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PPF DEPENDENT COMMON FIXED POINTS OF GENERALIZED WEAKLY CONTRACTIVE TYPE MULTI-VALUED MAPPINGS

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Abstract. In this paper, we introduce the notion of generalized weakly contractive type multi-valued mapping with respect to a single-valued mapping and prove the existence of PPF dependent coincidence points in Banach spaces. Further, we introduce the notion of generalized weakly contractive type multi-valued mappings for a pair of multi-valued mappings and prove the existence of PPF dependent common fixed points in Banach spaces. We draw some corollaries and provide nontrivial examples to illustrate our results.

1. Introduction

The Banach contraction principle is one of the fundemental and useful result in fixed point theory and it plays an important role in solving problems related to non-linear functional analysis. In 1969, Nadler [20] extended Banach contraction principle to the context of set valued mapping. For more works on the existence of fixed points of multi-valued maps, we refer Kaneko [16] and Mizoguchi and Takahashi [19]. In 1997, Alber and Gurre-Delabriere [1] introduced weakly contractive map which is a generalization of contraction map and obtained fixed point results in the setting of Hilbert spaces. Rhoades [22] extended this concept to metric spaces and Bae [6] considered these type of multi-valued mappings. Bose and Roychowdhury [9,10] considered some generalized versions of these mappings and proved some fixed point theorems.

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Let (X, d) be a metric space and $K(X)$, the family of all non-empty compact subsets of X and H represents the Hausdorff distance induced by the metric d. i.e.,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\}$$

for any $A, B \in K(X)$, where $d(a,B) = \inf_{b\in B} d(a,b)$ and $d(A,b) = \inf_{a\in A} d(a,b)$.

Definition 1. *[6] A point $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to K(X)$ if $x \in Tx$.*

Definition 2. *A point $x \in X$ is said to be a coincidence point of two mappings $f, g : X \to X$ if $f(x) = g(x)$.*

Definition 3. *[9] A mapping $T : X \to X$ is said to be a generalized weakly contractive map with respect to $f : X \to X$ if*

$$\psi(d(Tx,Ty)) \le \psi(d(fx,fy)) - \phi(d(fx,fy))$$

for all $x, y \in X$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous such that $\psi(t), \phi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is non-decreasing and ψ is monotonically increasing (strictly).

If $\psi(t) = t$ for all $t \in [0,\infty)$, and f is the identity map in Definition 3 then we say that $T : X \to X$ is said to be a weakly contractive map.

Definition 4. *[9] A multi-valued mapping $T : X \to K(X)$ is said to be a generalized weakly contractive map with respect to $f : X \to X$ if*

$$\psi(H(Tx,Ty)) \le \psi(d(fx,fy)) - \phi(d(fx,fy)),$$

for all $x, y \in X$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous such that $\psi(t), \phi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is non-decreasing and ψ is monotonically increasing (strictly).

If f is the identity mapping then the multi-valued mapping $T : X \to K(X)$ is said to be generalized weakly contractive. If $\psi(t) = t$ for all $t \in [0,\infty)$, then the multi-valued mapping $T : X \to K(X)$ is said to be weakly contractive with respect to f.

In 1977, Bernfeld, Lakshmikantham and Reddy [8] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point. Furthermore, they introduced the notation of Banach type contraction for a non-self mappings and proved the existence of PPF dependent fixed points of Banach type contractive mappings in the Razumikhin class. Several mathematicians proved the existence of PPF dependent fixed points of single-valued mapppings and multi-valued mappings, for more details we refer to [2–5, 7, 13, 15, 18]. In 2016, Farajzadeh, Kaewcharoen and Plubtieng [14] introduced the concept of PPF dependent fixed point of multi-valued mappings which is an extension of PPF dependent fixed point of single valued mapping and proved the existence of PPF dependent fixed point for multi-valued mappings.

Motivated by the research work of Bose and Roychowdhury [9] on weakly contractive maps, we extend the above said results for the case of PPF dependent coincidence points and PPF dependent common fixed points.

In this paper, we introduce the notion of generalized weakly contractive type multi-valued mapping with respect to a single-valued mapping and prove the existence of PPF dependent coincidence points in Banach spaces. Further, we introduce the notion of generalized weakly contractive type multi-valued mappings for a pair of multi-valued mappings and prove the existence of PPF dependent common fixed points in Banach spaces. We draw some corollaries and provide examples to illustrate our main results.

2. Preliminaries

In this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0, \infty)$, the set of all natural numbers by $\mathbb{N}$. Let $(E, ||.||_E)$ be a Banach space and we denote it by simply by E. Let $I = [a, b] \subseteq \mathbb{R}$ and $E_0 = C(I, E)$, the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a \le t \le b} ||\phi(t)||_E$ for $\phi \in E_0$.

In our discussion, let $CB(E)$ be the collection of all non-empty closed and bounded subsets of E. Then the Hausdorff metric H_E on $CB(E)$ induced by the norm $||.||_E$ is defined by

$$H_E(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

for any $A, B \in CB(E)$, where $d(a, B) = \inf\limits_{b\in B} ||a - b||_E$ and $d(A, b) = \inf\limits_{a\in A} ||a - b||_E$.

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \left\{\phi \in E_0 \mid ||\phi||_{E_0} = ||\phi(c)||_E\right\}$ and $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$. Clearly every constant function from I to E belongs to R_c so that R_c is a non-empty subset of E_0 .

Definition 5. *[8] Let R_c be the Razumikhin class of continuous functions in E_0. Then, we say that*

(i) *the class R_c is algebraically closed with respect to the difference if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.*
(ii) *the class R_c is topologically closed if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.*

The Razumikhin class of functions R_c has the following properties.

Theorem 1. [2] *Let R_c be the Razumikhin class of functions in E_0. Then*

(i) *for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*
(ii) *the Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.*
(iii) $\bigcap\limits_{c\in[a,b]} R_c = \{\phi \in E_0 \mid \phi : I \to E \text{ is constant}\}$.

Definition 6. *[8] Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $T\phi = \phi(c)$ for some $c \in I$.*

Definition 7. *[8] Let $T : E_0 \to E$ be a mapping. Then T is called a Banach type contraction if there exists a constant $k \in [0, 1)$ such that*

$$||T\phi - T\psi||_E \leq k\,||\phi - \psi||_{E_0}$$

for any $\phi, \psi \in E_0$.

Theorem 2. [8] *Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be an algebraically closed with respect to the difference and topologically closed. Then, T has a unique PPF dependent fixed point in R_c.*

Farajzadeh, Kaewcharoen and Plubtieng [14] introduced the concept of PPF dependent fixed points of multi-valued mappings as follows.

