_4) - \mathcal{R}ic(X_3, \mathcal{R}(X_1, X_2)X_4),$$

$$\mathcal{Q}(g, \mathcal{R}ic)(X_3, X_4; X_1, X_2) = -\mathcal{R}ic((X_1 \wedge_g X_2)X_3, X_4) - \mathcal{R}ic(X_3, (X_1 \wedge_g X_2)X_4)$$

and

$$(X_1 \wedge_g X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2$$

for all $X_1, X_2, X_3, X_4 \in \chi(M)$. $\mathcal{R}$ indicates the curvature tensor of M.
An α-cosymplectic manifold is known to be $\mathcal{W}_8$-Ricci pseudosymmetric if its curvature tensor satisfies

$$(\mathcal{W}_8(X_1, X_2).\mathcal{R}ic)(X_3, X_4) = L_{\mathcal{R}ic}\mathcal{Q}(g, \mathcal{R}ic)(X_3, X_4; X_1, X_2), \tag{8.1}$$

holds on $X_{4\mathcal{R}ic} = \{x \in M : \mathcal{R}ic \neq \frac{scal}{n} g \quad at \quad x\}$, in which $L_{\mathcal{R}ic}$ is some function on $X_{4\mathcal{R}ic}$. By using eq. (8.1), one obtains

$$\begin{aligned} &\mathcal{R}ic(\mathcal{W}_8(X_1, X_2)X_3, X_4) + \mathcal{R}ic(X_3, \mathcal{W}_8(X_1, X_2)X_4) \\ &= L_{\mathcal{R}ic}[g(X_2, X_3)\mathcal{R}ic(X_1, X_4) - g(X_1, X_3)\mathcal{R}ic(X_2, X_4) \\ &+ g(X_2, X_4)\mathcal{R}ic(X_1, X_3) - g(X_1, X_4)\mathcal{R}ic(X_2, X_3)]. \end{aligned} \tag{8.2}$$

Taking into account of $X_3 = \xi$ in (8.2) and by using the eqs. (2.1), (2.2), (3.2) and making the necessary simplifications, we get

$$\begin{aligned} &2\alpha^2[\mathcal{R}ic(X_2, X_4)\eta(X_1) - \mathcal{R}ic(X_1, X_2)\eta(X_4)] \\ &+ \alpha^4(n-1)[g(X_4, X_2)\eta(X_1) + g(X_1, X_4)\eta(X_2)] \\ &= L_{\mathcal{R}ic}[\mathcal{R}ic(X_2, X_4)\eta(X_1) - \mathcal{R}ic(X_1, X_4)\eta(X_2) \\ &- \alpha^2(n-1)(g(X_1, X_4)\eta(X_2) + g(X_2, X_4)\eta(X_1))]. \end{aligned} \tag{8.3}$$

Putting $X_2 = \xi$ in (8.3) and by using (2.2) and making the necessary simplifications, we obtain

$$\mathcal{R}ic(X_1, X_4) = \frac{(n-1)(\alpha^4 - \alpha^2 L_{\mathcal{R}ic})}{L_{\mathcal{R}ic}} g(X_1, X_4) - \frac{\alpha^4(n-1)}{L_{\mathcal{R}ic}}\eta(X_1)\eta(X_4).$$

In conclusion, one has the theorem given below:

Theorem 8.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8$-Ricci pseudosymmetric condition, then the manifold is an η-Einstein manifold.*

9. α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{Q} = 0$ condition

At this part, we examine α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{Q} = 0$ condition. Then, we obtain

$$\mathcal{W}_8(X_1, X_2)\mathcal{Q}X_3 - \mathcal{Q}(\mathcal{W}_8(X_1, X_2)X_3) = 0. \tag{9.1}$$

When $X_2 = \xi$ is put in eq. (9.1), one gets

$$\mathcal{W}_8(X_1, \xi)\mathcal{Q}X_3 - \mathcal{Q}(\mathcal{W}_8(X_1, \xi)X_3) = 0. \tag{9.2}$$

