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The Application of Euler-Rodrigues Formula over Hyper-Dual Matrices

Çağla Ramis* , Yusuf Yaylı and İrem Zengin

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ABSTRACT

The Lie group over the hyper-dual matrices and its corresponding Lie algebra are first introduced in this study. One of Euler's strategies called the Euler-Rodrigues formula is applied to the matrices of hyper-dual rotations. The fundamental relationship between the hyper-dual numbers and the dual numbers allows us to apply the formula on dual lines and two intersecting real lines in the three dimensional dual and Euclidean spaces, respectively.

Keywords: *Dual numbers, hyper-dual numbers, Euler-Rodrigues formula, Euler's theorem.*
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1. Introduction

The theory of numbers was an isolated serial despite the achieved brilliant results up to the nineteenth century. Gauss's Disquisitiones Arithmeticae initiated a new era, but over the years scientists have agreed that the structure of the real number system must be probed deeper and clarified on behalf of analysis. Moreover, considering the needs of natural and computer sciences, the necessity of developing the system has emerged. After some knowledge explosions occurred about the extension of the real number system, we were introduced with dual numbers by Clifford in the late nineteenth century [5].

A dual number (DN) is expressed in the form

$$\hat{a} = a + \varepsilon a^*; \quad a, a^* \in \mathbb{R}$$

where the dual unit ε satisfies the condition $\varepsilon^2 = 0$ and lies outside the domain of real numbers, namely $\varepsilon \neq 0$. Since a pure dual number has no inverse, dual numbers not a field. However, their algebra is a commutative ring which allows the computation of derivatives. The common practice of dual numbers arises in automatic differentiation which is a technique to implement the chain rule for computing derivatives [3, 13, 25, 28]. On the other hand, since Study's theorem leads the one-to-one correspondence between a unit dual vector and a line in 3-dimensional Euclidean space, the main contribution of dual numbers is originally demonstrated in kinematics by the operator of point-line displacements [27, 29]. When the geometric problems are studied in physics, quantum mechanics, robotics, computing or visualization, line geometry occurs naturally and provides the most elegant and efficient solutions [2, 14, 18, 19, 23, 26].

Recently, dual numbers were extended to the hyper-dual numbers by Fike and Alonso within the context of the second-order numerical differentiation, principally due to reduced computational time and errors [11]. Furthermore, the matrix representation of them is constructed to obtain higher-order derivatives [16]. A hyper dual number (HDN) $\tilde{a}$ is determined by two distinct non-real components with the properties $\varepsilon_1^2 = \varepsilon_2^2 = (\varepsilon_1\varepsilon_2)^2 = 0$ and takes the form

$$\tilde{a} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3; \quad a_i \in \mathbb{R}.$$

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*** Corresponding author***

The representation of the HDN is obtained as a combination of two dual numbers by Cohen and Shoham and this leads to arrive the meaning of HDN for physical systems [6]. By defining the hyper-dual velocity, the hyper-dual momentum and the hyper dual inertia operator, the first applications of HDN to rigid body motions are obtained by utilizing of the automatic differentiation feature of DN [7]. Other mathematical systems are built in connection with the hyper-dual quaternions and the hyper-dual split quaternions to represent rigid body's kinematics in three-dimensional Euclidean and Lorentzian spaces [1, 8].

The geometric interpretations of rigid body dynamics lead naturally to the study of Lie algebras and their corresponding Lie groups. Specifically, the sentence of fixed points has been investigated since the late of eighteenth century. Leonhard Euler realized that any displacement of a rigid body such that a point on the rigid body remains fixed in the three dimensional space [10, 21]. This stated theorem, named after Euler, tells us the existence of the axis of rotation. Finding the orthogonal matrix corresponded to a rotation about its axis was solved by Cayley's formula, and what's more following this method the matrix representation of a rotation was endowed with respect to the rotation angle and axis which is referred to as Euler-Rodrigues formula [9, 19]. In point-line geometry, the applications of Euler-Rodrigues formula were expanded with the dual axis and the dual rotation angle and called the dual Euler-Rodrigues formula by Kahveci et al. [17].

The historical process of rigid body transformations is motivated us to study on the Lie group $\widetilde{SO}(3)$ consisting of 3×3 hyper-dual matrices satisfying the properties $\widetilde{\mathbf{X}}^T\widetilde{\mathbf{X}} = \widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T = I_3$ and $\det \widetilde{\mathbf{X}} = 1$. By defining hyper-dual Euler-Rodrigues formula, we obtain the hyper-dual rotation matrices in terms of the hyper-dual angle and the hyper-dual axis. The fundamental relationship between HDN and the two DNs of order 1 allows us to interpret a hyper-dual rotation be a screw motion in three dimensional dual space. Later, we shall use one to one correspondence between a hyper-dual vector and a dual line and also two real lines, then the hyper-dual rotations act to rotate dual lines and the double real lines at the same time. This means further explaining the efficacy of hyper-dual rotations in the point-line geometry. For a regular matrix representation of hyper-dual rotations, we essentially prove the Euler's theorem for them and this naturally yields a method to obtain the rotation axis.

2. Basic Concepts and Notions

In this section we give a brief summary on dual numbers, hyper-dual numbers (HDN) and their vectors of order 3. The following list could provide us suitable mathematical tools for the whole study.

