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On Dynamics and Solutions Expressions of Higher-Order Rational Difference Equations

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Abstract — The principle goal of this paper is to look at some of the qualitative behavior of the critical point of the rational difference equation

$$\Psi_{n+1} = \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3} + \delta\Psi_{n-6}}, \quad n = 0, 1, 2, ...,$$

where α, β, γ and δ are arbitrary positive real numbers. We also used the proposed equation to get the general solution for particular cases and provided numerical examples to demonstrate our results.

Subject Classification (2020): 39A10.

1. Introduction

One of the most important scientific topics is difference equations, often known as discrete dynamical systems. The study of the qualitative properties of rational difference equations has sparked a lot of attention recently.

Many researchers have opted to utilize difference equations in mathematical models to explain the problems in various sciences, including allowing scientists to introduce their study's predictions and producing more precise results.

It is particularly fascinating to look into the behavior of the solutions to a system of nonlinear differential equations and examine the local asymptotic stability of their equilibrium points. Numerous studies have been conducted on the technique of identifying the general form of the solution for some special cases of the problem. The systems and behavior of rational difference equations have been the subject of numerous works (can be obtained in the references).

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Alayachi et al. [3] studied the qualitative properties of:

$$y_{n+1} = Ay_{n-1} + \frac{By_{n-1}y_{n-3}}{Cy_{n-3} + Dy_{n-5}}.$$

Almatrafi et al. [6] studied the global behavior of:

$$\chi_{n+1} = \alpha\chi_n + \frac{\beta\chi_n^2 + \gamma\chi_n\chi_{n-1} + \delta\chi_{n-1}^2}{\lambda\chi_n^2 + \mu\chi_n\chi_{n-1} + \sigma\chi_{n-1}^2}.$$

Alzubaidi and Elsayed [8] examined the dynamics behavior and gave the general form of:

$$\varphi_{n+1} = \alpha\varphi_{n-2} \pm \frac{\beta\varphi_{n-1}\varphi_{n-2}}{\gamma\varphi_{n-2} \pm \delta\varphi_{n-4}}.$$

Ibrahim et al. [26] investigated the global stability and boundedness of solutions for:

$$\Upsilon_{n+1} = \alpha + \sum_{i=0}^{k} a_i\Upsilon_{n-i} + \frac{\Upsilon_n\Upsilon_{n-k}}{\beta + \sum_{j=0}^{k} b_j\Upsilon_{n-j}}.$$

Kara and Yazlik [27] found the exact formulas for the solutions of the system:

$$x_n = \frac{x_{n-2}z_{n-3}}{z_{n-1}(a_n + b_n x_{n-2}z_{n-3})},$$
$$y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}(\alpha_n + \beta_n y_{n-2}x_{n-3})},$$
$$z_n = \frac{z_{n-2}y_{n-3}}{y_{n-1}(A_n + B_n z_{n-2}y_{n-3})}.$$

Karatas et al. [28] investigated the solutions of:

$$U_{n+1} = \frac{U_{n-5}}{1 + bU_{n-2}U_{n-5}}.$$

Abdul Khaliq et al. [30] investigated the asymptotic behavior of the solutions of:

$$\omega_{n+1} = \omega_{n-p}\left(\alpha + \frac{\beta\omega_n}{\gamma\omega_n + \delta\omega_{n-r}}\right).$$

In [35] Muna and Mohammad deal with:

$$V_{n+1} = \frac{(\alpha + \beta V_n)}{(A + BV_n + CV_{n-k})}.$$

The goal of this paper is to find a general solution to some special cases of the fractional recursive equation

$$\Psi_{n+1} = \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3} + \delta\Psi_{n-6}}, \quad n = 0, 1, 2, ..., \tag{1}$$

where α, β, γ and δ are arbitrary positive real numbers.

2. Local Stability of the Critical Point

The critical point of Eq.(1), is given by

$$\bar{\Psi} = \alpha\bar{\Psi} + \frac{\beta\bar{\Psi}^2}{\gamma\bar{\Psi} + \delta\bar{\Psi}},$$

$$(1-\alpha)\bar{\Psi} = \frac{\beta\bar{\Psi}^2}{(\gamma+\delta)\bar{\Psi}} \Rightarrow (1-\alpha)(\gamma+\delta)\bar{\Psi}^2 = \beta\bar{\Psi}^2.$$

Thus,

$$\left[(1-\alpha)(\gamma+\delta) - \beta\right]\bar{\Psi}^2 = 0.$$

If $(1-\alpha)(\gamma+\delta) \neq \beta$ then the unique critical point is $\bar{\Psi} = 0$.

