http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 2, Pages 500–511 (2023)
DOI:10.31801/cfsuasmas.1127250
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:June 07, 2022; Accepted: October 31, 2022

IDENTIFICATION OF THE TIME-DEPENDENT LOWEST TERM IN A FOURTH ORDER IN TIME PARTIAL DIFFERENTIAL EQUATION

Ibrahim TEKIN
Department of Fundamental Sciences, Rafet Kayis Faculty of Engineering,
Alanya Alaaddin Keykubat University, Antalya, TÜRKİYE

ABSTRACT. In this article, identification of the time-dependent lowest term in a fourth order in time partial differential equation (PDE) from knowledge of a boundary measurement is studied by means of contraction mapping.

1. Introduction

Fourth order derivative in time arises in various fields. For instance, in the Taylor series expansion of the Hubble law [22], in the study of chaotic hyper jerk systems [2] and in the kinematic performance of long-dwell mechanisms of linkage type [8]. The fourth order in time equation, that is our motivation point, was introduced and investigated by Dell'Oro and Pata [5] for the first time

$$\partial_{\tau\tau\tau\tau}y(x,\tau)+\alpha\partial_{\tau\tau\tau}y(x,\tau)+\beta\partial_{\tau\tau}y(x,\tau)-\gamma\partial_{xx\tau\tau}y(x,\tau)-\delta\partial_{xx\tau}y(x,\tau)-\rho\partial_{xx}y(x,\tau)=0$$

where $\alpha,\beta,\gamma,\delta,\rho$ are real numbers. This model is obtained from the third-order Moore–Gibson–Thompson equation with memory, which has been extensively studied in the literature, [7,13,14]. More recently, this model has attracted the attention of many authors, see [3,15,16,18,19].

Consider the third order in time nonlinear partial differential equation model in abstract form

$$\partial_{\tau\tau\tau}y(x,\tau)+\alpha\partial_{\tau\tau}y(x,\tau)-c^2\partial_{xx}y(x,\tau)-b\partial_{xx\tau}y(x,\tau)=G(x,\tau,y,y_\tau,y_{\tau\tau}) \tag{1}$$

where $G(x,t,y,y_\tau,y_{\tau\tau})$ is a non-linear or linear function and $\alpha,c,b>0$ are given parameters. This type of model is often called Moore-Gibson-Thompson equation and appeared in many scientific fields such as nonlinear acoustics, medical ultrasound, viscoelasticity and thermoelasticity, [4,6,10–12,20].

2020 *Mathematics Subject Classification.* 35R30, 35G05, 45D05.
Keywords. Inverse problems for PDEs, fourth order in time PDE, existence and uniqueness.
ibrahim.tekin@alanya.edu.tr 0000-0001-6725-5663
.

Taking the subtraction
$$\partial_\tau(1) - \alpha(1),$$
we obtain
$$\partial_{\tau\tau\tau\tau} y(x,\tau) - \alpha^2 \partial_{\tau\tau} y(x,\tau) - b\partial_{xx\tau\tau} y(x,\tau) + \alpha c^2 \partial_{xx} y(x,\tau) + (\alpha b - c^2)\partial_{xx\tau} y(x,\tau) = \partial_\tau G - \alpha G. \quad (2)$$

Taking into account that the critical parameter ($C.P. \equiv \alpha - \frac{c^2}{b}$) of the third order in time equation (1) is zero. i.e. the energy is conservative and no decay of the energy occurs. Then $\alpha b - c^2 = 0$. In this case, the fourth order in time equation (2) reads
$$\partial_{\tau\tau\tau\tau} y(x,\tau) + \beta \partial_{\tau\tau} y(x,\tau) - \gamma \partial_{xx\tau\tau} y(x,\tau) - \rho \partial_{xx} y(x,\tau) = F(x,\tau,y,y_\tau,y_{\tau\tau}), \quad (3)$$
where $F(x,\tau,y,y_\tau,y_{\tau\tau}) = \partial_\tau G - \alpha G$, $\beta = -\alpha^2$, $\gamma = b$ and $\rho = -\alpha c^2$.

In this paper, we choose the right hand side of the fourth order in time PDE (3) as a linear function such that $F(x,\tau,y,y_\tau,y_{\tau\tau}) = a(\tau)y(x,\tau) + f(x,\tau)$. Our aim is to investigate the solvability of the inverse problem of simultaneous identification of the solely time-dependent lowest term ($a(\tau)$) and displacement function ($y(x,\tau)$) in the fourth order in time PDE
$$\partial_{\tau\tau\tau\tau} y(x,\tau) + \beta \partial_{\tau\tau} y(x,\tau) - \gamma \partial_{xx\tau\tau} y(x,\tau) - \rho \partial_{xx} y(x,\tau) = a(\tau)y(x,\tau) + f(x,\tau), \quad (4)$$
for $(x,\tau) \in D_T$ subject to the initial conditions
$$y(x,0) = \xi_0(x),\ y_\tau(x,0) = \xi_1(x), y_{\tau\tau}(x,0) = \xi_2(x), y_{\tau\tau\tau}(x,0) = \xi_3(x),\ x \in [0,1], \quad (5)$$
and the boundary conditions
$$y(0,\tau) = y_x(1,\tau) = 0,\ \tau \in [0,T], \quad (6)$$
and the additional condition
$$y(1,\tau) = h(\tau),\ \tau \in [0,T], \quad (7)$$
where $D_T = \{(x,\tau) : 0 \le x \le 1,\ 0 \le \tau \le T\}$ for some fixed $T > 0$, $\beta, \gamma, \rho > 0$ are given constants, $f(x,\tau)$ is the force function, $\xi_0(x)$, $\xi_1(x)$, $\xi_2(x)$, $\xi_3(x)$ are initial displacements, and $h(\tau)$ is the extra measurement to obtain the solution of the inverse problem.