Definition 8. *[14] Let $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $\phi(c) \in T\phi$ for some $c \in I$.*

Definition 9. *[14] Let $f : E_0 \to E_0$ be a single-valued mapping and $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of f and T if $f\phi(c) \in T\phi$ for some $c \in I$.*

Here we observe that $f\phi$ is not a composition of ϕ and f.

Definition 10. *[14] Let $S, T : E_0 \to E$ be two single-valued mappings. A point $\phi \in E_0$ is said to be a PPF dependent common fixed point of S and T if $S\phi = T\phi = \phi(c)$ for some $c \in I$.*

We denote

$\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \psi$ is continuous, monotonically increasing and $\psi(t) = 0 \iff t = 0\}$

and

$\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \phi$ is continuous and $\phi(t) = 0 \iff t = 0\}$.

We use the following results in our subsequent discussions.

Proposition 1. *If $\{a_n\}$ and $\{b_n\}$ are two real sequences, $\{b_n\}$ is bounded, then $\liminf(a_n + b_n) \leq \liminf a_n + \limsup b_n$.*

Lemma 1. [20] Let A and B be two non-empty compact subsets of a metric space X. If $a \in A$ then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.

Lemma 2. [3] *Let $\{\phi_n\}$ be a sequence in E_0 such that $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ as $n \to \infty$. If $\{\phi_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_k - 1}||_{E_0} < \epsilon$ and*

(i) $\lim_{k \to \infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon$,

(ii) $\lim_{k \to \infty} ||\phi_{n_k+1} - \phi_{m_k}||_{E_0} = \epsilon$,

(iii) $\lim_{k \to \infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon$,

(iv) $\lim_{k \to \infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon$.

3. Existence of PPF Dependent Coincidence Points

In this section, we introduce the concept of PPF dependent coincidence point of $f : E \to E$ and $T : E_0 \to E$.

Definition 11. *Let $f : E \to E$ and $T : E_0 \to E$ be two mappings. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of f and T if $T\phi = (f \circ \phi)(c)$ for some $c \in I$, where $f \circ \phi$ denotes the composition of ϕ and f.*

We observe that if f is the identity mapping then PPF dependent coincidence point of f and T becomes PPF dependent fixed point of T.

Motivated by this idea, in the following, we now introduce the concept of PPF dependent coincidence point of $f : E \to E$ and $T : E_0 \to CB(E)$.

Definition 12. *Let $f : E \to E$ be a single-valued mapping and $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of f and T if $(f \circ \phi)(c) \in T\phi$ for some $c \in I$, where $f \circ \phi$ denotes the composition of ϕ and f.*

We observe that, if f is an identity mapping then ϕ is a PPF dependent fixed point of the multi-valued mapping T.

Notation: Let $c \in I$. Let $f : E \to E$ and $\phi \in E_0$. We denote $(f \circ \phi)(c)$ by $f\phi(c)$.

In the following, we introduce the notion of generalized weakly contractive type multi-valued mappings.

Definition 13. *Let $T : E_0 \to CB(E)$. Let $f : E \to E$ be a continuous function. Then, T is said to be a generalized weakly contractive type multi-valued mapping with respect to f if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(H_E(T\alpha, T\beta)) \leq \psi(||f\alpha - f\beta||_{E_0}) - \phi(||f\alpha - f\beta||_{E_0}) \tag{1}$$

for any $\alpha, \beta \in E_0$.

We observe the following:

(i) if f is the identity mapping in (1) then the mapping $T : E_0 \to CB(E)$ is said to be generalized weakly contractive type multi-valued mapping;

(ii) if $\psi(t) = t$ for any $t \in \mathbb{R}^+$ in (1) then the mapping $T : E_0 \to CB(E)$ is said to be weakly contractive type multi-valued mapping with respect to f;

(iii) if both f is the identity mapping and $\psi(t) = t$ for any $t \in \mathbb{R}^+$ in (1) then the mapping $T : E_0 \to CB(E)$ is said to be weakly contractive type multi-valued mapping.

Theorem 3. *Let $T : E_0 \to CB(E)$ and $f : E \to E$ be functions that satisfy the following conditions:*

(i) *T is a generalized weakly contractive type multi-valued mapping with respect to f,*

(ii) *$T\phi \subseteq f(R_c)(c) = \{f\phi(c) \mid \phi \in R_c\}$ for any $\phi \in E_0$,*
(iii) *R_c is algebraically closed with respect to the difference,*
(iv) *$f(R_c)$ is complete and*
(v) *$f(R_c) \subseteq R_c$.*

Then, T and f have a PPF dependent coincidence point in R_c.

Proof. Let $\phi_0 \in R_c$. Then, $T\phi_0 \subseteq E$. Let $x_1 \in E$ be such that $x_1 \in T\phi_0$. Since $T\phi_0 \subseteq f(R_c)(c)$, we choose ϕ_1 in R_c such that $x_1 = f\phi_1(c) \in T\phi_0$. From (1), we have

$$\psi(H_E(T\phi_0, T\phi_1)) \leq \psi(||f\phi_0 - f\phi_1||_{E_0}) - \phi(||f\phi_0 - f\phi_1||_{E_0}).$$

Since $x_1 \in T\phi_0$, by Lemma 1 there exists $x_2 \in T\phi_1$ such that

$$||x_1 - x_2||_E \leq H_E(T\phi_0, T\phi_1). \tag{2}$$

Since $x_2 \in T\phi_1$ and $T\phi_1 \subseteq f(R_c)(c)$, we choose ϕ_2 in R_c such that $x_2 = f\phi_2(c) \in T\phi_1$.
If $\phi_1 = \phi_2$ then ϕ_1 is a PPF dependent coincidence point of f and T.
Suppose that $\phi_1 \neq \phi_2$.
From (2), we have

$$||f\phi_1(c) - f\phi_2(c)||_E \leq H_E(T\phi_0, T\phi_1).$$

Since R_c is algebraically closed with respect to the difference, we have

$$||f\phi_1 - f\phi_2||_{E_0} \leq H_E(T\phi_0, T\phi_1). \tag{3}$$

From (1), we have

$$\psi(H_E(T\phi_1, T\phi_2)) \leq \psi(||f\phi_1 - f\phi_2||_{E_0}) - \phi(||f\phi_1 - f\phi_2||_{E_0}).$$