Assume $\Phi : (0,\infty)^3 \to (0,\infty)$ be a C^1 function defined by

$$\Phi(w_1, w_2, w_3) = \alpha w_1 + \frac{\beta w_1 w_2}{\gamma w_2 + \delta w_3}. \tag{2}$$

In consequence,

$$\frac{\partial\Phi}{\partial w_1} = \alpha + \frac{\beta w_2}{\gamma w_2 + \delta w_3}, \quad \frac{\partial\Phi}{\partial w_2} = \frac{\beta\delta w_1 w_3}{(\gamma w_2 + \delta w_3)^2}, \quad \frac{\partial\Phi}{\partial w_3} = \frac{-\beta\delta w_1 w_2}{(\gamma w_2 + \delta w_3)^2}. \tag{3}$$

At $\bar{\Psi} = 0$, we see that

$$\begin{aligned} \frac{\partial\Phi}{\partial w_1}(\bar{\Psi},\bar{\Psi},\bar{\Psi}) &= \alpha + \frac{\beta}{\gamma+\delta} = \gamma_1, \\ \frac{\partial\Phi}{\partial w_2}(\bar{\Psi},\bar{\Psi},\bar{\Psi}) &= \frac{\beta\delta}{(\gamma+\delta)^2} = \gamma_2, \\ \frac{\partial\Phi}{\partial w_3}(\bar{\Psi},\bar{\Psi},\bar{\Psi}) &= \frac{-\beta\delta}{(\gamma+\delta)^2} = \gamma_3. \end{aligned} \tag{4}$$

Hence,

$$Z_{n+1} - \left(\alpha + \frac{\beta}{\gamma+\delta}\right) Z_{n-2} - \left(\frac{\beta\delta}{(\gamma+\delta)^2}\right) Z_{n-3} + \left(\frac{\beta\delta}{(\gamma+\delta)^2}\right) Z_{n-6} = 0.$$

Theorem 2.1. The critical point $\bar{\Psi} = 0$ is locally asymptotically stable if

$$\beta(\gamma + 3\delta) < (1-\alpha)(\gamma+\delta)^2.$$

Proof.

By using the values in the Eq.(4) and by Lemma 1 in [30], ensures that Eq.(1) is asymptotically stable if

$$|\gamma_1| + |\gamma_2| + |\gamma_3| < 1,$$

$$\left|\alpha + \frac{\beta}{\gamma+\delta}\right| + \left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| + \left|\frac{-\beta\delta}{(\gamma+\delta)^2}\right| < 1,$$

or

$$\alpha + \frac{\beta(\gamma+\delta)}{(\gamma+\delta)^2} + \frac{\beta\delta}{(\gamma+\delta)^2} + \frac{\beta\delta}{(\gamma+\delta)^2} < 1,$$

$$\frac{\beta\gamma+3\beta\delta}{(\gamma+\delta)^2}<(1-\alpha),$$

therefore,

$$\beta(\gamma+3\delta)<(1-\alpha)(\gamma+\delta)^2.$$

3. Global Attractive of the Critical Point

In this section, we aim to investigate the global asymptotic stability of the positive solutions of Eq.(1).

Theorem 3.1. The critical point $\bar{\Psi}=0$ of Eq.(1) is a global attracting if

$$\gamma(1-\alpha)\neq\beta.$$

Proof.
From Eq.(3), we note that, the function $\Phi(w_1,w_2,w_3)$ is increasing in w_1 and w_2 and is decreasing in w_3. Assume that whenever (H,h) is a solution of the system

$$H=\Phi(H,H,h),$$
$$h=\Phi(h,h,H),$$

then, we have

$$H=\alpha H+\frac{\beta H^2}{\gamma H+\delta h}, \ \Rightarrow \ (1-\alpha)H=\frac{\beta H^2}{\gamma H+\delta h},$$

$$\gamma(1-\alpha)H^2+\delta(1-\alpha)hH=\beta H^2. \tag{5}$$

$$h=\alpha h+\frac{\beta h^2}{\gamma h+\delta H}, \ \Longrightarrow \ (1-\alpha)h=\frac{\beta h^2}{\gamma h+\delta H}.$$

$$\gamma(1-\alpha)h^2+\delta(1-\alpha)hH=\beta h^2. \tag{6}$$

By substrate Eq.(5) from Eq.(6) we obtain

$$\left[\gamma(1-\alpha)-\beta\right]\left(H^2-h^2\right)=0.$$

In consequence, $H=h$ if $\gamma(1-\alpha)\neq\beta$. It follows by Theorem 1 in [30] the equilibrium point $\bar{\Psi}=0$ of Eq.(1) is a global attractor.

4. Boundedness of solutions

Here, we demonstrate how the positive solutions to Eq.(1) have boundedness.