The inverse problems of determining time or space dependent coefficients for the higher order in time (more than 2) PDEs attract many scientists. The inverse problem of recovering the solely space dependent and solely time dependent coefficients for the third order in time PDEs are studied by [1] and [21], respectively. More recently, in [9] authors studied the inverse problem of determining time dependent potential and time dependent force terms from the third order in time partial differential equation by considering the critical parameter equal to zero.

Main purpose of this paper is the simultaneous identification of the time-dependent lowest coefficient $a(\tau)$, and $y(x,\tau)$, for the first time, from the equation (4), initial conditions (5), homogeneous boundary conditions (6) and additional condition (7) under the assumption on the parameters.

The article is organized as following: In Section 2, we first present the eigenvalues and eigenfunctions of the corresponding Sturm-Liouville spectral problem for equation (4). Then two Banach spaces are introduced and roots of the fourth order polynomial (quartic) are investigated. In Section 3, we transform the inverse problem into the system of integral equations which are Volterra type by using the eigenfunction expansion method. Then, the theorem of the existence and uniqueness of the solution of the inverse problem is proved via Banach fixed point theorem for sufficiently small times under some conformity and consistency conditions on the initial and boundary data.

2. Auxiliary Spectral Problem and Preliminaries

The corresponding spectral problem of the inverse problem (4)-(7) is

$$\begin{cases} W''(x) + \lambda W(x) = 0, \quad 0 \le x \le 1, \\ \\ W(0) = W'(1) = 0. \end{cases} \tag{8}$$

The eigenvalues and corresponding eigenfunctions of these eigenvalues of the spectral problem (8) are $\lambda_n = \left(\frac{2n+1}{2}\pi\right)^2$ and $W_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n = 0, 1, 2, ...$, respectively. The system of eigenfunctions $W_n(x)$ are biorthonormal on $[0, 1]$, i.e.:

$$\int_0^1 W_n(x)W_m(x)dx = \begin{cases} 1 & , m = n \\ 0 & , m \neq n \end{cases} .$$

Also the system $W_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n = 0, 1, 2, ...$ forms a Riesz basis in $L_2[0, 1]$.

Now, let us introduce two Banach spaces that are connected with the eigenvalues and eigenfunctions of the auxiliary spectral problem (8):

i:

$$B_T = \Big\{ y(x,\tau) = \sum_{n=0}^{\infty} y_n(\tau)W_n(x) : \; y_n(\tau) \in C[0,T],$$
$$J_T(y) = \left(\sum_{n=0}^{\infty} (\lambda_n^{5/2} \left\| y_n(\tau) \right\|_{C[0,T]})^2 \right)^{1/2} < +\infty \Big\}, \tag{9}$$

where $J_T(y) := \left\| y(x,\tau) \right\|_{B_T}$ is the norm of the function $y(x,\tau)$.

ii: $E_T = B_T \times C[0,T]$ is a Banach space with the norm

$$\left\| \nu(x,\tau) \right\|_{E_T} = \left\| y(x,\tau) \right\|_{B_T} + \left\| a(\tau) \right\|_{C[0,T]},$$

where $\nu(x,\tau) = \{y(x,\tau), a(\tau)\}$ is a vector function.

These spaces are suitable to investigate the solution of the inverse problem (4)-(7).

Consider the quartic polynomial $P(k)$

$$P(k) = k^4 + (\beta + \gamma\lambda_n)k^2 + \rho\lambda_n.$$

Let us denote $\Delta_n = (\beta+\gamma\lambda_n)^2 - 4\rho\lambda_n$, and consider $\Delta_n > 0$. Therefore, the roots of the quartic polynomial $P(k)$ are

$$\begin{aligned} k_{1,2} &= \pm\sqrt{-s_n}, \\ k_{3,4} &= \pm\sqrt{-\overline{s}_n}, \end{aligned}$$

where $s_n = \frac{\beta+\gamma\lambda_n-\sqrt{\Delta_n}}{2}$, and $\overline{s}_n = \frac{\beta+\gamma\lambda_n+\sqrt{\Delta_n}}{2}$. Since β, γ, ρ, and λ_n are strictly positive, s_n, and $\overline{s}_n$ are also positive. Thus we have four complex conjugate roots

$$\begin{aligned} k_{1,2} &= \pm i p_n, \\ k_{3,4} &= \pm i r_n, \end{aligned}$$

where $p_n = \sqrt{\frac{\beta+\gamma\lambda_n-\sqrt{\Delta_n}}{2}}$, $r_n = \sqrt{\frac{\beta+\gamma\lambda_n+\sqrt{\Delta_n}}{2}}$ and $s_n = p_n^2$, $\overline{s}_n = r_n^2$.