Nomenclature

$\mathbf{a}$, a	Scalar, Vector	ε	Dual unit
$\hat{a}$	Dual number	δ	Hyper-dual unit
$\hat{A}$	Dual vector	θ	Angle
$\tilde{a}$	Hyper-dual number	$\hat{\theta}$	Dual angle
$\tilde{A}$	Hyper-dual vector	$\tilde{\theta}$	Hyper-dual angle

The set of dual numbers, denoted by $\mathcal{D}$, is given as follows:

$$\mathcal{D}=\left\{\hat{a} = \mathbf{a}_0 + \varepsilon \mathbf{a}_1 \mid \mathbf{a}_0, \mathbf{a}_1 \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0\right\}. \tag{2.1}$$

The operations on the dual numbers are defined by

$$\begin{aligned} \hat{a} + \hat{b} &= (\mathbf{a}_0 + \mathbf{b}_0) + \varepsilon(\mathbf{a}_1 + \mathbf{b}_1), \\ \hat{a}\hat{b} &= \mathbf{a}_0\mathbf{b}_0 + \varepsilon(\mathbf{a}_0\mathbf{b}_1 + \mathbf{a}_1\mathbf{b}_0), \end{aligned} \tag{2.2}$$

where $\hat{a} = \mathbf{a}_0 + \varepsilon\mathbf{a}_1$ and $\hat{b} = \mathbf{b}_0 + \varepsilon\mathbf{b}_1$.

The set of dual vectors, denoted by $\mathcal{D}^3$,

$$\mathcal{D}^3=\left\{\hat{A} = a + \varepsilon a^* \mid a, a^* \in \mathbb{R}^3\right\} \tag{2.3}$$

has a module structure on the ring $\mathcal{D}$. The dot product $\odot_d : \mathcal{D}^3 \times \mathcal{D}^3 \to \mathcal{D}$ and the cross product $\otimes_d : \mathcal{D}^3 \times \mathcal{D}^3 \to \mathcal{D}^3$ of the dual vectors $\hat{A} = a + \varepsilon a^*$ and $\hat{B} = b + \varepsilon b^*$ are defined by

$$\begin{aligned} \hat{A} \odot_d \hat{B} &= a \cdot b + \varepsilon (a \cdot b^* + a^* \cdot b) \\ \hat{A} \otimes_d \hat{B} &= a \times b + \varepsilon (a \times b^* + a^* \times b) \end{aligned} \tag{2.4}$$

where "$\cdot$" and "$\times$" denote the usual scalar and vector products on $\mathbb{R}^3$, respectively.

The triple product expansion on dual vectors holds

$$\hat{A} \otimes_d (\hat{B} \otimes_d \hat{C}) = \left(\hat{A} \odot_d \hat{C}\right) \hat{B} - \left(\hat{A} \odot_d \hat{B}\right) \hat{C}. \tag{2.5}$$

Hence the modulus of $\hat{A}$ is given by

$$\left|\hat{A}\right|_d = \sqrt{\hat{A} \odot_d \hat{A}} = |a| + \varepsilon \frac{a \cdot a^*}{|a|}, \quad |a| \neq 0, \tag{2.6}$$

where "$|,|$" denotes the modulus in $\mathbb{R}^3$. If $\left|\hat{A}\right|_d = 1$, $\hat{A}$ is called a unit dual vector. The set of all unit dual vectors yields the unit dual sphere $\mathbb{S}_d$ as follows:

$$\mathbb{S}_d = \left\{ \hat{A} = a + \varepsilon a^* \mid \left|\hat{A}\right|_d = 1,\ \hat{A} \in \mathcal{D}^3 \right\}. \tag{2.7}$$

Theorem 2.1. *Each point of $\mathbb{S}_d$ corresponds to a directed line in $\mathbb{R}^3$ [27].*

Lemma 2.1. *Let $\hat{A}, \hat{B} \in \mathbb{S}_d$ and $\hat{\theta} = \theta + \varepsilon\theta^*$ be a dual angle between them, the followings are true:*

i. $\hat{A} \odot_d \hat{B} = \cos\hat{\theta} = \cos\theta - \varepsilon\theta^* \sin\theta$,

ii. $\left|\hat{A} \otimes_d \hat{B}\right| = \sin\hat{\theta} = \sin\theta + \varepsilon\theta^* \cos\theta$.

Definition 2.1 (Dual line)**.** Let $\hat{U}$ and $\hat{P}$ be a unit dual vector and a point in $\mathcal{D}^3$, respectively, then a dual line $\hat{L}$ in $\mathcal{D}^3$ is described by a linear equation of the form

$$\hat{L} = \hat{P} + \hat{t}\hat{U}$$

where the parameter $\hat{t} \in \mathcal{D}$.

For more properties of the theory of dual numbers, the reader is referred to [12, 19, 20].

Now we observe the properties of the dual numbers of order 2, briefly called the hyper-dual numbers. Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2$ be two dual units with the multiplication rules:

$$\begin{gathered} \varepsilon_1^2 = \varepsilon_2^2 = (\varepsilon_1\varepsilon_2)^2 = 0, \\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_1\varepsilon_2 \neq 0, \end{gathered} \tag{2.8}$$

the hyper-dual numbers have the following form:

$$\tilde{x} = \mathbf{a}_0 + \varepsilon_1\mathbf{a}_1 + \varepsilon_2\mathbf{a}_2 + \varepsilon_1\varepsilon_2\mathbf{a}_3. \tag{2.9}$$

In order to facilitate the rest of the paper, we call that $\varepsilon_1 = \varepsilon$ (the dual unit) and $\varepsilon_2 = \delta$ (the hyper-dual unit) and then the set $\mathcal{HD}$ of all the hyper-dual numbers can be stated as follows:

$$\mathcal{HD} = \left\{ \tilde{x} = \hat{x}_0 + \delta\hat{x}_1 \mid \hat{x}_0, \hat{x}_1 \in \mathcal{D};\ \varepsilon^2 = \delta^2 = 0;\ \varepsilon, \delta, \varepsilon\delta \neq 0 \right\} \tag{2.10}$$

where $\hat{x}_0 = \mathbf{a}_0 + \varepsilon\mathbf{a}_1$ and $\hat{x}_1 = \mathbf{a}_2 + \varepsilon\mathbf{a}_3$ are called the dual part and the hyper part of $\tilde{x}$.