Since $x_2 \in T\phi_1$, by Lemma 1 there exists $x_3 \in T\phi_2$ such that

$$||x_2 - x_3||_E \leq H_E(T\phi_1, T\phi_2). \tag{4}$$

Since $x_3 \in T\phi_2$ and $T\phi_2 \subseteq f(R_c)(c)$, we choose ϕ_3 in R_c such that $x_3 = f\phi_3(c) \in T\phi_2$.
If $\phi_2 = \phi_3$ then ϕ_2 is a PPF dependent coincident point of f and T.
Suppose that $\phi_2 \neq \phi_3$.
From (4), we have

$$||f\phi_2(c) - f\phi_3(c)||_E \leq H_E(T\phi_1, T\phi_2).$$

Since R_c is algebraically closed with respect to the difference, we have

$$||f\phi_2 - f\phi_3||_{E_0} \leq H_E(T\phi_1, T\phi_2). \tag{5}$$

On continuing this process, we get a sequence $\{f\phi_n\}$ in R_c such that

$$x_n = f\phi_n(c) \in T\phi_{n-1}, ||f\phi_n - f\phi_{n+1}||_{E_0} \leq H_E(T\phi_{n-1}, T\phi_n) \text{ for all } n \in \mathbb{N}. \tag{6}$$

Clearly,

$$\begin{aligned}\psi(||f\phi_n - f\phi_{n+1}||_{E_0}) &\leq \psi(H_E(T\phi_{n-1}, T\phi_n) \\ &\leq \psi(||f\phi_{n-1} - f\phi_n)||_{E_0}) - \phi(||f\phi_{n-1} - f\phi_n||_{E_0})\end{aligned} \tag{7}$$

$< \psi(||f\phi_{n-1} - f\phi_n||_{E_0}).$

Since ψ is monotonically increasing function, we have

$||f\phi_n - f\phi_{n+1}||_{E_0} \leq ||f\phi_{n-1} - f\phi_n||_{E_0}.$

Therefore, the sequence $\{||f\phi_n - f\phi_{n+1}||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$ and hence it is convergent.

Let $\left|\left|f\phi_n - f\phi_{n+1}\right|\right|_{E_0} \to r$ as $n \to \infty$.

From (7), we have

$\psi(||f\phi_n - f\phi_{n+1}||_{E_0}) \leq \psi(||f\phi_{n-1} - f\phi_n||_{E_0}) - \phi(||f\phi_{n-1} - f\phi_n||_{E_0}).$

On applying limits as $n \to \infty$ on both sides, we get

$\psi(r) \leq \psi(r) - \phi(r)$ and hence $r = 0$.

Therefore,

$$\lim_{n\to\infty} \left|\left|f\phi_n - f\phi_{n+1}\right|\right|_{E_0} = 0. \tag{8}$$

We now show that $\{f\phi_n\}$ is a Cauchy sequence.

Suppose that $\{f\phi_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ and two subsequences $\{f\phi_{m_k}\}$ and $\{f\phi_{n_k}\}$ of $\{f\phi_n\}$ such that for any $k \in \mathbb{N}, m_k > n_k > k$ such that

$$||f\phi_{n_k} - f\phi_{m_k}||_{E_0} \geq \epsilon. \tag{9}$$

Let m_k be the smallest positive integer greater than n_k satisfying (9).

Then, $||f\phi_{n_k} - f\phi_{m_k}||_{E_0} \geq \epsilon$ and $||f\phi_{n_k} - f\phi_{m_k-1}||_{E_0} < \epsilon$.

By Lemma 2 we have

$\lim\limits_{k\to\infty} \left|\left|f\phi_{n_k+1} - f\phi_{m_k}\right|\right|_{E_0} = \epsilon = \lim\limits_{k\to\infty} \left|\left|f\phi_{n_k} - f\phi_{m_k+1}\right|\right|_{E_0} = \lim\limits_{k\to\infty} \left|\left|f\phi_{n_k} - f\phi_{m_k}\right|\right|_{E_0}.$

Now, we show that $\lim\limits_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} = \epsilon$ for any $l_1, l_2 \in \mathbb{N}$.

Let $l_1, l_2 \in \mathbb{N}$. Now we consider

$$\begin{aligned} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} \leq\ & ||f\phi_{n_k+l_1} - f\phi_{n_k+l_1-1}||_{E_0} + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1-2}||_{E_0} \\ & + ... + ||f\phi_{n_k+1} - f\phi_{n_k}||_{E_0} + ||f\phi_{n_k} - f\phi_{m_k+1}||_{E_0} \\ & + ||f\phi_{m_k+1} - f\phi_{m_k+2}||_{E_0} + ... + ||f\phi_{m_k+l_2-1} - f\phi_{m_k+l_2}||_{E_0}. \end{aligned}$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} \leq \epsilon. \tag{10}$$

Now, we consider

$$\begin{aligned} ||f\phi_{n_k} - f\phi_{m_k+1}||_{E_0} \leq\ & ||f\phi_{n_k} - f\phi_{n_k+1}||_{E_0} + ||f\phi_{n_k+1} - f\phi_{n_k+2}||_{E_0} + ... \\ & + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1}||_{E_0} + ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} \\ & + ||f\phi_{m_k+l_2} - f\phi_{m_k+l_2-1}||_{E_0} + ... + ||f\phi_{m_k+2} - f\phi_{m_k+1}||_{E_0}. \end{aligned}$$

Now, by applying Proposition 1 with $a_k = ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0}$ and $b_k = (||f\phi_{n_k} - f\phi_{n_k+1}||_{E_0} + ||f\phi_{n_k+1} - f\phi_{n_k+2}||_{E_0} + ... + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1}||_{E_0} + ||f\phi_{m_k+l_2} - f\phi_{m_k+l_2-1}||_{E_0} + ... + ||f\phi_{m_k+2} - f\phi_{m_k+1}||_{E_0})$ we have

$$\begin{aligned} \epsilon \leq \liminf_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} + \limsup_{k\to\infty} (||f\phi_{n_k} - f\phi_{n_k+1}||_{E_0} \\ + ||f\phi_{n_k+1} - f\phi_{n_k+2}||_{E_0} + ... + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1}||_{E_0} + ||f\phi_{m_k+l_2} - f\phi_{m_k+l_2-1}||_{E_0} \\ + ... + ||f\phi_{m_k+2} - f\phi_{m_k+1}||_{E_0}). \end{aligned}$$

Hence

$$\epsilon \leq \liminf_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0}. \tag{11}$$

From (10) and (11), we get

$$\lim_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} = \epsilon \text{ for any } l_1, l_2 \in \mathbb{N}. \tag{12}$$