Theorem 4.1. Every solution of Eq.(1) is bounded if

$$\left(\alpha+\frac{\beta}{\gamma}\right)<1.$$

Proof.
Assume that $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(1), then

$$\begin{aligned}\Psi_{n+1} &= \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3}+\delta\Psi_{n-6}}\\ &\leq \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3}}\\ &= \left(\alpha+\frac{\beta}{\gamma}\right)\Psi_{n-2}.\end{aligned}$$

Hence,

$$\Psi_{n+1} \leq \Psi_{n-2}, \quad for\ all\ n \geq 0.$$

This implies that the subsequences are bounded from above by

$$\Psi_{\max} = \max\{\Psi_{-6},\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1},\Psi_0\}.$$

5. General Solution for Special Cases

In this section, we will find expressions of solution for some special cases of Eq.(1)

5.1. First Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha=\beta=\delta=\gamma=1,$ so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}, \quad n=0,1,2,..., \tag{7}$$

where the initial conditions $\Psi_{-6},\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.1. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(7). Thus for n=0,1,2,...,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+7}\sigma+\mathscr{F}_{6i+6}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i+6}\sigma+\mathscr{F}_{6i+5}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-1} = \lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+7}\lambda+\mathscr{F}_{6i+6}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i+6}\lambda+\mathscr{F}_{6i+5}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n} = \eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+7}\eta+\mathscr{F}_{6i+6}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i+6}\eta+\mathscr{F}_{6i+5}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)},$$

$$\Psi_{12n+1} = \sigma\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i-3}\eta+\mathscr{F}_{6i-4}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i-4}\eta+\mathscr{F}_{6i-5}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n+2} = \lambda\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i-1}\zeta+\mathscr{F}_{6i-2}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i-2}\zeta+\mathscr{F}_{6i-3}\kappa)},$$

$$\Psi_{12n+3} = \eta\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n+4}=\sigma\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n+5}=\lambda\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n+6}=\eta\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n+7}=\sigma\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n+8}=\lambda\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n+9}=\eta\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)},$$

where $\Psi_{-6}=\kappa,\Psi_{-5}=\tau,\Psi_{-4}=\mu,\Psi_{-3}=\zeta,\Psi_{-2}=\sigma,\Psi_{-1}=\lambda,\Psi_{0}=\eta$ and $\{\mathscr{F}_i\}_{i=-5}^{\infty}=\{1,1,1,1,1,1,1,1,2,3,5,8,13,21,...,\}$.

Proof.

For $n=0$ the result holds. Now suppose that $n>0$ and our assumption holds for $n-1$, that is

$$\Psi_{12n-14}=\sigma\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+7}\sigma+\mathscr{F}_{6i+6}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i+6}\sigma+\mathscr{F}_{6i+5}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-13}=\lambda\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+7}\lambda+\mathscr{F}_{6i+6}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i+6}\lambda+\mathscr{F}_{6i+5}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n-12}=\eta\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{6i+7}\eta+\mathscr{F}_{6i+6}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i+6}\eta+\mathscr{F}_{6i+5}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)},$$

$$\Psi_{12n-11}=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-3}\eta+\mathscr{F}_{6i-4}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i-4}\eta+\mathscr{F}_{6i-5}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-10}=\lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i-1}\zeta+\mathscr{F}_{6i-2}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i-2}\zeta+\mathscr{F}_{6i-3}\kappa)},$$

$$\Psi_{12n-9}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n-8}=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-7}=\lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n-6}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n-5}=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-4}=\lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n-3}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)}.$$

Now, we prove that the results are holds for n. From Eq.(7), it follows that

$$\begin{aligned}\Psi_{12n-2}&=\Psi_{12n-5}+\frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6}+\Psi_{12n-9}}\\&=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)}\end{aligned}$$

$$\left[1+\frac{\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)}}{\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)}+\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)}}\right]$$

$$=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)}$$

$$\left[1+\frac{\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)}{(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)}}{\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)}{(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)}+\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)}{(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)}}\right]$$

$$=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)}$$

$$\left[1+\frac{\frac{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n-2}\sigma+\mathscr{F}_{6n-3}\tau)}}{\frac{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n-2}\sigma+\mathscr{F}_{6n-3}\tau)}+1}\right]$$

$$=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)}$$

$$\left[1+\frac{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)+(\mathscr{F}_{6n-2}\sigma+\mathscr{F}_{6n-3}\tau)}\right]$$

$$=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)}$$

$$\left[1+\frac{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n}\sigma+\mathscr{F}_{6n-1}\tau)}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}$$
$$\left[\frac{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau) + (\mathscr{F}_{6n-1}\sigma + \mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau)}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(F_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)} \left[\frac{(\mathscr{F}_{6n+1}\sigma + \mathscr{F}_{6n}\tau))}{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau)}\right].$$

Hence, we get

$$\Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+7}\sigma + \mathscr{F}_{6i+6}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i+6}\sigma + \mathscr{F}_{6i+5}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}.$$

Other expressions can be investigated in the same way. The proof has been completed.