Addition and multiplication in $\mathcal{HD}$ are defined by

$$\begin{aligned} \tilde{x}+\tilde{y} &= (\hat{x}_0+\hat{y}_0)+\delta(\hat{x}_1+\hat{y}_1), \\ \tilde{x}\tilde{y} &= \hat{x}_0\hat{y}_0+\delta(\hat{x}_0\hat{y}_1+\hat{x}_1\hat{y}_0), \end{aligned} \tag{2.11}$$

where $\tilde{x}=\hat{x}_0+\delta\hat{x}_1$ and $\tilde{y}=\hat{y}_0+\delta\hat{y}_1$, respectively.

The set of hyper-dual vectors, denoted by $\mathcal{HD}^3$, is given by

$$\mathcal{HD}^3=\left\{\tilde{X}=\hat{X}_0+\delta\hat{X}_1 \mid \hat{X}_0,\hat{X}_1\in\mathcal{D}^3\right\}. \tag{2.12}$$

The scalar product $\odot_h:\mathcal{HD}^3\times\mathcal{HD}^3\to\mathcal{HD}$ and the vector product $\otimes_h:\mathcal{HD}^3\times\mathcal{HD}^3\to\mathcal{HD}^3$ of any two hyper-dual vectors $\tilde{X}=\hat{X}_0+\delta\hat{X}_1$ and $\tilde{Y}=\hat{Y}_0+\delta\hat{Y}_1$ are defined by

$$\begin{aligned} \tilde{X}\odot_h\tilde{Y} &= \hat{X}_0\odot_d\hat{Y}_0+\delta\left(\hat{X}_0\odot_d\hat{Y}_1+\hat{X}_1\odot_d\hat{Y}_0\right), \\ \tilde{X}\otimes_h\tilde{Y} &= \hat{X}_0\otimes_d\hat{Y}_0+\delta\left(\hat{X}_0\otimes_d\hat{Y}_1+\hat{X}_1\otimes_d\hat{Y}_0\right), \end{aligned} \tag{2.13}$$

respectively. Furthermore, the Lagrange's formula holds

$$\tilde{X}\otimes_h(\tilde{Y}\otimes_h\tilde{Z})=\left(\tilde{X}\odot_h\tilde{Z}\right)\tilde{Y}-\left(\tilde{X}\odot_h\tilde{Y}\right)\tilde{Z}. \tag{2.14}$$

and the modulus of $\tilde{X}$ can be expressed by

$$\left|\tilde{X}\right|_h=\sqrt{\tilde{X}\odot_h\tilde{X}}=\left|\hat{X}_0\right|_d+\delta\frac{\hat{X}_0\odot_d\hat{X}_1}{\left|\hat{X}_0\right|_d} \tag{2.15}$$

where $\left|\hat{X}_0\right|_d\neq 0$. If $\hat{X}_0$ is a unit dual vector and $\hat{X}_0\odot_d\hat{X}_1=0$, then $\left|\tilde{X}\right|_h=1$ meaning that $\tilde{X}$ is a unit hyper-dual vector. The set $\mathbb{S}_h$ of all unit hyper-dual vectors,

$$\mathbb{S}_h=\left\{\tilde{X}=\hat{X}_0+\delta\hat{X}_1 \mid \left|\tilde{X}\right|_h=1,\ \tilde{X}\in\mathcal{HD}^3\right\} \tag{2.16}$$

is defined as the unit hyper-dual sphere.

Theorem 2.2. *Each point of $\mathbb{S}_h$ corresponds to a directed dual line in $\mathcal{D}^3$ [1].*

Theorem 2.3. *Let $\mathcal{S}_h\subset\mathbb{S}_h$ be a subset whose elements are unit hyper-dual vectors with the unit hyper parts $\left(\text{i.e. } \left|\tilde{X}\right|_h=\left|\hat{X}_1\right|_d=1\right)$. Then the points of $\mathcal{S}_h$ correspond to two intersecting perpendicular directed lines in $\mathbb{R}^3$ [1].*

Lemma 2.2. *Let $\tilde{X},\tilde{Y}\in\mathbb{S}_h$ and $\tilde{\theta}=\hat{\theta}_0+\delta\hat{\theta}_1$ be a hyper-dual angle between them, the followings are stated:*

i. $\tilde{X}\odot_h\tilde{Y}=\cos\tilde{\theta}=\cos\hat{\theta}_0-\delta\hat{\theta}_1\sin\hat{\theta}_0$,

ii. $\left|\tilde{X}\otimes_h\tilde{Y}\right|=\sin\tilde{\theta}=\sin\hat{\theta}_0+\delta\hat{\theta}_1\cos\hat{\theta}_0$.

3. Hyper Dual Euler-Rodrigues Formula and Its Application

We consider that a hyper-dual matrix $\widetilde{\mathbf{X}}=\left[\widetilde{\mathbf{X}}_{ij}\right]$; $\widetilde{\mathbf{X}}_{ij}=\hat{X}_{ij}+\delta\hat{\mathcal{X}}_{ij}\in\mathcal{HD}$ can be written as $\widetilde{\mathbf{X}}=\hat{X}+\delta\hat{\mathcal{X}}$ where $\hat{X},\hat{\mathcal{X}}$ are dual matrices. Let $\tilde{O}(3)$ denote the set of all hyper-dual orthogonal matrices of the form

$$\tilde{O}(3)=\left\{\widetilde{\mathbf{X}}\in\mathcal{HD}_3^3 \mid \widetilde{\mathbf{X}}^T\widetilde{\mathbf{X}}=\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}^T=I_3\right\}. \tag{3.1}$$

If the hyper-dual orthogonal matrix $\widetilde{\mathbf{X}}$ satisfies the property $\det \widetilde{\mathbf{X}} = 1$, then $\tilde{O}(3)$ becomes a Lie group called special hyper-dual orthogonal group and denoted by $\widetilde{SO}(3)$. Hence the Lie algebras along with the bracket operation over matrices is obtained

$$\widetilde{so}(3) = \left\{\tilde{\Omega} \in \mathcal{HD}_3^3 \mid \tilde{\Omega}^T = -\tilde{\Omega}\right\} \tag{3.2}$$

which consists of hyper-dual skew symmetric matrices (see [15]).