We choose $l_1, l_2 \in \mathbb{N}$ such that $(m_k + l_2) - (n_k + l_1) = 1$.
From (7), we have

$$\psi(||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0}) \leq$$
$$\psi(||f\phi_{n_k+l_1-1} - f\phi_{m_k+l_2-1}||_{E_0}) - \phi(||f\phi_{n_k+l_1-1} - f\phi_{m_k+l_2-1}||_{E_0}).$$

On applying limits as $k \to \infty$ on both sides and by using (12), we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \eta(\epsilon),$$

a contradiction.
Therefore, $\{f\phi_n\}$ is a Cauchy sequence in $f(R_c)$. Since $f(R_c)$ is complete, we have $f\phi_n \to \eta$ as $n \to \infty$ for some $\eta \in f(R_c)$ and hence there exists $\phi^* \in R_c$ such that $\eta = f\phi^*$ and $\lim_{n\to\infty} f\phi_n = f\phi^*$.
Now, for any $n \in \mathbb{N}$

$$d(f\phi_{n+1}(c), T\phi^*) \leq H_E(T\phi_n, T\phi^*),$$

and hence

$$\psi(d(f\phi_{n+1}(c), T\phi^*)) \leq \psi(H_E(T\phi_n, T\phi^*))$$
$$\leq \psi(||f\phi_n - f\phi^*||_{E_0}) - \phi(||f\phi_n - f\phi^*||_{E_0}).$$

On applying limits as $n \to \infty$ on both sides, we get

$$\psi(d(f\phi^*(c), T\phi^*)) \leq \psi(0) - \phi(0) \text{ and hence } \psi(d(f\phi^*(c), T\phi^*)) = 0.$$

Therefore, $f\phi^*(c) \in T\phi^*$ and hence T and f have a PPF dependent coincidence point in R_c. □

4. Existence of PPF Dependent Common Fixed Points

In this section, we introduce the concept of PPF dependent common fixed points for a pair of multi-valued mappings.

Definition 14. *Let $S, T : E_0 \to CB(E)$ be two multi-valued mappings. A point $\phi \in E_0$ is said to be a PPF dependent common fixed point of S and T if $\phi(c) \in S\phi$ and $\phi(c) \in T\phi$ for some $c \in I$.*

In the following we define generalized weakly contractive type mappings for a pair of multi-valued mappings.

Definition 15. *Let $S, T : E_0 \to CB(E)$ be two multi-valued functions. The pair (S, T) is said to be a pair of generalized weakly contractive type multi-valued mappings on E_0 if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(H_E(T\alpha, S\beta)) \leq \psi(M(\alpha, \beta)) - \phi(M(\alpha, \beta)) \tag{13}$$

for any $\alpha, \beta \in E_0$, *where*
$M(\alpha,\beta) = \max\{||\alpha-\beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), S\beta), \frac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), S\beta)]\}$.

Theorem 4. *Let* $S, T : E_0 \to CB(E)$ *be two multi-valued mappings such that:*

(i) *the pair* (S, T) *is a pair of generalized weakly contractive type multi-valued mappings on* E_0,
(ii) R_c *is algebraically closed with respect to the difference and*
(iii) $T\phi \subseteq R_c(c)$ *and* $S\phi \subseteq R_c(c)$ *for any* $\phi \in E_0$.

Then, S *and* T *have a PPF dependent common fixed point in* R_c.

Proof. Let $\phi_0 \in R_c$. Then, $T\phi_0 \subseteq E$. Let $x_1 \in E$ be such that $x_1 \in T\phi_0$.
Since $T\phi_0 \subseteq R_c(c)$, we choose ϕ_1 in R_c such that $x_1 = \phi_1(c) \in T\phi_0$.
From (13), we have

$$\psi(H_E(T\phi_0, S\phi_1)) \leq \psi(M(\phi_0,\phi_1)) - \phi(M(\phi_0,\phi_1)).$$

If $M(\phi_0,\phi_1) = 0$ then $\phi_0 = \phi_1$ and hence ϕ_0 is a PPF dependent common fixed point of S and T.
Suppose that $M(\phi_0,\phi_1) > 0$. By Lemma 1 there exists $x_2 \in S\phi_1$ such that

$$||x_1 - x_2||_E \leq H_E(T\phi_0, S\phi_1). \tag{14}$$

Since $x_2 \in S\phi_1$ and $S\phi_1 \subseteq R_c(c)$, we choose ϕ_2 in R_c such that $x_2 = \phi_2(c) \in S\phi_1$.
From (13), we have

$$\psi(H_E(S\phi_1, T\phi_2)) = \psi(H_E(T\phi_2, S\phi_1)) \leq \psi(M(\phi_2,\phi_1)) - \phi(M(\phi_2,\phi_1)).$$

If $M(\phi_2,\phi_1) = 0$ then $\phi_1 = \phi_2$ and hence ϕ_1 is a PPF dependent common fixed point of S and T.
Suppose that $M(\phi_2,\phi_1) > 0$. By Lemma 1 there exists $x_3 \in T\phi_2$ such that

$$||x_2 - x_3||_E \leq H_E(S\phi_1, T\phi_2). \tag{15}$$

Since $x_3 \in T\phi_2$ and $T\phi_2 \subseteq R_c(c)$, we choose ϕ_3 in R_c such that $x_3 = \phi_3(c) \in T\phi_2$.
Again from (13), we have

$$\psi(H_E(T\phi_2, S\phi_3)) \leq \psi(M(\phi_2,\phi_3)) - \phi(M(\phi_2,\phi_3)).$$

If $M(\phi_2,\phi_3) = 0$ then $\phi_2 = \phi_3$ and hence ϕ_2 is a PPF dependent common fixed point of S and T.
Suppose that $M(\phi_2,\phi_3) > 0$. On continuing this process, we get a sequence $\{\phi_n\}$ in R_c such that

$$\phi_{2n+1}(c) \in T\phi_{2n},\ \phi_{2n+2}(c) \in S\phi_{2n+1} \tag{16}$$

and

$$M(\phi_n,\phi_{n+1}) > 0 \tag{17}$$

with $||\phi_{2n+1}(c) - \phi_{2n+2}(c)||_E \leq H_E(T\phi_{2n}, S\phi_{2n+1})$
and $||\phi_{2n+2}(c) - \phi_{2n+3}(c)||_E \leq H_E(S\phi_{2n+1}, T\phi_{2n+2})$ for all $n \in \mathbb{N} \cup \{0\}$.
Since R_c is algebraically closed with respect to the difference, for all $n \in \mathbb{N} \cup \{0\}$ we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq H_E(T\phi_{2n}, S\phi_{2n+1}) \tag{18}$$

and

$$||\phi_{2n+2} - \phi_{2n+3}||_{E_0} \leq H_E(S\phi_{2n+1}, T\phi_{2n+2}) = H_E(T\phi_{2n+2}, S\phi_{2n+1}). \tag{19}$$