5.2. Second Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha = \gamma = \beta = 1$ and $\delta = -1$, so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3} - \Psi_{n-6}}, \quad n = 0, 1, 2, ..., \tag{8}$$

where the initial conditions $\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.2. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(8). Thus for n=0,1,2,...,

$$\Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta - \mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta - \mathscr{F}_{3i-1}\kappa)},$$

$$\Psi_{12n-1} = \lambda \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta - \mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma - \mathscr{F}_{3i-1}\tau)(\mathscr{F}_{3i+2}\zeta - \mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n} = \eta \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda - \mathscr{F}_{3-1}\mu)(\mathscr{F}_{3i+2}\sigma - \mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n+1} = \frac{\sigma(2\zeta - \kappa)}{(\zeta - \kappa)} \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta - \mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n+2} = \frac{\lambda(2\sigma - \tau)}{(\sigma - \tau)} \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+6}\sigma - \mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta - \mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta - \mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n+3} = \frac{\eta(2\lambda - \mu)}{(\lambda - \mu)} \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda - \mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma - \mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n+4} = \frac{\sigma(2\eta - \zeta)(2\zeta - \kappa)}{(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)\\ (\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)\\ (\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)\end{array}},$$

$$\Psi_{12n+5} = \frac{\lambda(2\sigma-\tau)(3\zeta-\kappa)}{\zeta(\sigma-\tau)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta-\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)\\(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}},$$

$$\Psi_{12n+6} = \frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)\end{array}},$$

$$\Psi_{12n+7} = \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}},$$

$$\Psi_{12n+8} = \frac{\lambda(3\eta-\zeta)(2\sigma-\tau)(3\zeta-\kappa)}{\eta\zeta(\sigma-\tau)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+7}\eta-\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta-\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)\\(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}},$$

$$\Psi_{12n+9} = \frac{\eta(2\lambda-\mu)(3\sigma-\tau)(5\zeta-2\kappa)}{\sigma(\lambda-\mu)(2\zeta-\kappa)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+8}\zeta-\mathscr{F}_{3i+6}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}},$$

where $\Psi_{-6}=\kappa, \Psi_{-5}=\tau, \Psi_{-4}=\mu, \Psi_{-3}=\zeta, \Psi_{-2}=\sigma, \Psi_{-1}=\lambda, \Psi_0=\eta$ and $\{\mathscr{F}_i\}_{i=-1}^{\infty}=\{1,0,1,1,2,3,5,8,13, 21,...\}$.

Proof.

For $n=0$ the result holds. Now suppose that $n>0$ and our assumption holds for $n-1$, that is

$$\Psi_{12n-14} = \sigma \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta-\mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta-\mathscr{F}_{3i-1}\kappa)},$$

$$\Psi_{12n-13} = \lambda \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma-\mathscr{F}_{3i-1}\tau)(\mathscr{F}_{3i+2}\zeta-\mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n-12} = \eta \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda-\mathscr{F}_{3-1}\mu)(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n-11} = \frac{\sigma(2\zeta-\kappa)}{(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta-\mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-10} = \frac{\lambda(2\sigma-\tau)}{(\sigma-\tau)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta-\mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n-9} = \frac{\eta(2\lambda-\mu)}{(\lambda-\mu)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda - \mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma - \mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n-8} = \frac{\sigma(2\eta-\zeta)(2\zeta-\kappa)}{(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-7} = \frac{\lambda(2\sigma-\tau)(3\zeta-\kappa)}{\zeta(\sigma-\tau)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+6}\sigma - \mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta - \mathscr{F}_{3i+5}\kappa)}{(\mathscr{F}_{3i+2}\eta - \mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n-6} = \frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda - \mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+7}\sigma - \mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n-5} = \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda - \mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-4} = \frac{\lambda(3\eta-\zeta)(2\sigma-\tau)(3\zeta-\kappa)}{\eta\zeta(\sigma-\tau)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+7}\eta - \mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+6}\sigma - \mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta - \mathscr{F}_{3i+5}\kappa)}{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n-3} = \frac{\eta(2\lambda-\mu)(3\sigma-\tau)(5\zeta-2\kappa)}{\sigma(\lambda-\mu)(2\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda - \mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+7}\sigma - \mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+8}\zeta - \mathscr{F}_{3i+6}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)}.$$

Now, we prove that the results are holds for n. From Eq.(8), it follows that

$$
\begin{aligned}
\Psi_{12n-2} &= \Psi_{12n-5} + \frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6}-\Psi_{12n-9}} \\
&= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}
\end{aligned}
$$

$$
\left[1+\frac{\frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}}{\begin{array}{c}\frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}- \\ \frac{\eta(2\lambda-\mu)}{(\lambda-\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}\end{array}}\right]
$$

$$
= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}
$$

$$
\left[1+\frac{\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)}}{\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)}-\prod_{i=1}^{n-2}\frac{(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)}{(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)}}\right]
$$

$$
= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}
$$

$$
\left[1+\frac{\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma-\mathscr{F}_{3n-3}\tau)}}{\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma-\mathscr{F}_{3n-3}\tau)}-1}\right]
$$

$$
= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}
$$

$$
\left[1+\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)-(\mathscr{F}_{3n-1}\sigma-\mathscr{F}_{3n-3}\tau)}\right]
$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}} \left[1+\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}} \left[\frac{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)+(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}} \left[\frac{(\mathscr{F}_{3n+2}\sigma-\mathscr{F}_{3n}\tau)}{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)}\right].$$

Therefore,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta-\mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta-\mathscr{F}_{3i-1}\kappa)}.$$

The following cases can be proved using a similar technique.