3. Existence and Uniqueness

In this section, our aim is to set and prove the main theorem that is about the unique solvability of the inverse problem for the fourth order in time PDE. Before giving these let us define the classical solution of the inverse problem.

Let the pair of functions $\{y(x,\tau), a(\tau)\}$ be from the class $C^{2,4}(D_T) \times C[0,T]$ and satisfies the equation (4) and conditions (5)-(7). Then we call that the pair $\{y(x,\tau), a(\tau)\}$ is the classical solution of the inverse problem (4)-(7).

The existence and uniqueness theorem of the solution of the inverse problem is as follows:

Theorem 1. *Let the assumptions*

A$_1$: $\xi_0(x) \in C^4[0,1]$, $\xi_0^{(5)}(x) \in L_2[0,1]$,
$\xi_0(0) = \xi_0'(1) = \xi_0''(0) = \xi_0'''(1) = \xi_0^{(4)}(0) = 0$,
A$_2$: $\xi_1(x) \in C^3[0,1]$, $\xi_1^{(4)}(x) \in L_2[0,1]$,
$\xi_1(0) = \xi_1'(1) = \xi_1''(0) = \xi_1'''(1) = 0$,
A$_3$: $\xi_2(x) \in C^2[0,1]$, $\xi_2'''(x) \in L_2[0,1]$,
$\xi_2(0) = \xi_2'(1) = \xi_2''(0) = 0$,
A$_4$: $\xi_3(x) \in C^1[0,1]$, $\xi_3''(x) \in L_2[0,1]$,
$\xi_3(0) = \xi_3'(1) = 0$,
A$_5$: $h(\tau) \in C^4[0,T]$, $h(\tau) \neq 0$, $\forall \tau \in [0,T]$,
$h(0) = \xi_0(1)$, $h'(0) = \xi_1(1), h''(0) = \xi_2(1), h'''(0) = \xi_3(1)$,
A$_6$: $f(x,\tau) \in C(\overline{D}_T)$, $f_x, f_{xx}, f_{xxx} \in C[0,1]$, $\forall \tau \in [0,T]$,
$f(0,\tau) = f_x(1,\tau) = f_{xx}(0,\tau) = 0$,

be satisfied, $\beta, \gamma, \rho > 0$, and $\Delta_n = (\beta+\gamma\lambda_n)^2 - 4\rho\lambda_n > 0$. Then, the inverse problem (4)-(7) has a unique solution for small T.

Proof. Let $a(\tau) \in C[0,T]$ be an arbitrary function. Thus,we will use the Fourier (Eigenfunction expansion) method to construct the formal solution of the inverse problem (4)-(7). In keeping with this aim, let us consider

$$y(x,\tau)=\sum_{n=0}^{\infty} y_n(\tau)W_n(x), \tag{10}$$

is a formal solution of the inverse problem (4)-(7).

Since $y(x,\tau)$ is the formal solution of the inverse problem (4)-(7), we get the following Cauchy problems with respect to $y_n(\tau)$ from the equation (4) and initial conditions (5);

$$\begin{cases} y_n^{(4)}(\tau)+(\beta+\gamma\lambda_n)y_n''(\tau)+\rho\lambda_n y_n(\tau)=F_n(\tau;a,y), \\ y_n(0)=\xi_{0n},\ y_n'(0)=\xi_{1n},\ y_n''(0)=\xi_{2n}, y_n'''(0)=\xi_{3n},\ n=0,1,2,.... \end{cases} \tag{11}$$

Here

$$F_n(\tau;a,y)=a(\tau)y_n(\tau)+f_n(\tau),$$
$$y_n(\tau)=\sqrt{2}\int_0^1 y(x,\tau)\sin(\sqrt{\lambda_n}x)dx,$$
$$f_n(\tau)=\sqrt{2}\int_0^1 f(x,\tau)\sin(\sqrt{\lambda_n}x)dx,$$

and

$$\xi_{in}=\sqrt{2}\int_0^1 \xi_i(x)\sin(\sqrt{\lambda_n}x)dx,\ i=0,1,2,3,\ n=0,1,2,....$$

These Cauchy problems have the quartic characteristic polynomial

$$P(k)=k^4+(\beta+\gamma\lambda_n)k^2+\rho\lambda_n.$$

Since $\Delta_n=(\beta+\gamma\lambda_n)^2-4\rho\lambda_n>0$, solving (11) by using the roots of this characteristic polynomial that are investigated in previous section, we obtain

$$\begin{aligned} y_n(t)=&\frac{r_n^2\cos(p_n\tau)-p_n^2\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{0n}+\frac{r_n^3\sin(r_n\tau)-p_n^3\sin(p_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{1n} \\ &+\frac{\cos(p_n\tau)-\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{2n}+\frac{r_n\sin(r_n\tau)-p_n\sin(p_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{3n} \\ &+\int_0^\tau\left[\frac{p_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(r_n(\tau-\eta))-\frac{r_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(p_n(\tau-\eta))\right]F_n(\eta;a,y)d\eta. \end{aligned} \tag{12}$$