By appealing to the matrix geometric algebra, we could obtain an isomorphism between $\widetilde{so}(3)$ and $\mathcal{HD}^3$ given by

$$f : \widetilde{so}(3) \to \mathcal{HD}^3$$
$$\widetilde{\mathbf{S}} = \begin{pmatrix} 0 & -\tilde{s}_z & \tilde{s}_y \\ \tilde{s}_z & 0 & -\tilde{s}_x \\ -\tilde{s}_y & \tilde{s}_x & 0 \end{pmatrix} \to f(\widetilde{\mathbf{S}}) = \tilde{S} = (\tilde{s}_x, \tilde{s}_y, \tilde{s}_z) \tag{3.3}$$

where for any $\tilde{X} \in \mathcal{HD}^3$ the following is true:

$$\widetilde{\mathbf{S}}\tilde{X} = \tilde{S} \otimes_h \tilde{X}. \tag{3.4}$$

Let us now see how to define a hyper-dual rotation $\widetilde{\mathbf{R}}$ through the hyper-dual angle $\tilde{\theta}$ and the non-pure hyper-dual unit axis $\tilde{S}$. If we illustrate the method used in [17] to get "hyper-dual Euler-Rodrigues formula" for the hyper-dual matrix $\widetilde{\mathbf{R}} \in \widetilde{SO}(3)$, then we obtain

$$\widetilde{\mathbf{R}} = I_3 + \sin\tilde{\theta}\, \widetilde{\mathbf{S}} + (1 - \cos\tilde{\theta})\widetilde{\mathbf{S}}^2 \tag{3.5}$$

where $\widetilde{\mathbf{S}}$ is the corresponding skew symmetric matrix of the axis $\tilde{S} = (\tilde{s}_x, \tilde{s}_y, \tilde{s}_z)$ defined by (3.3). Thus we have

$$\widetilde{\mathbf{R}} = \begin{pmatrix} 1 - C_\theta(\tilde{s}_y^2 + \tilde{s}_z^2) & C_\theta\tilde{s}_x\tilde{s}_y - S_\theta\tilde{s}_z & C_\theta\tilde{s}_x\tilde{s}_z + S_\theta\tilde{s}_y \\ C_\theta\tilde{s}_x\tilde{s}_y + S_\theta\tilde{s}_z & 1 - C_\theta(\tilde{s}_x^2 + \tilde{s}_z^2) & C_\theta\tilde{s}_y\tilde{s}_z - S_\theta\tilde{s}_x \\ C_\theta\tilde{s}_x\tilde{s}_z - S_\theta\tilde{s}_y & C_\theta\tilde{s}_y\tilde{s}_z + S_\theta\tilde{s}_x & 1 - C_\theta(\tilde{s}_x^2 + \tilde{s}_y^2) \end{pmatrix} \tag{3.6}$$

where $S_\theta = \sin\tilde{\theta}$ and $C_\theta = 1 - \cos\tilde{\theta}$.

Theorem 3.1. *Let the hyper-dual angle and the hyper-dual vector be $\tilde{\theta} = \hat{\theta}_0 + \delta\hat{\theta}_1 \in \mathcal{HD}$ and $\tilde{S} = \hat{S}_0 + \delta\hat{S}_1 \in \mathbb{S}_h$, respectively, the hyper-dual rotation $\widetilde{\mathbf{R}}$ is expressed in the form*

$$\widetilde{\mathbf{R}} = \hat{\mathbf{A}} + \delta\hat{\mathbf{B}} \tag{3.7}$$

where the dual part of $\widetilde{\mathbf{R}}$ is a dual rotation through the dual angle $\hat{\theta}_0$ and the dual axis $\hat{S}_0$, and the hyper part of $\widetilde{\mathbf{R}}$ is defined by a dual skew-symmetric matrix $\hat{\mathbf{W}}$ as $\hat{\mathbf{B}} = \hat{\mathbf{W}}\hat{\mathbf{A}}$.

Proof. Under the isomorphism f, we have $f(\widetilde{\mathbf{S}}) = \tilde{S}$ which implies that $f(\hat{\mathbf{S}}_0) = \hat{S}_0$, $f(\hat{\mathbf{S}}_1) = \hat{S}_1$ and $\widetilde{\mathbf{S}} = \hat{\mathbf{S}}_0 + \delta\hat{\mathbf{S}}_1$. Consider the equations of trigonometric functions of $\tilde{\theta}$ given in lemma 2.2. Applying the equalities of $\tilde{\theta}$ and $\tilde{S}$ in (3.6), it is clear that $\widetilde{\mathbf{R}}$ is written the sum of two dual matrices in the form $\hat{\mathbf{A}} + \delta\hat{\mathbf{B}}$. The property of $\widetilde{\mathbf{R}} \in \widetilde{SO}(3)$ gives

$$\hat{\mathbf{A}}\hat{\mathbf{A}}^T + \delta\left(\hat{\mathbf{A}}\hat{\mathbf{B}}^T + \hat{\mathbf{B}}\hat{\mathbf{A}}^T\right) = I_3. \tag{3.8}$$

It follows that

$$\hat{\mathbf{A}}\hat{\mathbf{A}}^T = I_3 \text{ and } \hat{\mathbf{A}}\hat{\mathbf{B}}^T + \hat{\mathbf{B}}\hat{\mathbf{A}}^T = 0. \tag{3.9}$$

This implies that $\hat{\mathbf{A}}$ is a dual orthogonal matrix and (3.5) also gives that

$$\hat{\mathbf{A}} = I_3 + \sin\hat{\theta}_0\, \hat{S}_0 + (1 - \cos\hat{\theta}_0)\hat{S}_0^2$$

which defines a rotation through the angle $\hat{\theta}_0$ and the axis $\hat{S}_0$.