5.3. Third Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha=\gamma=\delta=1$ and $\beta=-1$, so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}, \quad n=0,1,2,..., \tag{9}$$

where the initial conditions $\Psi_{-6},\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.3. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(9). Thus for n=0,1,2,...,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i}\zeta+\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-1} = \lambda\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i}\sigma+\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i}\lambda+\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n+1}=\frac{\sigma\kappa}{(\zeta+\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)},$$

$$\Psi_{12n+2}=\frac{\lambda\tau}{(\sigma+\tau)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n+3}=\frac{\eta\mu}{(\lambda+\mu)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n+4}=\frac{\sigma\zeta\kappa}{(\eta+\zeta)(\zeta+\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}},$$

$$\Psi_{12n+5}=\frac{\lambda\tau(\zeta+\kappa)}{(\sigma+\tau)(\zeta+2\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}},$$

$$\Psi_{12n+6}=\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}},$$

$$\Psi_{12n+7}=\frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}},$$

$$\Psi_{12n+8}=\frac{\lambda\tau(\eta+\zeta)(\zeta+\kappa)}{(\eta+2\zeta)(\sigma+\tau)(\zeta+2\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+5}\eta+\mathscr{F}_{3i+6}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}},$$

$$\Psi_{12n+9}=\frac{\eta\mu(\sigma+\tau)(\zeta+2\kappa)}{(\lambda+\mu)(\sigma+2\tau)(2\zeta+3\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+6}\zeta+\mathscr{F}_{3i+7}\kappa)\end{array}},$$

where $\Psi_{-6}=\kappa,\Psi_{-5}=\tau,\Psi_{-4}=\mu,\Psi_{-3}=\zeta,\Psi_{-2}=\sigma,\Psi_{-1}=\lambda,\Psi_0=\eta$ and $\{\mathscr{F}_i\}_{i=0}^{\infty}=\{0,1,1,2,3,5,8,13,21,...,\}$.

Proof.

For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is

$$\Psi_{12n-14} = \sigma\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i}\zeta+\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-13} = \lambda\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i}\sigma+\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n-12} = \eta\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i}\lambda+\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n-11} = \frac{\sigma\kappa}{(\zeta+\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)},$$

$$\Psi_{12n-10} = \frac{\lambda\tau}{(\sigma+\tau)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n-9} = \frac{\eta\mu}{(\lambda+\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n-8} = \frac{\sigma\zeta\kappa}{(\eta+\zeta)(\zeta+\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}},$$

$$\Psi_{12n-7} = \frac{\lambda\tau(\zeta+\kappa)}{(\sigma+\tau)(\zeta+2\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}},$$

$$\Psi_{12n-6} = \frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}},$$

$$\Psi_{12n-5} = \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}},$$

$$\Psi_{12n-4} = \frac{\lambda\tau(\eta+\zeta)(\zeta+\kappa)}{(\eta+2\zeta)(\sigma+\tau)(\zeta+2\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+5}\eta+\mathscr{F}_{3i+6}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}},$$

$$\Psi_{12n-3} = \frac{\eta\mu(\sigma+\tau)(\zeta+2\kappa)}{(\lambda+\mu)(\sigma+2\tau)(2\zeta+3\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+6}\zeta+\mathscr{F}_{3i+7}\kappa)\end{array}}.$$

Now, we prove that the results are holds for n. From Eq.(9), it follows that

$$\begin{aligned}
\Psi_{12n-2} &= \Psi_{12n-5} - \frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6}+\Psi_{12n-9}} \\
&= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}} \\
&\quad \left[1 - \frac{\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}}{\begin{array}{c}\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)} + \\ \frac{\eta\mu}{(\lambda+\mu)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}\end{array}}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}} \\
&\quad \left[1 - \frac{\prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)}}{\prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)} + \prod_{i=1}^{n-2} \frac{(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)}{(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)}}\right]
\end{aligned}$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$
$$\left[1-\frac{\frac{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}}{\frac{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}+1}\right]$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$
$$\left[1-\frac{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)+(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}\right]$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$
$$\left[\frac{(\mathscr{F}_{3n}\sigma+\mathscr{F}_{3n+1}\tau)-(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n}\sigma+\mathscr{F}_{3n+1}\tau)}\right]$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$
$$\left[\frac{(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}{(\mathscr{F}_{3n}\sigma+\mathscr{F}_{3n+1}\tau)}\right].$$

Thus,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i}\zeta+\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}.$$

Other relations can be proved in the same way.