We consider

$$\begin{aligned} M(\phi_{2n}, \phi_{2n+1}) &= \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, d(\phi_{2n}(c), T\phi_{2n}), d(\phi_{2n+1}(c), S\phi_{2n+1}), \\ &\qquad \tfrac{1}{2}[d(\phi_{2n+1}(c), T\phi_{2n}) + d(\phi_{2n}(c), S\phi_{2n+1})]\}, \\ &\leq \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n}(c) - \phi_{2n+1}(c)||_E, ||\phi_{2n+1}(c) - \phi_{2n+2}(c)||_E, \\ &\qquad \tfrac{1}{2}[0 + ||\phi_{2n}(c) - \phi_{2n+2}(c)||_E\} \\ &= \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}, \tfrac{1}{2}[||\phi_{2n} - \phi_{2n+2}||_{E_0}]\} \\ &\leq \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}, \\ &\qquad \tfrac{1}{2}[||\phi_{2n} - \phi_{2n+1}||_{E_0} + ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}]\} \\ &= \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\}, \end{aligned}$$

and hence

$$M(\phi_{2n}, \phi_{2n+1}) \leq \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\}. \tag{20}$$

Suppose that $\max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\} = ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}$.
Now, from (20), we have

$$M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n+1} - \phi_{2n+2}||_{E_0},$$

and hence

$$\psi(M(\phi_{2n}, \phi_{2n+1})) \leq \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}).$$

Now, from (18), we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq H_E(T\phi_{2n}, S\phi_{2n+1}),$$

and hence

$$\begin{aligned} \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) &\leq \psi(H_E(T\phi_{2n}, S\phi_{2n+1})) \\ &\leq \psi(M(\phi_{2n}, \phi_{2n+1})) - \phi(M(\phi_{2n}, \phi_{2n+1})) \qquad (21) \\ &\leq \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) - \phi(M(\phi_{2n}, \phi_{2n+1})). \end{aligned}$$

Therefore, $f(M(\phi_{2n}, \phi_{2n+1})) = 0$ and hence $M(\phi_{2n}, \phi_{2n+1}) = 0$,
a contradiction.
Therefore,

$$\max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\} = ||\phi_{2n} - \phi_{2n+1}||_{E_0}. \tag{22}$$

Now, from (20), we have

$$M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n} - \phi_{2n+1}||_{E_0}. \tag{23}$$

Now, from (18), we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq H_E(T\phi_{2n}, S\phi_{2n+1}),$$

and hence

$$\begin{aligned} \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) &\leq \psi(H_E(T\phi_{2n}, S\phi_{2n+1})) \\ &\leq \psi(M(\phi_{2n}, \phi_{2n+1})) - \phi(M(\phi_{2n}, \phi_{2n+1})) \\ &< \psi(M(\phi_{2n}, \phi_{2n+1})) \text{ (by using (17)} \\ &\leq \psi(||\phi_{2n} - \phi_{2n+1}||_{E_0}). \text{ (by using (23))} \end{aligned}$$

Since ψ is monotonically increasing function, we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n} - \phi_{2n+1}||_{E_0}. \tag{24}$$

Similarly we have $||\phi_{2n+2}-\phi_{2n+3}||_{E_0} \le M(\phi_{2n+2},\phi_{2n+1}) \le ||\phi_{2n+2}-\phi_{2n+1}||_{E_0}$

$$= ||\phi_{2n+1}-\phi_{2n+2}||_{E_0}. \tag{25}$$

From (24) and (25), we have $||\phi_{n+1}-\phi_n||_{E_0} \le ||\phi_n-\phi_{n-1}||_{E_0}$ for all $n\in\mathbb{N}$. Therefore, the sequence $\{||\phi_{n+1}-\phi_n||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$, and hence convergent.

Let $\lim\limits_{n\to\infty}||\phi_{n+1}-\phi_n||_{E_0}=r$(say).

From (24), we have

$$||\phi_{2n+1}-\phi_{2n+2}||_{E_0} \le M(\phi_{2n},\phi_{2n+1}) \le ||\phi_{2n}-\phi_{2n+1}||_{E_0}.$$

On applying limits as $n\to\infty$, we get

$$r \le \lim_{n\to\infty} M(\phi_{2n},\phi_{2n+1}) \le r \text{ and hence } \lim_{n\to\infty} M(\phi_{2n},\phi_{2n+1}) = r.$$

From (21), we have

$$\psi(||\phi_{2n+1}-\phi_{2n+2}||_{E_0}) \le \psi(M(\phi_{2n},\phi_{2n+1})) - \phi(M(\phi_{2n},\phi_{2n+1})).$$

On applying limits as $n\to\infty$, we get $\psi(r)\le\psi(r)-\phi(r)$ and which implies that $r=0$.

Therefore,

$$\lim_{n\to\infty}||\phi_{n+1}-\phi_n||_{E_0}=0. \tag{26}$$

Now, we show that $\{\phi_n\}$ is a Cauchy sequence.

From (26), to prove $\{\phi_n\}$ is a Cauchy sequence it is enough to prove that $\{\phi_{2n}\}$ is a Cauchy sequence.

Suppose that $\{\phi_{2n}\}$ is not a Cauchy sequence.

Then, there exists $\epsilon>0$ and two subsequences $\{\phi_{2m_k}\}$ and $\{\phi_{2n_k}\}$ of $\{\phi_{2n}\}$ such that for any $k\in\mathbb{N}, m_k>n_k>k$ such that

$$||\phi_{2n_k}-\phi_{2m_k}||_{E_0}\ge\epsilon. \tag{27}$$

Let m_k be the smallest positive integer greater than n_k that is satisfying (27).

Then, $||\phi_{2n_k}-\phi_{2m_k}||_{E_0}\ge\epsilon$ and $||\phi_{2n_k}-\phi_{2m_k-2}||_{E_0}<\epsilon$.

We now show that $\lim\limits_{k\to\infty}||\phi_{2n_k}-\phi_{2m_k+1}||_{E_0}=\epsilon$.