If eq. (3.2) is used in eq. (9.2), then one gets

$$\begin{aligned} &\mathcal{R}(X_1, \xi)\mathcal{Q}X_3 + \tfrac{1}{n-1}(\mathcal{R}ic(X_1, \xi)\mathcal{Q}X_3 - \mathcal{R}ic(\xi, \mathcal{Q}X_3)X_1) \\ &- \mathcal{Q}[\mathcal{R}(X_1, \xi)X_3 + \tfrac{1}{n-1}(\mathcal{R}ic(X_1, \xi)X_3 - \mathcal{R}ic(\xi, X_3)X_1)] = 0. \end{aligned} \tag{9.3}$$

Taking into account of (2.2), (2.4), (2.5) in eq. (9.3) and making the necessary simplifications, we get

$$\alpha^2\mathcal{R}ic(X_1, X_3)\xi + \alpha^4(n-1)g(X_1, X_3)\xi = 0. \tag{9.4}$$

Taking inner product with ξ in (9.4) and making the necessary simplifications, we obtain

$$\mathcal{R}ic(X_1, X_3) = -\alpha^2(n-1)g(X_1, X_3).$$

In conclusion, one has the theorem given below:

Theorem 9.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{Q} = 0$ condition, then the manifold is an Einstein manifold.*

10. η-Ricci solitons on α-cosymplectic manifolds satisfying $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$

At this part, we examine η-Ricci solitons on α-cosymplectic manifold satisfying $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$. The condition that must be satisfied by $\mathcal{R}ic$ is [5]:

$$\mathcal{R}ic(\mathcal{W}_8(\xi, X_1)X_2, X_3) + \mathcal{R}ic(X_2, \mathcal{W}_8(\xi, X_1)X_3) = 0 \tag{10.1}$$

for all X_1, X_2 and $X_3 \in \chi(M)$. By using (2.5), (2.8), (2.9) and (3.2) in (10.1), we have

$$\begin{aligned}&[\tfrac{(\alpha+\lambda)(2\alpha+\lambda-\mu)}{n-1} - \alpha^2(\alpha-\mu)][g(X_1,X_2)\eta(X_3) + g(X_1,X_3)\eta(X_2)]\\&+\tfrac{2(\alpha+\lambda)(\lambda+\mu)}{n-1} g(X_2,X_3)\eta(X_1) + 2\alpha^2(\alpha-\mu)\eta(X_1)\eta(X_2)\eta(X_3) = 0\end{aligned} \tag{10.2}$$

for all X_1, X_2 and $X_3 \in \chi(M)$. Putting $X_1 = \xi$ in (10.2), we obtain

$$(2\alpha+\lambda-\mu)\eta(X_2)\eta(X_3) + (\lambda+\mu)g(X_2,X_3) = 0$$

for all $X_2, X_3 \in \chi(M)$. But $\alpha+\lambda = n-1$, so $2\alpha+\lambda-\mu = 0$ and $\lambda = -\mu$ and we can state:

Theorem 10.1. *If (φ, ξ, η, g) is an almost contact metric structure on the n-dimensional α-cosymplectic manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$, then $\lambda = -\alpha + n - 1$ and $\mu = \alpha - n + 1$.*

11. η-Ricci solitons on α-cosymplectic manifolds satisfying $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$

At this part, we examine η-Ricci solitons on α-cosymplectic manifold satisfying $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$. The condition to be satisfied by $\mathcal{R}ic$ is [5]:

$$\begin{aligned}&\mathcal{R}ic(X_1, \mathcal{W}_8(X_2,X_3)X_4)\xi - \mathcal{R}ic(\xi, \mathcal{W}_8(X_2,X_3)X_4)X_1\\&+\mathcal{R}ic(X_1,X_2)\mathcal{W}_8(\xi,X_3)X_4 - \mathcal{R}ic(\xi,X_2)\mathcal{W}_8(X_1,X_3)X_4\\&+\mathcal{R}ic(X_1,X_3)\mathcal{W}_8(X_2,\xi)X_4 - \mathcal{R}ic(\xi,X_3)\mathcal{W}_8(X_2,X_1)X_4\\&+\mathcal{R}ic(X_1,X_4)\mathcal{W}_8(X_2,X_3)\xi - \mathcal{R}ic(\xi,X_4)\mathcal{W}_8(X_2,X_3)X_1 = 0\end{aligned} \tag{11.1}$$