Substitute the expression (12) into (10) to determine $y(x,\tau)$. Then we get

$$y(x,\tau)=\sum_{n=0}^{\infty}\left[\frac{r_n^2\cos(p_n\tau)-p_n^2\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{0n}+\frac{r_n^3\sin(r_n\tau)-p_n^3\sin(p_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{1n}\right.$$

$$+\frac{\cos(p_n\tau)-\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{2n}+\frac{r_n\sin(r_n\tau)-p_n\sin(p_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{3n}$$
$$+\int_0^\tau\left[\frac{p_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(r_n(\tau-\eta))-\frac{r_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(p_n(\tau-\eta))\right]F_n(\eta;a,y)d\eta\Bigg]$$
$$\times W_n(x). \tag{13}$$

Let us derive the equation of $a(\tau)$. If we evaluate the equation (4) at $x=1$ and consider the additional condition (7), then we have:

$$a(\tau)=\frac{1}{h(\tau)}\left[h^{(4)}(\tau)+\beta h''(\tau)-f(1,\tau)+\sum_{n=0}^{\infty}(-1)^{n+1}\lambda_n\left(\gamma y_n''(\tau)+\rho y_n(\tau)\right)\right] \tag{14}$$

where $y_n(\tau)$ is defined in (12) and

$$y_n''(\tau)=\frac{p_n^2r_n^2\left(\cos(r_n\tau)-\cos(p_n\tau)\right)}{\sqrt{\Delta_n}}\xi_{0n}+\frac{r_n^5\sin(p_n\tau)-p_n^5\sin(r_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{1n}$$
$$+\frac{r_n^2\cos(r_n\tau)-p_n^2\cos(p_n\tau)}{\sqrt{\Delta_n}}\xi_{2n}+\frac{p_n^3\sin(p_n\tau)-r_n^3\sin(r_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{3n}$$
$$+\int_0^\tau\left[\frac{p_n^2r_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(p_n(\tau-\eta))-\frac{r_n^2p_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(r_n(\tau-\eta))\right]F_n(\eta;a,y)d\eta. \tag{15}$$

We convert the inverse problem (4)-(7) into the system of Volterra integral equations (13)-(14) with respect to $y(x,\tau)$ and $a(\tau)$ by considering

$$y_n(\tau)=\int_0^1 y(x,\tau)W_n(x)dx,\ n=0,1,2,...$$

is the solution of the system of differential equations (11). Analogously, we can prove that if $\{y(x,\tau),a(\tau)\}$ is a solution of the inverse problem (4)-(7), then $y_n(\tau),\ n=0,1,2,...$satisfy the system of differential equations (11). For proof of this assertion please see ([17]). From this assertion we can conclude that proving the uniqueness of the solution of the inverse problem (4)-(7), It is enough to prove the unique solvability of the system (13)-(14).

To prove the existence of a unique solution of the system (13) and (14) we need to rewrite this system into operator form and to show that this operator a contraction operator. Consider $\nu(x,\tau)=[y(x,\tau),a(\tau)]^T$ is a 2×1 inverse problem's solution vector function. Thus, we can rewrite the system of equations (13) and (14) into the operator equation form as

$$\nu=\underline{\mathbf{O}}(\nu) \tag{16}$$

where $\underline{\mathbf{O}}(\nu)\equiv[O_1,O_2]^T$ and ϕ_1 and ϕ_2 are

$$O_1(\nu)=\sum_{n=0}^{\infty}\Bigg[\frac{r_n^2\cos(p_n\tau)-p_n^2\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{0n}+\frac{r_n^3\sin(r_n\tau)-p_n^3\sin(p_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{1n}$$

$$+\frac{\cos(p_n\tau)-\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{2n}+\frac{r_n\sin(r_n\tau)-p_n\sin(p_n\tau)}{p_nr_n\sqrt{\Delta_n}}\xi_{3n}$$
$$\left.+\int_0^\tau\left[\frac{p_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(r_n(\tau-\eta))-\frac{r_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(p_n(\tau-\eta))\right]F_n(\eta;a,y)d\eta\right]$$
$$\times X_n(x),$$

and

$$O_2(\nu)=\frac{1}{h(\tau)}\left[h^{(4)}(\tau)+\beta h''(\tau)-f(1,\tau)+\sum_{n=0}^{\infty}(-1)^{n+1}\lambda_n\left(\gamma y_n''(\tau)+\rho y_n(\tau)\right)\right].$$

We can easily obtain following equalities

$$\xi_{0n}=\frac{1}{\lambda_n^{5/2}}\alpha_{0n},\ \xi_{1n}=\frac{1}{\lambda_n^{2}}\alpha_{1n},\ \xi_{2n}=\frac{1}{\lambda_n^{3/2}}\alpha_{2n},\ \xi_{3n}=\frac{1}{\lambda_n}\alpha_{3n},\ f_n(\tau)=\frac{1}{\lambda_n^{3/2}}\omega_n(\tau),$$

using integration by parts under consideration of the assumptions $(A_1)-(A_6)$, where