Clearly

$$\epsilon \le ||\phi_{2n_k}-\phi_{2m_k}||_{E_0} \le ||\phi_{2n_k}-\phi_{2m_k+1}||_{E_0} + ||\phi_{2m_k+1}-\phi_{2m_k}||_{E_0}.$$

Now, by applying Proposition 1 with $a_k=||\phi_{2n_k}-\phi_{2m_k+1}||_{E_0}$ and $b_k=||\phi_{2m_k+1}-\phi_{2m_k}||_{E_0}$ we have

$$\epsilon \le \liminf_{k\to\infty}||\phi_{2n_k}-\phi_{2m_k+1}||_{E_0} + \limsup_{k\to\infty}||\phi_{2m_k+1}-\phi_{2m_k}||_{E_0},$$

and hence

$$\epsilon \le \liminf_{k\to\infty}||\phi_{2n_k}-\phi_{2m_k+1}||_{E_0}. \tag{28}$$

Clearly

$$\begin{aligned}||\phi_{2n_k}-\phi_{2m_k+1}||_{E_0} &\le ||\phi_{2n_k}-\phi_{2m_k-2}||_{E_0} + ||\phi_{2m_k-2}-\phi_{2m_k-1}||_{E_0}\\ &\quad + ||\phi_{2m_k-1}-\phi_{2m_k}||_{E_0} + ||\phi_{2m_k}-\phi_{2m_k+1}||_{E_0}\\ &< \epsilon + ||\phi_{2m_k-2}-\phi_{2m_k-1}||_{E_0}\end{aligned}$$

$$+ ||\phi_{2m_k-1} - \phi_{2m_k}||_{E_0} + ||\phi_{2m_k} - \phi_{2m_k+1}||_{E_0}.$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} \leq \epsilon. \tag{29}$$

From (28) and (29), we get

$$\lim_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} = \epsilon. \tag{30}$$

We now show that $\lim_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} = \epsilon$ for any $l_1, l_2 \in \mathbb{N}$.
Let $l_1, l_2 \in \mathbb{N}$.
We now consider

$$\begin{aligned}||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} \leq\ & ||\phi_{2n_k+l_1} - \phi_{2n_k+l_1-1}||_{E_0} + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_2-2}||_{E_0} \\ & + ... + ||\phi_{2n_k+1} - \phi_{2n_k}||_{E_0} + ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} \\ & + ||\phi_{2m_k+1} - \phi_{2m_k+2}||_{E_0} + ... + ||\phi_{2m_k+l_2-1} - \phi_{2m_k+l_2}||_{E_0}.\end{aligned}$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} \leq \epsilon. \tag{31}$$

We now consider

$$\begin{aligned}||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} \leq\ & ||\phi_{2n_k} - \phi_{2n_k+1}||_{E_0} + ||\phi_{2n_k+1} - \phi_{2n_k+2}||_{E_0} + ... \\ & + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_1}||_{E_0} + ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} \\ & + ||\phi_{2m_k+l_2} - \phi_{2m_k+l_2-1}||_{E_0} + ... + ||\phi_{2m_k+2} - \phi_{2m_k+1}||_{E_0}.\end{aligned}$$

Now, by applying Proposition 1 with $a_k = ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0}$ and
$b_k = (||\phi_{2n_k} - \phi_{2n_k+1}||_{E_0} + ||\phi_{2n_k+1} - \phi_{2n_k+2}||_{E_0} + ... + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_1}||_{E_0} + ||\phi_{2m_k+l_2} - \phi_{2m_k+l_2-1}||_{E_0} + ... + ||\phi_{2m_k+2} - \phi_{2m_k+1}||_{E_0})$
we have

$$\begin{aligned}\epsilon \leq & \liminf_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} + \limsup_{k\to\infty}(||\phi_{2n_k} - \phi_{2n_k+1}||_{E_0} \\ & + ||\phi_{2n_k+1} - \phi_{2n_k+2}||_{E_0} + ... + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_1}||_{E_0} \\ & + ||\phi_{2m_k+l_2} - \phi_{2m_k+l_2-1}||_{E_0} + ... + ||\phi_{2m_k+2} - \phi_{2m_k+1}||_{E_0}).\end{aligned}$$

Hence

$$\epsilon \leq \liminf_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0}. \tag{32}$$

From (31) and (32), we get that for any $l_1, l_2 \in \mathbb{N}$

$$\lim_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} = \epsilon. \tag{33}$$

Now, we choose $l_1, l_2 \in \mathbb{N}$ such that $2n_k + l_1$ is even, $2m_k + l_2$ is odd and $(2m_k + l_2) - (2n_k + l_1) = 1$.
From (24), we have

$$||\phi_{2n_k+l_1+1} - \phi_{2m_k+l_2+1}||_{E_0} \leq M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2}) \leq ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0}.$$

On applying limits as $k \to \infty$, we get

$$\epsilon \leq \lim_{k\to\infty} M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2}) \leq \epsilon \text{ and hence } \lim_{k\to\infty} M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2}) = \epsilon.$$

From (21), we have

$$\psi(||\phi_{2n_k+l_1+1} - \phi_{2m_k+l_2+1}||_{E_0}) \leq \psi(M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2})) - \phi(M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2})).$$

On applying limits as $k \to \infty$ we get,

$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon)$ and hence $\epsilon = 0$,

a contradiction.

Therefore, the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.

Since E_0 is complete, we have $\phi_n \to \phi^*$ as $n \to \infty$ for some $\phi^* \in E_0$.

Since R_c is topologically closed, we have $\phi^* \in R_c$.

Now, we show that ϕ^* is a PPF dependent common fixed point of S and T.

We now consider,

$$\begin{aligned} d(\phi^*(c), S\phi^*) \le M(\phi_{2k}, \phi^*) \\ &= \max\{||\phi_{2k} - \phi^*||_{E_0}, d(\phi_{2k}(c), T\phi_{2k}), d(\phi^*(c), S\phi^*), \\ &\qquad \tfrac{1}{2}[d(\phi^*(c), T\phi_{2k}) + d(\phi_{2k}(c), S\phi^*)]\} \\ &\le \max\{||\phi_{2k} - \phi^*||_{E_0}, ||\phi_{2k}(c) - \phi_{2k+1}(c)||_E + d(\phi_{2k+1}(c), T\phi_{2k}), d(\phi^*(c), S\phi^*), \\ &\qquad \tfrac{1}{2}[||\phi^*(c) - \phi_{2k+1}(c)||_E + d(\phi_{2k+1}(c), T\phi_{2k}) \\ &\qquad + ||\phi_{2k}(c) - \phi^*(c)||_E + d(\phi^*(c), S\phi^*)]\} \\ &= \max\{||\phi_{2k} - \phi^*||_{E_0}, ||\phi_{2k} - \phi_{2k+1}||_{E_0}, d(\phi^*(c), S\phi^*), \\ &\qquad \tfrac{1}{2}[||\phi^* - \phi_{2k+1}||_{E_0} + ||\phi_{2k} - \phi^*||_{E_0} + d(\phi^*(c), S\phi^*)]\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get

$$\begin{aligned} d(\phi^*(c), S\phi^*) &\le \lim_{k\to\infty} M(\phi_{2k}, \phi^*) \\ &\le \max\{0, 0, d(\phi^*(c), S\phi^*), \tfrac{1}{2}[d(\phi^*(c), S\phi^*)]\} \\ &= d(\phi^*(c), S\phi^*). \end{aligned}$$

Hence $\lim_{k\to\infty} M(\phi_{2k}, \phi^*) = d(\phi^*(c), S\phi^*)$.