Suppose that $\hat{\mathbf{B}}\hat{\mathbf{A}}^T = \hat{\mathbf{W}}$, then the second equality of (3.9) holds $\hat{\mathbf{W}} = -\hat{\mathbf{W}}^T$ which completes the proof. □

Remark 3.1. If $\hat{\theta}_1$ and $\hat{S}_1$ are zero, then we have dual Euler-Rodrigues formula. After this if we further restrict $\tilde{\theta}$ and $\tilde{S}$ with real parts, (3.5) gives the formula in Euclidean space (see [9, 17]).

Note that $\widetilde{\mathbf{R}}$ inherits the property that $\widetilde{\mathbf{R}}(\tilde{X}) = \tilde{Y}$ where $\tilde{X}$ and $\tilde{Y}$ are any two hyper-dual vectors in $\mathcal{HD}^3$ such that $\left|\tilde{X}\right|_h = \left|\tilde{Y}\right|_h$. Then we can give the following.

Lemma 3.1. *If $\tilde{X}$ is a hyper-dual unit vector, then the hyper-dual rotation $\widetilde{\mathbf{R}}(\tilde{X}) = \tilde{Y}$ means the transforming of the corresponding dual line of $\tilde{X}$ to the corresponding dual line of $\tilde{Y}$.*

Proof. Theorem 2.2 gives that if $\tilde{X}$ is a hyper-dual unit vector, then corresponds to a directed dual line. Then $\widetilde{\mathbf{R}}(\tilde{X}) = \tilde{Y}$ is a dual line under the condition that $\left|\tilde{X}\right|_h = \left|\tilde{Y}\right|_h = 1$. That proves the idea. □

The previous lemma naturally leads to the interpretation of hyper-dual rotation in the dual 3-space. To accomplish this we firstly define a screw motion as a hyper-dual displacement about a hyper-dual axis. The hyper-dual displacement consists of a dual rotation by the dual part of the hyper-dual angle about the hyper-dual axis and a translation by the hyper part of the hyper-dual angle along the hyper-dual axis.

Assume that $\tilde{X} = \hat{X}_0 + \delta\hat{X}_1$ and $\tilde{Y} = \tilde{Y}_0 + \delta\tilde{Y}_1$ are hyper-dual unit vectors satisfying $\widetilde{\mathbf{R}}(\tilde{X}) = \tilde{Y}$ with the hyper dual rotation $\widetilde{\mathbf{R}}$ through $\tilde{\theta} = \hat{\theta}_0 + \delta\hat{\theta}_1$ the hyper-dual angle and the hyper-dual unit axis $\tilde{S} = (\tilde{s}_x, \tilde{s}_y, \tilde{s}_z)$. If $\hat{L}_x$ and $\hat{L}_y$ are the corresponding dual lines to $\tilde{X}$ and $\tilde{Y}$, respectively in $\mathcal{D}^3$, then we conclude that $\hat{\theta}_0$ is the dual angle between the dual vectors $\hat{X}_0$ and $\tilde{Y}_0$, and $\hat{\theta}_1$ is the closest distance between $\hat{L}_x$ and $\hat{L}_y$ by lemma 2.2.

Now we are ready to state the following theorem:

Theorem 3.2. *The hyper-dual rotation $\widetilde{\mathbf{R}}$ composed by the hyper-dual angle $\tilde{\theta} = \hat{\theta}_0 + \delta\hat{\theta}_1$ and the unit hyper-dual axis $\tilde{S}$ is associated with a screw motion which is the composition of a dual rotation by angle $\hat{\theta}_0$ about $\hat{L}_s$ and a translation by $\hat{\theta}_1$ along $\hat{L}_s$ which $\hat{L}_s$ is the dual line corresponding of the hyper-dual unit axis of $\widetilde{\mathbf{R}}$ in $\mathcal{D}^3$.*

Consider $\tilde{X} = \hat{X}_0 + \delta\hat{X}_1$ is a unit hyper-dual vector with $\left|\hat{X}_1\right|_d = 1$, thus we could see $\tilde{X}$ as two intersecting perpendicular directed lines in $\mathbb{R}^3$ from theorem 2.3. If we have a unit hyper-dual vector $\tilde{Y}$ with unit hyper part through the hyper-dual rotation $\widetilde{\mathbf{R}}(\tilde{X}) = \tilde{Y}$, then we can identify $\widetilde{\mathbf{R}}$ in Euclidean 3-space by composing theorem 2.3 and theorem 3.2 and state the following result:

Theorem 3.3. *The hyper-dual rotation $\widetilde{\mathbf{R}}$ transforms any two intersecting perpendicular directed lines to two intersecting perpendicular directed lines in $\mathbb{R}^3$.*

The following examples depict the idea of rotation in $\mathcal{HD}^3$ and its geometric reflections in $\mathcal{D}^3$ and $\mathbb{R}^3$.

Example 3.1. Let us find a hyper-dual rotation $\widetilde{\mathbf{R}}$ with the property that $\widetilde{\mathbf{R}}(\tilde{X}) = \tilde{Y}$ where the hyper-dual vectors are $\tilde{X} = (1 - \delta\varepsilon, \varepsilon - \delta\varepsilon, \varepsilon + \delta)$ and $\tilde{Y} = (\varepsilon + \delta, -1 + \delta\varepsilon, -\varepsilon)$.

First, the rotation axis $\tilde{S} = \hat{S}_0 + \delta\hat{S}_1$ is calculated by the theory of [29]

$$\begin{aligned} \tilde{S} &= \tilde{X} \otimes_h \tilde{Y} \\ &= (\varepsilon + \delta,\ \varepsilon + 2\delta\varepsilon,\ -1 + \delta\varepsilon) \end{aligned}$$

where $\hat{S}_0 = (\varepsilon, \varepsilon, -1)$ and $\hat{S}_1 = (1, 2\varepsilon, \varepsilon)$.