$$\begin{aligned}
\omega_n(\tau) &= -\sqrt{2}\int_0^1 f_{xxx}(x,\tau)\cos(\sqrt{\lambda_n}x)dx,\\
\alpha_{0n} &= \sqrt{2}\int_0^1 \xi_0^{(5)}(x)\cos(\sqrt{\lambda_n}x)dx,\\
\alpha_{1n} &= \sqrt{2}\int_0^1 \xi_1^{(4)}(x)\sin(\sqrt{\lambda_n}x)dx,\\
\alpha_{2n} &= -\sqrt{2}\int_0^1 \xi_2'''(x)\cos(\sqrt{\lambda_n}x)dx,
\end{aligned}$$

and

$$\alpha_{3n}=-\sqrt{2}\int_0^1 \xi_3''(x)\sin(\sqrt{\lambda_n}x)dx.$$

Since $\sqrt{2}\sin(\sqrt{\lambda_n}x)$ (or $\sqrt{2}\cos(\sqrt{\lambda_n}x)$) forms a biorthonormal system of functions on $[0,1]$, by using Bessel's inequality we get the estimates

$$\sum_{n=0}^{\infty}|\alpha_{0n}|^2\le\left\|\xi_0^{(5)}\right\|^2_{L_2[0,1]},\ \sum_{n=0}^{\infty}|\alpha_{1n}|^2\le\left\|\xi_1^{(4)}\right\|^2_{L_2[0,1]},$$

$$\sum_{n=0}^{\infty}|\alpha_{2n}|^2\le\left\|\xi_2'''\right\|^2_{L_2[0,1]},\ \sum_{n=0}^{\infty}|\alpha_{3n}|^2\le\left\|\xi_3''\right\|^2_{L_2[0,1]},$$

$$\sum_{n=0}^{\infty}|\omega_n(\tau)|^2\le\|f_{xxx}(\cdot,\tau)\|^2_{L_2[0,1]}. \tag{17}$$

Also we can easily obtain the following estimates of the coefficients which arise in the operator equations $O_1(\nu)$ and $O_2(\nu)$:

$$|\chi_1(\tau)| \le d_1,\ |\chi_2(\tau)| \le \frac{d_2}{\sqrt{\lambda_n}},\ |\chi_3(\tau)| \le \frac{d_3}{\lambda_n},\ |\chi_4(\tau)| \le \frac{d_4}{\lambda_n^{3/2}},\ |\chi_5(\tau)| \le \frac{d_5}{\lambda_n},$$

$$|\Gamma_1(\tau)| \le \lambda_n D_1,\ |\Gamma_2(\tau)| \le \sqrt{\lambda_n} D_2,\ |\Gamma_3(\tau)| \le D_3,\ |\Gamma_4(\tau)| \le \frac{D_4}{\sqrt{\lambda_n}},\ |\Gamma_5(\tau)| \le D_5, \tag{18}$$

where

$$\chi_1(\tau) = \frac{r_n^2\cos(p_n\tau) - p_n^2\cos(r_n\tau)}{\sqrt{\Delta_n}},\ \chi_2(\tau) = \frac{r_n^3\sin(r_n\tau) - p_n^3\sin(p_n\tau)}{p_n r_n\sqrt{\Delta_n}},$$
$$\chi_3(\tau) = \frac{\cos(p_n\tau) - \cos(r_n\tau)}{\sqrt{\Delta_n}},\ \chi_4(\tau) = \frac{r_n\sin(r_n\tau) - p_n\sin(p_n\tau)}{p_n r_n\sqrt{\Delta_n}},$$
$$\chi_5(t) = \frac{p_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(r_n(\tau-\eta)) - \frac{r_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(p_n(\tau-\eta)),$$

$\Gamma_i(\tau) = \chi_i''(\tau)$, $i = \overline{1,5}$, d_i and D_i, $i = \overline{1,5}$ are positive real constants. (These boundaries can be obtained by taking λ_n common multiplier)

Now we can show in two steps that $\underline{\mathbf{O}}$ is a contraction operator by considering the assumptions and estimates are given above.

I) First let us verify that $\underline{\mathbf{O}}$ is a continuous map which maps the space E_T onto itself continuously. That is to say, our aim is to show $O_1(\nu) \in B_T$ and $O_2(\nu) \in C[0,T]$ for arbitrary $\nu(x,\tau) = [y(x,\tau), a(\tau)]^T$ such that $y(x,\tau) \in B_T$, $a(\tau) \in C[0,T]$.

Let us start with $O_1(\nu) \in B_T$, i.e. we need to verify

$$J_T(O_1) = \left(\sum_{n=0}^{\infty}(\lambda_n^{5/2}\left\|O_{1,n}(\tau)\right\|_{C[0,T]})^2\right)^{1/2} < +\infty,$$

where

$$\begin{aligned} O_{1,n}(\tau) =& \frac{r_n^2\cos(p_n\tau) - p_n^2\cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{0n} + \frac{r_n^3\sin(r_n\tau) - p_n^3\sin(p_n\tau)}{p_n r_n\sqrt{\Delta_n}}\xi_{1n} \\ &+ \frac{\cos(p_n\tau) - \cos(r_n\tau)}{\sqrt{\Delta_n}}\xi_{2n} + \frac{r_n\sin(r_n\tau) - p_n\sin(p_n\tau)}{p_n r_n\sqrt{\Delta_n}}\xi_{3n} \\ &+ \int_0^\tau \left[\frac{p_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(r_n(\tau-\eta)) - \frac{r_n}{\sqrt{\Delta_n\rho\lambda_n}}\sin(p_n(\tau-\eta))\right] F_n(\eta; a, y)d\eta. \end{aligned}$$