Now,

$$\begin{aligned} d(\phi^*(c), S\phi^*)) &\le ||\phi^*(c) - \phi_{2k+1}(c)||_E + d(\phi_{2k+1}(c), S\phi^*) \\ &\le ||\phi^* - \phi_{2k+1}||_{E_0} + H_E(T\phi_{2k}, S\phi^*). \end{aligned}$$

Applying limits as $k \to \infty$, we get

$$d(\phi^*(c), S\phi^*)) \le \lim_{k\to\infty} H_E(T\phi_{2k}, S\phi^*),$$

and hence

$$\begin{aligned} \psi(d(\phi^*(c), S\phi^*)) &\le \lim_{k\to\infty} \psi(H_E(T\phi_{2k}, S\phi^*)) \\ &\le \lim_{k\to\infty} \psi(M(\phi_{2k}, \phi^*)) - \lim_{k\to\infty} \phi(M(\phi_{2k}, \phi^*)) \\ &= \psi(d(\phi^*(c), S\phi^*)) - \phi(d(\phi^*(c), S\phi^*)). \end{aligned}$$

Therefore, $\phi(d(\phi^*(c), S\phi^*)) = 0$ and hence $\phi^*(c) \in S\phi^*$.

Similarly we can prove that $\phi^*(c) \in T\phi^*$.

Therefore, ϕ^* is a PPF dependent common fixed point of S and T. □

5. Corollaries and Examples

Corollary 1. *Let $T : E_0 \to CB(E)$ and $f : E \to E$ be a function that satisfy the following conditions:*

(i) *T is weakly contractive type multi-valued mapping with respect to f,*
(ii) *$T\phi \subseteq f(R_c)(c)$ for any $\phi \in E_0$,*
(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *$f(R_c)$ is complete and*
(v) *$f(R_c) \subseteq R_c$.*

Then, T and f have a PPF dependent coincidence point in R_c.

Proof. Follows from Theorem 3 by choosing $\psi(t) = t,\ t \in \mathbb{R}^+$ in the inequality (1). □

By choosing $f = I,\ I$ the identity map in Theorem 3, we get the following corollary.

Corollary 2. *Let $T : E_0 \to CB(E)$ be a multi-valued mapping. Assume that T satisfy the following conditions:*

(i) *T is a generalized weakly contractive type multi-valued mapping,*
(ii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,*
(iii) *R_c is algebraically closed with respect to the difference.*

Then, T has a PPF dependent fixed point in R_c.

The following corollary follows by choosing $\psi(t) = t,\ t \in \mathbb{R}^+$ in Corollary 2.

Corollary 3. *Let $T : E_0 \to CB(E)$ be a mapping satisfy the following conditions:*

(i) *T is weakly contractive type multi-valued mapping,*
(ii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,*
(iii) *R_c is algebraically closed with respect to the difference.*

Then, T has a PPF dependent fixed point in R_c.

Corollary 4. *Let $T : E_0 \to CB(E)$ be a mapping satisfiy the following conditions:*

(i) *suppose that there exists $k \in [0, 1)$ such that*
$$H_E(T\alpha, T\beta)) \leq k\ \ ||\alpha - \beta||_{E_0} \text{ for all } \alpha, \beta \in E_0,$$
(ii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,*
(iii) *R_c is algebraically closed with respect to the difference.*

Then, T has a PPF dependent fixed point in R_c.

Proof. Follows by choosing $\phi(t) = (1 - k)t, t \in \mathbb{R}^+$ in Corollary 3. □

Corollary 5. *Let $S, T : E_0 \to CB(E)$ be two multi-valued mappings such that*

(i) *$H_E(T\alpha, S\beta)) \leq k \max\{||\alpha - \beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), S\beta), \frac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), S\beta)]\}$
for any $\alpha, \beta \in E_0$,*
(ii) *R_c is algebraically closed with respect to the difference and*
(iii) *$T\alpha \subseteq R_c(c)$ and $S\alpha \subseteq R_c(c)$ for all $\alpha \in E_0$.*

Then, S and T have a PPF dependent common fixed point in R_c.

Proof. Follows by choosing $\psi(t) = t$ and $\phi(t) = (1 - k)t$ for $t \in \mathbb{R}^+$ in Theorem 4. □

If $S = T$ in Theorem 4 and Corollary 5, we get the following corollaries.

Corollary 6. *Let $T : E_0 \to CB(E)$ be a multi-valued mapping. Assume that:*

(i) *there exist two functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(H_E(T\alpha, T\beta)) \leq \psi(M(\alpha, \beta)) - \phi(M(\alpha, \beta)) \tag{34}$$

for all $\alpha, \beta \in E_0$, where
$M(\alpha,\beta) = \max\{||\alpha-\beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), T\beta), \frac{1}{2}[d(\beta(c), T\alpha)+d(\alpha(c), T\beta)]\}$,

(ii) *R_c is algebraically closed with respect to the difference and*

(iii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$.*

Then, T has a PPF dependent fixed point in R_c.

Corollary 7. *Let $T : E_0 \to CB(E)$ be two multi-valued mappings such that*

(i) *$H_E(T\alpha, T\beta)) \leq k \max\{||\alpha-\beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), T\beta), \frac{1}{2}[d(\beta(c), T\alpha)+ d(\alpha(c), T\beta)]\}$*
for all $\alpha, \beta \in E_0$,

(ii) *R_c is algebraically closed with respect to the difference and*

(iii) *$T\alpha \subseteq R_c(c)$ for any $\alpha \in E_0$.*

Then, T has a PPF dependent fixed point in R_c.