5.4. Fourth Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha = \gamma = 1$, and $\beta = \delta = -1$, so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3} - \Psi_{n-6}}, \quad n = 0, 1, 2, ..., \tag{10}$$

where the initial conditions $\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.4. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(10). Thus for n=0,1,2,...,

$$\begin{aligned}
\Psi_{12n-6} &= \frac{(-1)^n \eta^n \mu^n (\sigma-\tau)^n}{\sigma^n \kappa^{n-1} (\lambda-\mu)^n}, \\
\Psi_{12n-5} &= \frac{\sigma^n \zeta^n \kappa^n (\lambda-\mu)^n}{\lambda^n \tau^{n-1} (\eta-\zeta)^n (\zeta-\kappa)^n}, \\
\Psi_{12n-4} &= \frac{(-1)^n \lambda^n \tau^n (\eta-\zeta)^n (\zeta-\kappa)^n}{\eta^n \zeta^n \mu^{n-1} (\sigma-\tau)^n}, \\
\Psi_{12n-3} &= \frac{(-1)^n \eta^n \zeta \mu^n (\sigma-\tau)^n}{\sigma^n \kappa^n (\lambda-\mu)^n}, \\
\Psi_{12n-2} &= \frac{\sigma^{n+1} \zeta^n \kappa^n (\lambda-\mu)^n}{\lambda^n \tau^n (\eta-\zeta)^n (\zeta-\kappa)^n}, \\
\Psi_{12n-1} &= \frac{(-1)^n \lambda^{n+1} \tau^n (\eta-\zeta)^n (\zeta-\kappa)^n}{\eta^n \zeta^n \mu^n (\sigma-\tau)^n}, \\
\Psi_{12n} &= \frac{(-1)^n \eta^{n+1} \mu^n (\sigma-\tau)^n}{\sigma^n \kappa^n (\lambda-\mu)^n}, \\
\Psi_{12n+1} &= -\frac{\sigma^{n+1} \zeta^n \kappa^{n+1} (\lambda-\mu)^n}{\lambda^n \tau^n (\eta-\zeta)^n (\zeta-\kappa)^{n+1}}, \\
\Psi_{12n+2} &= \frac{(-1)^{n+1} \lambda^{n+1} \tau^{n+1} (\eta-\zeta)^n (\zeta-\kappa)^n}{\eta^n \zeta^n \mu^n (\sigma-\tau)^{n+1}}, \\
\Psi_{12n+3} &= \frac{(-1)^{n+1} \eta^{n+1} \mu^{n+1} (\sigma-\tau)^n}{\sigma^n \kappa^n (\lambda-\mu)^{n+1}}, \\
\Psi_{12n+4} &= \frac{\sigma^{n+1} \zeta^{n+1} \kappa^{n+1} (\lambda-\mu)^n}{\lambda^n \tau^n (\eta-\zeta)^{n+1} (\zeta-\kappa)^{n+1}}, \\
\Psi_{12n+5} &= \frac{(-1)^{n+1} \lambda^{n+1} \tau^{n+1} (\eta-\zeta)^n (\zeta-\kappa)^{n+1}}{\eta^n \zeta^{n+1} \mu^n (\sigma-\tau)^{n+1}},
\end{aligned}$$

where $\Psi_{-6} = \kappa, \Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda, \Psi_0 = \eta$.

Proof.
For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is

$$
\begin{aligned}
\Psi_{12n-18} &= \frac{(-1)^{n-1}\eta^{n-1}\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-2}(\lambda-\mu)^{n-1}},\\
\Psi_{12n-17} &= \frac{\sigma^{n-1}\zeta^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-2}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}},\\
\Psi_{12n-16} &= \frac{(-1)^{n-1}\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-2}(\sigma-\tau)^{n-1}},\\
\Psi_{12n-15} &= \frac{(-1)^{n-1}\eta^{n-1}\zeta\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}},\\
\Psi_{12n-14} &= \frac{\sigma^{n}\zeta^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}},\\
\Psi_{12n-13} &= \frac{(-1)^{n-1}\lambda^{n}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n-1}},\\
\Psi_{12n-12} &= \frac{(-1)^{n-1}\eta^{n}\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}},\\
\Psi_{12n-11} &= -\frac{\sigma^{n}\zeta^{n-1}\kappa^{n}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n}},\\
\Psi_{12n-10} &= \frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}},\\
\Psi_{12n-9} &= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}},\\
\Psi_{12n-8} &= \frac{\sigma^{n}\zeta^{n}\kappa^{n}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n}(\zeta-\kappa)^{n}},\\
\Psi_{12n-7} &= \frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n}}{\eta^{n-1}\zeta^{n}\mu^{n-1}(\sigma-\tau)^{n}}.
\end{aligned}
$$