After some manipulations under the assumptions $(A_1)-(A_6)$, using the estimates (18) we obtain

$$
\begin{aligned}
(J_T(O_1))^2 =& \sum_{n=0}^{\infty} (\lambda_n^{5/2} \left\| O_{1,n}(\tau) \right\|_{C[0,T]})^2 \\
\leq& 6d_1^2 \sum_{n=0}^{\infty} |\alpha_{0n}|^2 + 6d_2^2 \sum_{n=0}^{\infty} |\alpha_{1n}|^2 + 6d_3^2 \sum_{n=0}^{\infty} |\alpha_{2n}|^2 + 6d_4^2 \sum_{n=0}^{\infty} |\alpha_{3n}|^2 \\
&+ 6d_5^2 T^2 \sum_{n=0}^{\infty} \left(\max_{0 \leq \tau \leq T} |\omega_n(\tau)| \right)^2 \\
&+ 6d_5^2 T^2 \left(\max_{0 \leq \tau \leq T} |a(\tau)| \right)^2 \sum_{n=0}^{\infty} \left(\lambda_n^{5/2} \left\| y_n(\tau) \right\|_{C[0,T]} \right)^2 .
\end{aligned}
$$

Since $y(x,\tau)$, $a(\tau)$ belong to the spaces B_T, and $C[0,T]$, respectively, the series at the right hand side of $(J_T(\phi_1))^2$ are convergent from the Bessel's inequality (considering the estimates (17)). $J_T(O_1)$ is convergent (i.e. $J_T(O_1) < +\infty$) because $(J_T(O_1))^2$ is bounded above. Thus we can conclude that $O_1(\nu)$ belongs to the space B_T.

Now let us prove that $O_2(\nu) \in C[0,T]$. By using the equation of $a(\tau)$ (14), we can write

$$
\begin{aligned}
|O_2(\nu)| \leq& \frac{1}{\min\limits_{0 \leq \tau \leq T} |h(\tau)|} \left[\left| h^{(4)}(\tau) \right| + \beta \left| h''(\tau) \right| + |f(1,\tau)| \right. \\
&\left. + \sum_{n=0}^{\infty} \lambda_n \left(\gamma \left| y_n''(\tau) \right| + \rho \left| y_n(\tau) \right| \right) \right].
\end{aligned}
$$

Taking into account the estimates (17) and (18) and using the Cauchy-Schwartz inequality, from the inequality for $|\phi_2(\nu)|$ we get

$$
\begin{aligned}
\max_{0 \leq \tau \leq T} |O_2(\nu)| \leq& \frac{1}{\min\limits_{0 \leq \tau \leq T} |h(\tau)|} \left[\left| h^{(4)}(\tau) \right| + \beta \left| h''(\tau) \right| + |f(1,\tau)| \right. \\
&+ m_1 \sum_{n=0}^{\infty} |\alpha_{0n}|^2 + m_2 \sum_{n=0}^{\infty} |\alpha_{1n}|^2 + m_3 \sum_{n=0}^{\infty} |\alpha_{2n}|^2 + m_4 \sum_{n=0}^{\infty} |\alpha_{3n}|^2 \\
&+ m_5 T \left(\max_{0 \leq \tau \leq T} |a(\tau)| \right)^2 \sum_{n=0}^{\infty} \left(\lambda_n^{5/2} \left\| y_n(\tau) \right\|_{C[0,T]} \right)^2 \\
&\left. + m_6 T \sum_{n=0}^{\infty} \left(\max_{0 \leq \tau \leq T} |\omega_n(\tau)| \right) \right],
\end{aligned} \tag{19}
$$

where $m_i = \gamma D_i \left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n}\right)^{1/2} + \rho d_i \left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n^3}\right)^{1/2}, \ i=\overline{1,5}$ and $m_6 = \gamma D_5 \left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n^3}\right)^{1/2} + \rho d_5 \left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n^5}\right)^{1/2}$. Considering the estimates (17) and $\sum_{n=0}^{\infty} \frac{1}{\lambda_n}$, $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^3}$ and $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^5}$ are convergent, the majorizing series (19) are convergent. According to Weierstrass M-test $O_2(\nu)$ is absolutely continuous. Thus, $O_2(\nu)$ belongs to the space $C[0,T]$.

Thereby, we have shown that $\underline{\mathbf{O}}$ is a continuous and onto map on E_T.