Example 1. *Let $E = \mathbb{R}$, $c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R}$, $E_0 = C(I, E)$.*
Let $k \geq 1$. We define $f : E \to E$ by $f(x) = kx$ for any $x \in E$.
Clearly, f is a continuous function.
By definition, $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$ and
$f(R_c)(c) = \{(f \circ \phi)(c) \mid \phi \in R_c\} = \{f(\phi(c)) \mid \phi \in R_c\} = \{k\phi(c) \mid \phi \in R_c\}$.
First we show that $f(R_c) = R_c$.
Let $\alpha \in R_c$. Then $\alpha = \beta$ for some $\beta \in R_c$.
Clearly, $\alpha = k\frac{1}{k}\beta = k\eta$ (by Theorem 1, $\eta = \frac{1}{k}\beta \in R_c$) so that
$\alpha(x) = k\eta(x) = f(\eta(x)) = (f \circ \eta)(x)$ for any $x \in I$.
Therefore, $\alpha = f \circ \eta \in f(R_c)$ and hence

$$R_c \subseteq f(R_c). \tag{35}$$

Now, let $\alpha \in f(R_c)$. Then $\alpha = f \circ \beta$ for some $\beta \in R_c$.
Clearly, $\alpha(x) = (f \circ \beta)(x) = f(\beta(x)) = k\beta(x) = (k\beta)(x)$ for any $x \in I$.
Therefore, $\alpha = k\beta \in R_c$ and hence

$$f(R_c) \subseteq R_c. \tag{36}$$

From (35) and (36), we get $f(R_c) = R_c$.
Since E_0 is complete and R_c is topologically closed we have $f(R_c) = R_c$ is complete.
For any $\gamma \in \mathbb{R}$, we define $\phi_\gamma : I \to E$ by

$$\phi_\gamma(x) = \begin{cases} \gamma x^2 & \textit{if } x \in [\frac{1}{2}, 1] \\ \frac{\gamma}{x^2} & \text{if } x \in [1, 2]. \end{cases}$$

Clearly $\phi_\gamma \in E_0, ||\phi_\gamma||_{E_0} = ||\phi_\gamma(c)||_E$ and hence $\phi_\gamma \in R_c$ for any $\gamma \in \mathbb{R}$.
Let $\mathrm{F}_0 = \{\phi_\gamma \mid \gamma \in \mathbb{R}\}$.
Then, F_0 is algebraically closed with respect to the difference and $\mathrm{F}_0 \subseteq R_c$.

We observe that $\mathbb{R} = \{\phi_\gamma(c) \mid \gamma \in \mathbb{R}\} = \mathrm{F}_0(c) \subseteq R_c(c)$.
Clearly, $R_c(c) \subseteq \mathbb{R}$ *and hence* $f(R_c)(c) = R_c(c) = \mathbb{R}$.
We define $T : E_0 \to CB(E)$ *by* $T\phi = [0,\ \frac{k}{4}||\phi(c)||_E]$ *for any* $\phi \in E_0$.
Clearly, $T\phi \subseteq \mathbb{R} = R_c(c) = f(R_c)(c)$.
We define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = \frac{t^2}{2}$ *and*

$$\phi(t) = \begin{cases} \frac{15\ t^3}{32} & \textit{if } t \in [0,1] \\ \frac{15\ t}{32} & \text{if } \ t \geq 1. \end{cases}$$

Clearly, $\psi \in \Psi$ *and* $\phi \in \Phi$.
From the definition of Hausdorff distance, it follows that, for any $\alpha, \beta \in E_0$

$$H_E(T\alpha, T\beta) = \frac{k}{4}\begin{cases} ||\alpha(c)||_E - ||\beta(c)||_E & \textit{if } ||\alpha(c)||_E \geq ||\beta(c)||_E \\ ||\beta(c)||_E - ||\alpha(c)||_E & \text{if } ||\beta(c)||_E \geq ||\alpha(c)||_E \end{cases}$$

$$\begin{aligned} &= \tfrac{k}{4}|\ ||\alpha(c)||_E - ||\beta(c)||_E\ | = \tfrac{1}{4}|\ ||k\alpha(c)||_E - ||k\beta(c)||_E\ | \\ &\leq \tfrac{1}{4}|k\alpha(c) - k\beta(c)| = \tfrac{1}{4}|(f\circ\alpha)(c) - (f\circ\beta)(c)| \\ &= \tfrac{1}{4}||(f\alpha - f\beta)(c)||_E \\ &\leq \tfrac{1}{4}||f\alpha - f\beta||_{E_0}. \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(H_E(T\alpha, T\beta)) &\leq \psi(\tfrac{1}{4}||f\alpha - f\beta||_{E_0}) = \tfrac{1}{32}[\ ||f\alpha - f\beta||_{E_0}\ ]^2 \\ &\leq \psi(||f\alpha - f\beta||_{E_0}) - \phi(||f\alpha - f\beta||_{E_0}). \end{aligned}$$

Therefore, T *and* f *satisfy all the hypotheses of Theorem 3 and* $\phi_0 \in R_c$ *is a PPF dependent coincidence point of* T *and* f.

Example 2. *Let* $E = \mathbb{R},\ c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R},\ E_0 = C(I, E)$.
On continuing the same procedure as in the Example 1, we get $R_c(c) = \mathbb{R}$.
We define $T : E_0 \to CB(E)$ *by* $T\phi = [0,\ \frac{1}{5}||\phi(c)||_E]$ *for any* $\phi \in E_0$.
Clearly $T\phi \subseteq R_c(c)$.
We define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = 2t$ *and* $\phi(t) = \frac{6t}{5}$ *for any* $t \in \mathbb{R}^+$.
Clearly, $\psi \in \Psi$ *and* $\phi \in \Phi$.
Clearly, for any $\alpha, \beta \in E_0$, *we have*

$$\begin{aligned} H_E(T\alpha, T\beta) &\leq \tfrac{1}{5}||\alpha - \beta||_{E_0} \\ &\leq \tfrac{1}{5}\max\{||\alpha - \beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), T\beta), \tfrac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), T\beta)]\} \\ &= \tfrac{1}{5}M(\alpha, \beta). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(H_E(T\alpha, T\beta)) &\leq \psi(\tfrac{1}{5}M(\alpha, \beta)) = \tfrac{2}{5}M(\alpha, \beta) \\ &\leq 2M(\alpha, \beta) - \tfrac{6}{5}M(\alpha, \beta) \\ &= \psi(M(\alpha, \beta)) - \phi(M(\alpha, \beta)). \end{aligned}$$

Therefore, T *satisfies all the hypotheses of Corollary 6 and* $\phi_0 \in R_c$ *is a PPF dependent fixed point of* T.

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