Now, we prove that the results are holds for n. From Eq.(10), it follows that

$$
\begin{aligned}
\Psi_{12n-6} &= \Psi_{12n-9} - \frac{\Psi_{12n-9}\Psi_{12n-10}}{\Psi_{12n-10}-\Psi_{12n-13}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}} - \frac{\frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}}\frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}}}{\frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}} - \frac{(-1)^{n-1}\lambda^{n}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n-1}}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}} - \frac{\frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}}\frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}}}{\frac{(-1)^{n-1}\lambda^{n}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}[-\tau-\sigma+\tau]}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}} - \frac{(-1)^{n}\eta^{n}\mu^{n}\tau(\sigma-\tau)^{n-1}}{\sigma^{n}\kappa^{n-1}(\lambda-\mu)^{n}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}[\sigma-\tau]}{\sigma^{n}\kappa^{n-1}(\lambda-\mu)^{n}}.
\end{aligned}
$$

So we have

$$
\Psi_{12n-6} = \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n}}{\sigma^{n}\kappa^{n-1}(\lambda-\mu)^{n}}.
$$

Similarly,

$$\begin{aligned}
\Psi_{12n-5} &= \Psi_{12n-8} - \frac{\Psi_{12n-8}\Psi_{12n-9}}{\Psi_{12n-9} - \Psi_{12n-12}} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} - \frac{\frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}\frac{(-1)^n\eta^n\mu^n(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n}}{\frac{(-1)^n\eta^n\mu^n(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n} - \frac{(-1)^{n-1}\eta^n\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}}} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} - \frac{\frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}\frac{(-1)^n\eta^n\mu^n(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n}}{\frac{(-1)^{n-1}\eta^n\mu^{n-1}(\sigma-\tau)^{n-1}[-\mu-\lambda+\mu]}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n}} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} - \frac{\sigma^n\zeta^n\mu\kappa^n(\lambda-\mu)^{n-1}}{\lambda^n\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}[\lambda-\mu]}{\lambda^n\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}.
\end{aligned}$$

Hence, we obtain

$$\Psi_{12n-5} = \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^n}{\lambda^n\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}.$$

Similarly, by using the same method, we can investigate other relations.

6. Numerical Examples

For our prior results, we present some numerical examples to explain the solution behavior of Eq.(1).

Example 1. In numerical simulation they assumed that for Eq.(7) the initial value are $\Psi_{-6} = 0.3, \Psi_{-5} = 0.6, \Psi_{-4} = 0.9, \Psi_{-3} = 1.2, \Psi_{-2} = 1.5, \Psi_{-1} = 1.8$ and $\Psi_0 = 2.1$. Then the solution appear in Figure 1.

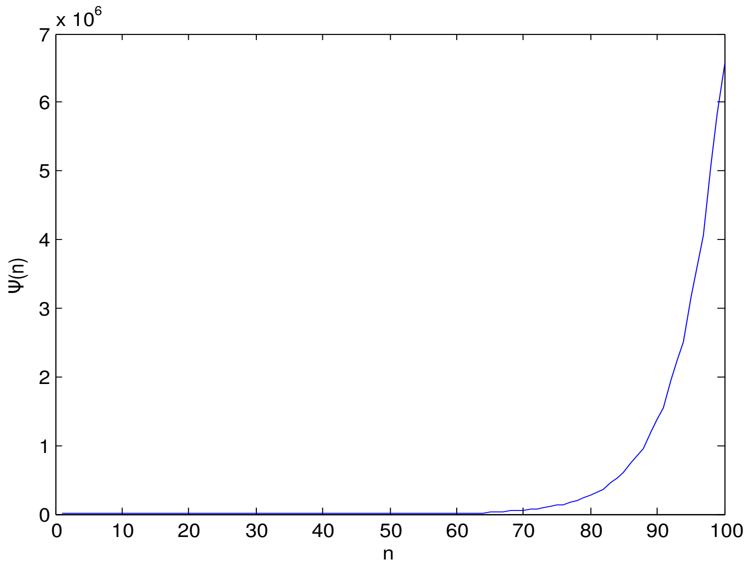

Figure 1. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}$.

Example 2. Numerically when the initial value are $\Psi_{-6} = 4.6, \Psi_{-5} = 2.5, \Psi_{-4} = 1.4, \Psi_{-3} = 3, \Psi_{-2} = 4.5, \Psi_{-1} = 6.3$ and $\Psi_0 = 3.5$. Figure 2 shows the results of Eq.(8).

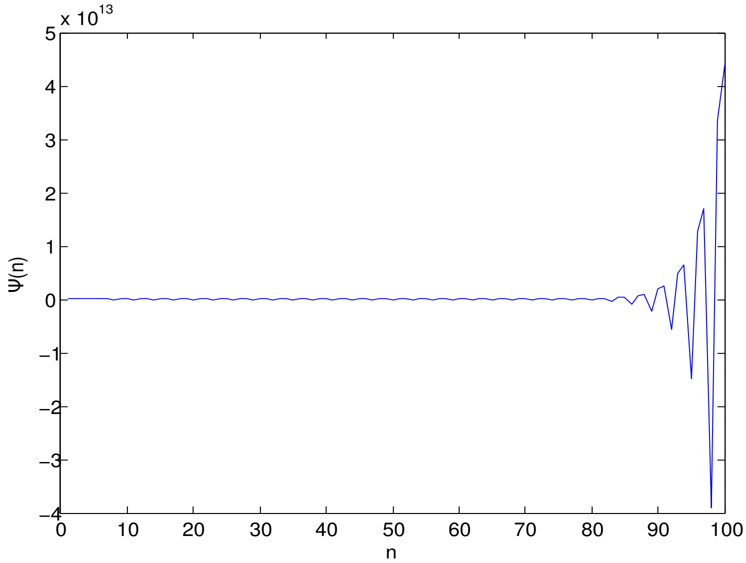

Figure 2. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}-\Psi_{n-6}}$.