II) Since $\underline{\mathbf{O}}$ in a continuous map onto E_T, let us prove that the operator $\underline{\mathbf{O}}$ is contraction mapping operator. Assume that let ν_1 and ν_2 be any two elements of E_T such that $\nu_i = \left[y^{(i)}(x,\tau), a^{(i)}(\tau)\right]^T$, $i=1,2$. From the definition of the space E_T, we have $\|\underline{\mathbf{O}}(\nu_1) - \underline{\mathbf{O}}(\nu_2)\|_{E_T} = \|O_1(\nu_1) - O_1(\nu_2)\|_{B_T} + \|O_2(\nu_1) - O_2(\nu_2)\|_{C[0,T]}$. For the convenience of this norm, let us consider the following differences

$$O_1(\nu_1) - O_1(\nu_2) = \sum_{n=0}^{\infty} \left[\int_0^{\tau} \left(\frac{p_n}{\sqrt{\Delta_n \rho \lambda_n}} \sin(r_n(\tau-\eta)) - \frac{r_n}{\sqrt{\Delta_n \rho \lambda_n}} \sin(p_n(\tau-\eta))\right)\right.$$
$$\left. \times \left(F_n(\eta; a^1, y^1) - F_n(\eta; a^2, y^2)\right) d\eta\right] W_n(x),$$

$$O_2(\nu_1) - O_2(\nu_2) = \frac{1}{h(\tau)} \left[\sum_{n=0}^{\infty} \lambda_n \left\{\gamma \int_0^{\tau} \left[\frac{p_n^2 r_n}{\sqrt{\Delta_n \rho \lambda_n}} \sin(p_n(\tau-\eta)) - \frac{r_n^2 p_n}{\sqrt{\Delta_n \rho \lambda_n}} \sin(r_n(\tau-\eta))\right]\right.\right.$$
$$\times \left(F_n(\eta; a^1, y^1) - F_n(\eta; a^2, y^2)\right) d\eta$$
$$+\rho \int_0^{\tau} \left[\frac{p_n}{\sqrt{\Delta_n \rho \lambda_n}} \sin(r_n(\tau-\eta)) - \frac{r_n}{\sqrt{\Delta_n \rho \lambda_n}} \sin(p_n(\tau-\eta))\right]$$
$$\left.\left. \times \left(F_n(\eta; a^1, y^1) - F_n(\eta; a^2, y^2)\right) d\eta\right\}\right].$$

After some manipulations in last equations under the assumptions (A$_1$)-(A$_6$) and using the estimates (17)-(18) , we obtain

$$\|O_1(\nu_1) - O_1(\nu_2)\|_{B_T} \le T \left[C_1 \left\|y^{(1)} - y^{(2)}\right\|_{B_T} + C_2 \left\|a^{(1)} - a^{(2)}\right\|_{C[0,T]}\right],$$
$$\|O_2(\nu_1) - O_2(\nu_2)\|_{C[0,T]} \le \frac{T}{\min\limits_{0\le\tau\le T} |h(\tau)|} \left[C_3 \left\|y^{(1)} - y^{(2)}\right\|_{B_T} + C_4 \left\|a^{(1)} - a^{(2)}\right\|_{C[0,T]}\right],$$

where C_k, $k=\overline{1,4}$ are the constants depend on the norms $\left\|a^{(1)}\right\|_{C[0,T]}$, $\left\|y^{(2)}\right\|_{B_T}$, m_5, and m_6 . From the last inequalities it follows that

$$\|\underline{\mathbf{O}}(\nu_1) - \underline{\mathbf{O}}(\nu_2)\|_{E_T} \le A(T) C(a^{(1)}, y^{(2)}, m_5, m_6) \|\nu_1 - \nu_2\|_{E_T}$$

where $A(T) = T\left(1 + \frac{1}{\min\limits_{0\le\tau\le T}|h(\tau)|}\right)$ and $C(a^{(1)}, y^{(2)}, m_5, m_6) = \max\{C_1, C_2, C_3, C_4\}$ is the constant depends on the norms $\left\|a^{(1)}\right\|_{C[0,T]}$, $\left\|y^{(2)}\right\|_{B_T}$, m_5, and m_6.

Since $h(\tau) \in C^4[0,T]$, $h(\tau) \neq 0$, $\forall \tau \in [0,T]$, $a^{(1)}(\tau) \in C[0,T]$, $y^{(2)}(x,\tau) \in B_T$ and m_5, m_6 are finite constants, $\frac{1}{\min\limits_{0\le\tau\le T}|h(\tau)|}$ and $C(a^{(1)}, y^{(2)}, m_5, m_6)$ are bounded above. Thus $A(T)C(a^{(1)}, y^{(2)}, m_5, m_6)$ tends to zero as $T \to 0$. In other words, for sufficiently small T we have $0 < A(T)C(a^{(1)}, y^{(2)}, m_5, m_6) < 1$. This means that the operator $\underline{\mathbf{O}}$ is a contraction mapping operator.

From the first and second steps, the operator $\underline{\mathbf{O}}$ is contraction mapping operator that is a continuous and onto map on E_T. Then according to Banach fixed point theorem the solution of the operator equation (16) exists and it is unique. □

4. Conclusion

The paper studies the inverse initial-boundary value problem of determining the time dependent lowest term together with the displacement function in a fourth order in time PDE from an additional observation. The unique solvability of the solution of the inverse problem on a sufficiently small time interval has been proved by using of the contraction principle. The proposed work is novel and has never been solved theoretically nor numerically before. Our results shed light on the methodology for the existence and uniqueness of the inverse problem for the fourth order in time PDEs in two dimensions.

Declaration of Competing Interests This work does not have any conflicts of interest.