By applying (3.3), its corresponding skew symmetric matrix is obtained

$$\widetilde{\mathbf{S}} = \begin{pmatrix} 0 & 1 - \delta\varepsilon & \varepsilon + 2\delta\varepsilon \\ -1 + \delta\varepsilon & 0 & -\varepsilon - \delta \\ -\varepsilon - 2\delta\varepsilon & \varepsilon + \delta & 0 \end{pmatrix}$$

so that

$$\widetilde{\mathbf{S}}^2 = \begin{pmatrix} -1 + 2\delta\varepsilon & \delta\varepsilon & -\varepsilon - \delta \\ \delta\varepsilon & -1 & -\varepsilon - 2\delta\varepsilon \\ -\varepsilon - \delta & -\varepsilon - 2\delta\varepsilon & -2\delta\varepsilon \end{pmatrix}.$$

Use the properties of lemma 2.2 to compute the hyper-dual angle $\tilde{\theta} = \hat{\theta}_0 + \delta\hat{\theta}_1$ between $\tilde{X}$ and $\tilde{Y}$ such that

$$\cos\tilde{\theta} = \delta \; and \; \sin\tilde{\theta} = 1.$$

Applying the hyper-dual Euler-Rodrigues formula, we could construct the desired rotation as follows:

$$\widetilde{\mathbf{R}} = I_3 + \widetilde{\mathbf{S}} + (1-\delta)\widetilde{\mathbf{S}}^2$$

$$\widetilde{\mathbf{R}} = \begin{pmatrix} \delta\,(1+2\varepsilon) & 1 & \delta(-1+3\varepsilon) \\ -1+2\delta\varepsilon & \delta & -2\varepsilon-\delta\,(1+\varepsilon) \\ -2\varepsilon-\delta(1+\varepsilon) & \delta(1-\varepsilon) & 1-2\delta\varepsilon \end{pmatrix}$$

and its two dual components are

$$\hat{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -2\varepsilon \\ -2\varepsilon & 0 & 1 \end{pmatrix} \text{ and } \hat{\mathbf{B}} = \begin{pmatrix} 1+2\varepsilon & 0 & -1+3\varepsilon \\ 2\varepsilon & 1 & -(1+\varepsilon) \\ -(1+\varepsilon) & 1-\varepsilon & -2\varepsilon \end{pmatrix}$$

where $\widetilde{\mathbf{R}} = \hat{\mathbf{A}}+\delta\hat{\mathbf{B}}$.

Remark here that $\widetilde{\mathbf{R}}(\tilde{S}) = \tilde{S}$; $\hat{\mathbf{A}}(\hat{S}_0) = \hat{S}_0$ and the dual rotation angle of $\hat{\mathbf{A}}$ is $\hat{\theta}_0 = \dfrac{\pi}{2}$; $\hat{\mathbf{B}} = \hat{\mathbf{W}}\hat{\mathbf{A}}$ is composed by the skew symmetric matrix

$$\hat{\mathbf{W}} = \begin{pmatrix} 0 & -1 & -1+\varepsilon \\ 1 & 0 & -(1+\varepsilon) \\ 1-\varepsilon & 1+\varepsilon & 0 \end{pmatrix}.$$

Noting that

$$\begin{aligned} \tilde{X} &= \hat{X}_0 + \delta\hat{X}_1 \text{ with } \hat{X}_0 = (1,\varepsilon,\varepsilon),\ \hat{X}_1 = (-\varepsilon,-\varepsilon,1) \in \mathcal{D}^3 \\ \tilde{Y} &= \hat{Y}_0 + \delta\hat{Y}_1 \text{ with } \hat{Y}_0 = (\varepsilon,-1,-\varepsilon),\ \hat{Y}_1 = (1,\varepsilon,0) \in \mathcal{D}^3 \\ \tilde{S} &= \hat{S}_0 + \delta\hat{S}_1 \text{ with } \hat{S}_0 = (\varepsilon,\varepsilon,-1),\ \hat{S}_1 = (1,2\varepsilon,\varepsilon) \in \mathcal{D}^3 \end{aligned}$$

where $\left|\hat{X}_0\right|_d = \left|\hat{Y}_0\right|_d = \left|\hat{S}_0\right|_d = 1$, $\hat{X}_0 \odot_d \hat{X}_1 = 0$, $\hat{Y}_0 \odot_d \hat{Y}_1 = 0$ and $\hat{S}_0 \odot_d \hat{S}_1 = 0$. This means $\tilde{X}, \tilde{Y}, \tilde{S} \in \mathbb{S}_h$ so that they each have the concept of a directed line in $\mathcal{D}^3$ given as follows:

$$\begin{aligned} \tilde{X} &\longleftrightarrow \hat{L}_{\tilde{X}} = (\varepsilon,-1,-\varepsilon) + \hat{t}_x(1,\varepsilon,\varepsilon), \\ \tilde{Y} &\longleftrightarrow \hat{L}_{\tilde{Y}} = (0,-\varepsilon,1) + \hat{t}_y(\varepsilon,-1,-\varepsilon), \\ \tilde{S} &\longleftrightarrow \hat{L}_{\tilde{S}} = (2\varepsilon,-1,-\varepsilon) + \hat{t}_s(\varepsilon,\varepsilon,-1), \end{aligned}$$

where $\hat{t}_x, \hat{t}_y, \hat{t}_s \in \mathcal{D}$.

Conclude that $\widetilde{\mathbf{R}}$ turns the directed line $\hat{L}_{\tilde{X}}$ to $\hat{L}_{\tilde{Y}}$ with respect to the rotation by the angle $\hat{\theta}_0$ about $\hat{L}_{\tilde{S}}$ and a translation by $\hat{\theta}_1 = -1$ along $\hat{L}_{\tilde{S}}$ in dual 3-space.