for all X_1, X_2, X_3 and $X_4 \in \chi(M)$. By taking an inner product with ξ in (11.1), we get

$$\begin{aligned}&\mathcal{R}ic(X_1, \mathcal{W}_8(X_2,X_3)X_4) - \mathcal{R}ic(\xi, \mathcal{W}_8(X_2,X_3)X_4)\eta(X_1)\\&+\mathcal{R}ic(X_1,X_2)\eta(\mathcal{W}_8(\xi,X_3)X_4) - \mathcal{R}ic(\xi,X_2)\eta(\mathcal{W}_8(X_1,X_3)X_4)\\&+\mathcal{R}ic(X_1,X_3)\eta(\mathcal{W}_8(X_2,\xi)X_4) - \mathcal{R}ic(\xi,X_3)\eta(\mathcal{W}_8(X_2,X_1)X_4)\\&+\mathcal{R}ic(X_1,X_4)\eta(\mathcal{W}_8(X_2,X_3)\xi) - \mathcal{R}ic(\xi,X_4)\eta(\mathcal{W}_8(X_2,X_3)X_1) = 0\end{aligned} \tag{11.2}$$

for all X_1, X_2, X_3 and $X_4 \in \chi(M)$.
By taking $X_2 = X_3 = \xi$ in (11.2) and by virtue (2.5), (2.8), (2.9) and (3.2) and making the necessary simplifications, we have

$$(\alpha+\lambda)[g(X_1,X_4) - \eta(X_1)\eta(X_4)] = 0 \tag{11.3}$$

or

$$(\alpha+\lambda)g(\varphi X_1, \varphi X_4) = 0$$

for all $X_1, X_4 \in \chi(M)$. But $\alpha+\mu = n-1$, so $(\alpha+\lambda) = 0$ and we can state:

Theorem 11.1. *If (φ, ξ, η, g) is an almost contact metric structure on the n-dimensional α-cosymplectic manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$, then $\lambda = -\alpha$ and $\mu = -\alpha + n - 1$.*

By using (11.3) from (2.8), we have

$$\mathcal{R}ic(X_1, X_2) = -(\lambda+\mu)\eta(X_1)\eta(X_2).$$

So, one has the corollary given below:

Corollary 11.1. *If (φ, ξ, η, g) is an almost contact metric structure on the n-dimensional α-cosymplectic manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$, then the manifold is a special type of η-Einstein manifold.*

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Aktan, N., Yıldırım, M., Murathan, C.: *Almost f-cosymplectic manifolds*. Mediterr. J. Math. *11* (2), 775-787 (2014).

[2] Amruthalakshmi, M. R., Prakasha, D. G., Turki, N. B. and Unal, I.: *η-Ricci tensor on α-cosymplectic manifolds*, Hindawi Adv. Math. Phys., Vol. Article ID 7939654, 11 pag., (2022).

[3] Beyendi, S., Ayar, G., Aktan, N.: *On a type of α-cosymplectic manifolds*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68, 852-861 (2019).

[4] Beyendi, S., Yıldırım, M.: *On generalized weakly symmetric α-cosymplectic manifolds* , Hacet. J. Math. Stat, Vol: *50* (6) 1745-1755 (2021).

[5] Blaga.: *η-Ricci solitons on para-Kenmotsu manifolds*. Balkan J. Geom. Appl. *20* (1), 1-13 (2015).

[6] Blair, D. E.: *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. 509, Springer-Verlag, Berlin, (1976).

[7] Chen, X.: *Notes on Ricci solitons in f-cosymplectic manifolds*, J. Math. Phys. Anal. Geom., Vol: 13, No. 3, pp., 242-253 (2017).

[8] Deszcz, R.: *On Ricci-pseudosymmetric warped products*, Demonstratio Math., 22, 1053-1065 (1989).

[9] Deszcz, R.: *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Ser. A, 44, 1-34 (1992).

[10] Erken, I. K.: *On a classification of almost α-cosymplectic manifolds*, Khayyam J. Math., 5, 1-10 (2019).

[11] Goldberg, S. I. and Yano, K.: *Integrability of almost cosymplectic structure*, Pac. J. Math. 31 373-381 (1969).