References

[1] Arancibia, R., Lecaros, R., Mercado, A., Zamorano, S., An inverse problem for Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Journal of Inverse and Ill-posed Problems*, 30(5) 82022, 659-675. https://doi.org/10.1515/jiip-2020-0090

[2] Chlouverakis, K. E., Sprott, J. C., Chaotic hyperjerk systems, *Chaos, Solitons & Fractals*, 28(3) (2006),739-746. https://doi.org/10.1016/j.chaos.2005.08.019

[3] Choucha, A., Boulaaras, S., Ouchenane, D., Abdalla, M., Mekawy, I., Benbella, A., Existence and uniqueness for Moore-Gibson-Thompson equation with, source terms, viscoelastic memory and integral condition, *AIMS Mathematics*, 6(7) (2021), 7585-7624.

[4] Conti, M., Pata, V., Quintanilla, R., Thermoelasticity of Moore-Gibson-Thompson type with history dependence in the temperature, *Asymptotic Analysis,* 120(1-2) (2020), 1-21. https://doi.org/10.3233/ASY-191576

[5] Dell'Oro, F., Pata, V., On a fourth-order equation of Moore-Gibson-Thompson type, *Milan J. Math.*, 85 (2017), 215-234.

[6] Dell'Oro, F., Pata, V., On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity, *Applied Mathematics & Optimization*, 7683 (2017), 641-655. https://doi.org/10.1007/s00245-016-9365-1

[7] Dell' Oro, F., Lasiecka, I., Pata, V., The Moore-Gibson-Thompson equation with memory in the critical case, *J Differ Equ.*, 261 (2016), 4188-4222. https://doi.org/10.1016/j.jde.2016.06.025

[8] Figliolini, G., Lanni, C., Jerk and jounce relevance for the kinematic performance of long-dwell mechanisms, *Mechanisms and Machine Science*, 73 (2019), 219-228. https://doi.org/10.1007/978-3-030-20131-9_22

[9] Huntul, M. J., Tekin, I., On an inverse problem for a nonlinear third order in time partial differential equation, *Results in Applied Mathematics*, 15 (2022), 100314. https://doi.org/10.1016/j.rinam.2022.100314

[10] Kaltenbacher, B., Lasiecka, I., Pospieszalska, M. K., Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Mathematical Models and Methods in Applied Sciences*, 22(11) (2012), 1250035. https://doi.org/10.1142/S0218202512500352

[11] Kaltenbacher, B., Nikolić, V., The Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time, *Mathematical Models and Methods in Applied Sciences*, 29(13) (2019), 2523-2556. https://doi.org/10.1142/S0218202519500532

[12] Kaltenbacher, B., Mathematics of nonlinear acoustics, *Evolution Equations & Control Theory,* 4(4) (2015), 447-491.

[13] Lasiecka, I., Wang, X., Moore-Gibson-Thompson equation with memory, part I: exponential decay of energy, *Zeitschrift für angewandte Mathematik und Physik,* 67(2) (2016), 1-23. https://doi.org/10.1007/s00033-015-0597-8

[14] Lasiecka, I., Wang, X., Moore-Gibson-Thompson equation with memory, part II: general decay of energy, *Journal of Differential Equations*, 259(12) (2015), 7610-7635. https://doi.org/10.1016/j.jde.2015.08.052

[15] Liu, W., Chen, Z., Tu, Z., New general decay result for a fourth-order Moore-Gibson-Thompson equation with memory, *Electronic Research Archive*, 28(1) (2020), 433. https://doi.org/10.3934/era.2020025

[16] Lizama, C., Murillo-Arcila, M., well-posedness for a fourth-order equation of Moore-Gibson-Thompson type, *Electronic Journal of Qualitative Theory of Differential Equations*, 81 (2021), 1-18. https://doi.org/10.14232/ejqtde.2021.1.81

[17] Mehraliyev, Y. T., On solvability of an inverse boundary value problem for a second order elliptic equation, *Bulletin of Tver State University, Series: Applied mathematics*, 23 (2011), 25-38.(in Russian)

[18] Mesloub, A., Zara, A., Mesloub, F., Cherif, B. B., Abdalla, M., The Galerkin method for fourth-Order equation of the Moore-Gibson-Thompson type with integral condition, *Advances in Mathematical Physics*, (2021), 2021. https://doi.org/10.1155/2021/5532691

[19] Murillo-Arcila, M., Well-posedness for the fourth-order Moore-Gibson-Thompson equation in the class of Banach-space-valued Hölder-continuous functions, *Mathematical Methods in the Applied Sciences*, (2022), 1-10. doi:10.1002/mma.8618

[20] Pellicer Sabadí, M., Said-Houari,B., Well posedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Applied Mathematics and Optimization*, 80(2) 82019), 447-478. https://doi.org/10.1007/s00245-017-9471-8

[21] Tekin, I., Inverse problem for a nonlinear third order in time partial differential equation, *Mathematical Methods in the Applied Sciences*, 44(11) (2021), 9571-9581. https://doi.org/10.1002/mma.7380

[22] Visser, M., Jerk, snap and the cosmological equation of state, *Classical and Quantum Gravity*, 21(11) (2004), 2603. https://doi.org/10.1088/0264-9381/21/11/006