For a further discussion of $\widetilde{\mathbf{R}}$, notice that $\left|\hat{X}_1\right|_d = \left|\hat{Y}_1\right|_d = 1$, then $\tilde{X}, \tilde{Y} \in \mathcal{S}_h$ and we have their correspondents as two intersecting perpendicular directed lines in $\mathbb{R}^3$ given by

$$\tilde{X} \longleftrightarrow \begin{cases} d_{\hat{X}_0} = (0,-1,1) + \lambda_0(1,0,0) \\ d_{\hat{X}_1} = (1,-1,0) + \lambda_1(0,0,1) \end{cases}$$

$$\tilde{Y} \longleftrightarrow \begin{cases} d_{\hat{Y}_0} = (1,0,1) + \mu_0(0,-1,0) \\ d_{\hat{Y}_1} = (0,0,1) + \mu_1(1,0,0) \end{cases}$$

where $\lambda_0, \lambda_1, \mu_0, \mu_1 \in \mathbb{R}$. Thus we could clearly see that the mapping $\widetilde{\mathbf{R}}(\tilde{X})$ sends the two intersecting perpendicular directed lines $\left\{d_{\hat{X}_0}, d_{\hat{X}_1}\right\}$ to $\left\{d_{\hat{Y}_0}, d_{\hat{Y}_1}\right\}$ in Euclidean 3-space.

Example 3.2. Let the dual orthogonal matrix $\hat{\mathbf{A}}$ be a rotation by the angle $\theta = \dfrac{\pi}{2}$ about the $Oz-$axis and a translation by $\theta^{*} = 1$ along the axis, then we have

$$\hat{\mathbf{A}} = \begin{pmatrix} -\varepsilon & -1 & 0 \\ 1 & -\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying the motion $\hat{\mathbf{A}}$ over the two intersecting perpendicular directed lines composed by the $Ox-$axis and the $Oy-$axis where the axes are thought as $\hat{l}_x = (1,0,0) + \varepsilon 0$ and $\hat{l}_y = (0,1,0) + \varepsilon 0$, we obtain their images by

$$\hat{k}_x = (-1, -\varepsilon, 0) \text{ and } \hat{k}_y = (-\varepsilon, 1, 0),$$

respectively.

Indeed, the dual motion $\hat{\mathbf{A}}$ allows us to utilize itself on the hyper-dual vector $\hat{l}_x + \delta \hat{l}_y$ as follows:

$$\begin{pmatrix} -\varepsilon & -1 & 0 \\ 1 & -\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \delta \\ 0 \end{pmatrix} = \begin{pmatrix} -\varepsilon - \delta \\ 1 - \delta\varepsilon \\ 0 \end{pmatrix} = \hat{k}_x + \delta \hat{k}_y.$$

Thus, we may well think that the dual orthogonal matrices can play the role by rotating the hyper-dual vectors.

4. Hyper-Dual Euler's Theorem

In spherical geometry, Euler's fixed point theorem states that if a sphere is moved around its centre, then it is always possible to find a diameter which remains fixed. Since there is a one to one correspondence between a spatial rotation and a 3×3 rotation matrix R given with $RR^T = R^T R = I_3$ and $\det R = 1$, the matrix expression of this theorem proves the existence of a non-zero vector s satisfying $Rs = s$. Now let us state the theorem in terms of hyper-dual rotation matrices which is called hyper-dual Euler's theorem and prove the only theorem of this section.

Theorem 4.1. *For any hyper-dual rotation matrix $\widetilde{\mathbf{R}} \in \widetilde{SO}(3)$ there exists a non-zero hyper-dual vector $\tilde{S} \in \mathcal{HD}^3$, such that $\widetilde{\mathbf{R}}\tilde{S} = \tilde{S}$.*

Proof. Let $\widetilde{\mathbf{R}}$ be the hyper-dual rotation matrix satisfying $\widetilde{\mathbf{R}}\widetilde{\mathbf{R}}^T = \widetilde{\mathbf{R}}^T\widetilde{\mathbf{R}} = I_3$, $\det \widetilde{\mathbf{R}} = 1$. $\widetilde{\mathbf{R}}$ can be expressed by the sum of a symmetric matrix and a skew-symmetric matrix as follows:

$$\widetilde{\mathbf{R}} = \frac{\widetilde{\mathbf{R}} + \widetilde{\mathbf{R}}^T}{2} + \frac{\widetilde{\mathbf{R}} - \widetilde{\mathbf{R}}^T}{2} \tag{4.1}$$

Let the skew-symmetric part of $\widetilde{\mathbf{R}}$ be $\widetilde{\mathbf{S}}$, then we can see that the following is true

$$\widetilde{\mathbf{R}}\widetilde{\mathbf{S}}\widetilde{\mathbf{R}}^T = \widetilde{\mathbf{S}}. \tag{4.2}$$

If we apply the isomorphism f given by (3.3) for $\widetilde{\mathbf{S}}$, there exists a corresponding hyper-dual vector $\tilde{S} \in \mathcal{HD}^3$ such that $f(\widetilde{\mathbf{S}}) = \tilde{S}$. Thus we have $\widetilde{\mathbf{S}} = f^{-1}_{\tilde{S}}$ and $f^{-1}_{\tilde{S}}(\tilde{\alpha}) = \tilde{S} \otimes_h \tilde{\alpha}$ for all $\tilde{\alpha} \in \mathcal{HD}^3$. The property of cross product yields that

$$\widetilde{\mathbf{R}}(\tilde{S} \otimes_h \tilde{\alpha}) = \widetilde{\mathbf{R}}\tilde{S} \otimes_h \widetilde{\mathbf{R}}\tilde{\alpha} \tag{4.3}$$

which means $\widetilde{\mathbf{R}} f^{-1}_{\tilde{S}} = f^{-1}_{\widetilde{\mathbf{R}}\tilde{S}}\widetilde{\mathbf{R}}$.