[12] Hui, S. K. and Lemence, R. S.: *Ricci pseudosymmetric generalized quasi-Einstein manifolds*, SUT. J. Math., 51, 195-213 (2015).

[13] Ingalahalli, G. Anil, S. C., Begewadi, C. S.: *A study on W_8-curvature tensor in Kenmotsu manifolds*, Int. J. Math. And Appl., *8* (2), 27-34 (2020).

[14] Jahanara, B., Haesen, S., Senturk, Z. and Verstraelen, L.: *On the parallel transport of the Ricci curvatures*, J. Geom. Phys., 57, 1771-1777 (2007).

[15] Kenmotsu, K.: *A class of almost contact Riemannian manifolds*. Tohoku Math. J. *24* (1), 93-103 (1972).

[16] Kim, T. W., Pak, H. K.: *Canonical foliations of certain classes of almost contact metric structures*, Acta Math, Sinica, Eng. Ser. Aug., *21* (4), 841-846 (2005).

[17] Li, J. and Liu, X.: *Ricci solitons on homogeneous almost α-cosymplectic three-manifolds*, Mediterr. J. Math., https://doi.org/10.1007/s00009-021-01947-7 1660-5446/22/010001-12, 19:26, (2022).

[18] Matsumoto, K., Ianus, S. and Milrai, I.: Publicationes Mathematicae Debrecen, 33(3-4), 199-204 (1986).

[19] Njori, P. W., Moindi, S. K. and Pokhariyal, G. P.: *A study on W_6-curvature tensors and W_8-curvature tensors in Kenmotsu manifolds admitting semisymmetric metric connection,* Int. J. Stat. Appl. Math., *6* (2), 01-05 (2021).

[20] Olszak, Z.: *On almost cosymplectic manifolds*, Kodai Math, *4* (2), 239- 250 (1981).

[21] Öztürk, H., Murathan, C., Aktan, N., Vanlı, A. T.: *Almost α-cosymplectic f-manifolds*, Analele stııntıfıce ale unıversıtatıı 'AI. I. Cuza' Dı ıaşı S. N., Matematica, Tomul LX, f. 1., (2014).

[22] Pokhariyal, G. P.: *Relativistic significance of curvature tensors*, Internat. J. Math. Math. Sci., *5* No. 1, 133-139 (1982).

[23] Pokhariyal, G. P., Mishra, R. S.: *The curvature tensor and their relativistic significance*, Yokohoma Math. J. 18, 105-108 (1970).

[24] Shaikh, A. A. and Hui, S. K.: *On locally φ-symmetric β-Kenmotsu manifolds*, Extracta Mathematicae, 24, 301-316 (2009).

[25] Tripathi, M. M. and Gupta, P.: *T-curvature tensor on a semi-Riemannian manifold*, J. Adv. Math. Studies, Vol: 4, No: 1, 117-129 (2011).

[26] Uwimbabazi Ruganzu, L. F., Moindi, S. K., Pokhariyal, G. P. and Katende, J.: *η-Ricci solitons defined with W_8-curvature tensor and cylic Ricci tensor on para-Kenmotsu manifolds*, Int. J. Stat. Appl. Math., *4* (5), 80-84 (2019).

[27] Wanjiru, W. K.: *A study of W_6-curvature tensor in sasakian manifold,* (IJRSI), Vol: *6* (5), 343-346 (2019).

[28] Yıldırım, M., Ayar, G.: *Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds*, Journal of Universal Math., *4* (2) 201-208 (2021).

[29] Yıldız, A., De, U. C.: *On a type of Kenmotsu manifolds*, Differential Geometry- Dynamical Systems, Vol: 12, 289-298 (2010).

[30] Yoldaş, H. I. : *Some results on α-cosymplectic manifolds* Bull. Transilv. Univ. Bra., Vol: *1* (63), No. 2, 115-128 (2021).

Affiliations

SELAHATTİN BEYENDİ
ADDRESS: Inonu University Faculty of Education 44000, Malatya, TÜRKİYE.
E-MAIL: selahattin.beyendi@inonu.edu.tr
ORCID ID: 0000-0002-1037-6410