Example 3. Figures 3 depict the behavior of Eq.(9), with initial conditions are $\Psi_{-6} = 2.8, \Psi_{-5} = 5.9, \Psi_{-4} = 8.5, \Psi_{-3} = 4.2, \Psi_{-2} = 7.4, \Psi_{-1} = 3.2$ and $\Psi_0 = 6.7$.

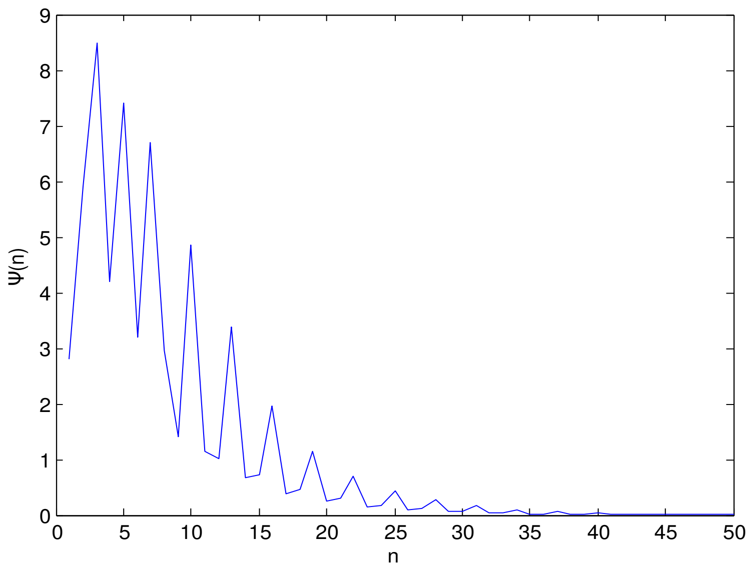

Figure 3. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}$.

Example 4. For Eq.(10) the initial conditions are set as follows: $\Psi_{-6} = 2.2, \Psi_{-5} = 3.9, \Psi_{-4} = 7.5, \Psi_{-3} = 4.2, \Psi_{-2} = 4.8, \Psi_{-1} = 3.2$ and $\Psi_0 = 6.7$, results shows in Figure 4.

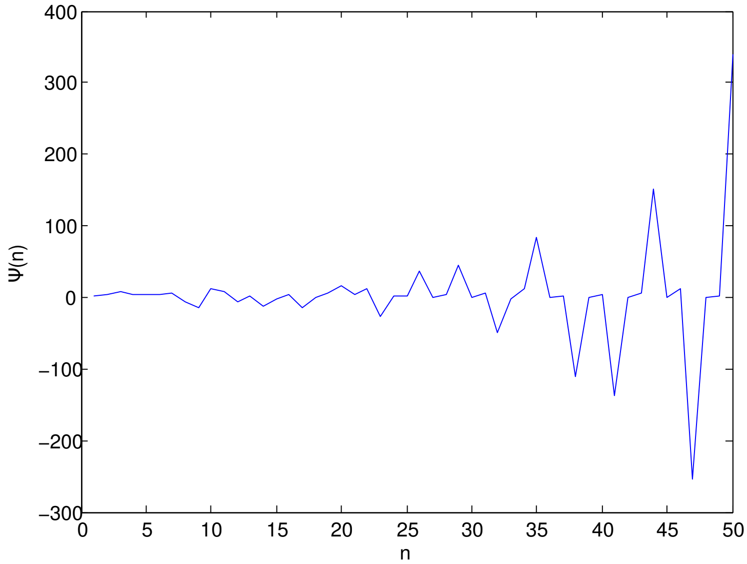

Figure 4. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}-\Psi_{n-6}}$.

7. Conclusions

Studying the dynamics of such equations is a very significant mathematical topic since these equations are strongly related to models in population dynamics and biological sciences. The basic goal of equations dynamics is to predict the global behavior of a equation based on the information of its current state. In this article, we have found general form of the solutions of rational difference equations and we investigated the dynamics of equilibrium point. In sections 2 and 3, we have investigated the existence and uniqueness of equilibrium point and the solutions qualitative behavior is explored, such as local and global stability. Also, we have proven that the solution is bounded in section 4. In section 5, we have obtained expressions of solutions of four special cases of the studied equations 7,8,9 and 10, as applications of Eq.(1). Finally, to support our theoretical discussion some illustrative examples are provided in section 6.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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