By using the property of $\widetilde{\mathbf{R}}$, we have

$$\widetilde{\mathbf{R}} f^{-1}_{\tilde{S}} \widetilde{\mathbf{R}}^T = f^{-1}_{\widetilde{\mathbf{R}}\tilde{S}} \tag{4.4}$$

Combining (4.2) and (4.4), we can state

$$f^{-1}_{\widetilde{\mathbf{R}}\tilde{S}} = \widetilde{\mathbf{S}} \tag{4.5}$$

which illuminates $\widetilde{\mathbf{R}}\tilde{S} = \tilde{S}$.

Note that if $\widetilde{\mathbf{R}} = \widetilde{\mathbf{R}}^T$, then $\widetilde{\mathbf{S}} = 0$ so that $\widetilde{\mathbf{R}}^2 = I_3$ and we can clearly see that $\widetilde{\mathbf{R}}\left(I_3 + \widetilde{\mathbf{R}}\right) = I_3 + \widetilde{\mathbf{R}}$. This means that each column of $I_3 + \widetilde{\mathbf{R}}$ remains fixed and the non-zero of them becomes as the fixed vector in this case.

This completes the proof. $\square$

Although the previous proof is given for the hyper-dual Euler's theorem, we actually obtain a method to calculate the axis of any hyper-dual rotation. Let us see how the method works on hyper-dual orthogonal matrices.

Example 4.1. Consider the hyper-dual matrix

$$\widetilde{\mathbf{R}} = \begin{pmatrix} 2\varepsilon & -1 & \delta(4\varepsilon - 2) \\ 1 & 2\varepsilon & 2\delta\varepsilon \\ 2\delta\varepsilon & \delta(4\varepsilon - 2) & 1 \end{pmatrix} \in \widetilde{SO}(3).$$

The skew-symmetric part of $\widetilde{\mathbf{R}}$ is

$$\widetilde{\mathbf{S}} = \begin{pmatrix} 0 & -1 & \delta(\varepsilon - 1) \\ 1 & 0 & \delta(1 - \varepsilon) \\ \delta(1 - \varepsilon) & \delta(\varepsilon - 1) & 0 \end{pmatrix}$$

and the corresponding vector of $\widetilde{\mathbf{S}}$ is

$$\tilde{S} = (\delta(\varepsilon - 1),\ \delta(\varepsilon - 1),\ 1)$$

where see that $\widetilde{\mathbf{R}}\tilde{S} = \tilde{S}$.

Example 4.2. Consider the hyper-dual orthogonal matrix $\widetilde{\mathbf{R}} = \begin{pmatrix} -\varepsilon & 1 & 0 \\ 1 & \varepsilon & 0 \\ 0 & 0 & -1 \end{pmatrix}$ satisfying $\widetilde{\mathbf{R}} = \widetilde{\mathbf{R}}^T$. Then the fixed vector $\tilde{S}$ will be obtained by the non-zero column of $I_3 + \widetilde{\mathbf{R}}$ which is $\tilde{S} = (1 - \varepsilon, 1, 0)$.

5. Conclusion

Several methods have been developed to describe dynamics of a rigid body in three dimensions. The calculation techniques have been also changed for past 20 or 30 years. The innovations, especially in the digital platforms, compel us to comply with mathematics in the new systems. In 2011, one of them introduced by Fike et al. and called the hyper-dual numbers (HDN) [11]. The highlights of HDN are observed in applied mathematics for numerical derivatives, in robotics for rigid body motions, etc. [1, 6, 7, 8, 28]. To contribute to the HDN theory in a geometric sense, we examine the rigid body transformations in terms of orthogonal matrices in this study. We firstly define the hyper-dual Euler-Rodrigues formula and build the algorithm to obtain a rotation through the hyper-dual angle and the hyper-dual vector. Since a hyper-dual number can be explained by two dual numbers, this means what is done in hyper-dual space should have a meaning in dual space. This question leads to discuss the geometric meaning of the hyper-dual rotation in dual space, then we see that it implies a dual screw motion. Moreover, one-to-one correspondence between the elements of unit hyper-dual sphere and a dual line, and also two real lines illuminate the geometric effects of hyper dual rotations in dual and Euclidean spaces.

In this paper, hyper-dual Euler-Rodrigues formula is given as a generalization of Euler-Rodrigues formulas over dual matrices and real matrices. The formula yields rotation matrices with specified rotation axes and rotation angles depending on the vector space. The inverse problem is that if a rotation matrix is given, then the rotation axis and the rotation angle can be found? We answered this in the last section by proving the Euler's theorem for the hyper-dual rotation matrices. For a further discussion, notice that a hyper-dual rotation corresponds a spatial displacement in dual space and screw motions of Euclidean space are represented by dual orthogonal matrices. Consequently, we naturally obtain the screw axis and prove the Chasles' theorem in section four that plays exactly the same role in theory of given by [4, 22, 24].

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Affiliations

ÇAĞLA RAMİS
ADDRESS: Department of Mathematics, Faculty of Science and Arts, Nevşehir Hacı Bektaş Veli University, Nevşehir- Türkiye.
E-MAIL: cramis@nevsehir.edu.tr
ORCID ID: 0000-0002-2809-8324

YUSUF YAYLI
ADDRESS: Department of Mathematics, Faculty of Science, Ankara University, Ankara-Türkiye.
E-MAIL: yayli@science.ankara.edu.tr
ORCID ID: 0000-0003-4398-3855

İREM ZENGİN
ADDRESS: Department of Mathematics, Faculty of Science and Arts, İzmir Democracy University, İzmir-Türkiye.
E-MAIL: irem.zengin@idu.edu.tr
ORCID ID: 0000-0003-1642-5559