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Research Article

Generalized k-Order Fibonacci Hybrid Quaternions

Kübra GÜL[1*]

[1]Department of Computer Engineering, Faculty of Engineering and Architecture, Kafkas University, Kars, Turkey.

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Abstract

In this study, the generalized k-order Fibonacci hybrid quaternions are defined. We give the recurrence relation, generating function, the summation formula and some properties for these quaternions. Furthermore, the matrix representation for the generalized k-order Fibonacci hybrid quaternions is determined. The Q_k matrix defined for k-order Fibonacci numbers is given for the generalized k-order Fibonacci hybrid quaternions. By the means of this matrix and other defined matrices, several identities of these quaternions are also obtained.

Keywords: Fibonacci sequence, k-order Fibonacci sequence, hybrid numbers, quaternions.

Genelleştirilmiş k Mertebeden Fibonacci Hibrit Kuaterniyonlar

Öz

Bu çalışmada, genelleştirilmiş k mertebeden Fibonacci hibrit kuaterniyonlar tanımlanmıştır. Bu kuaterniyonlar için yineleme bağıntısı, üreteç fonksiyonu, toplam formülü ve bazı özellikler verilmiştir. Ayrıca, genelleştirilmiş k mertebeden Fibonacci hibrit kuaterniyonlar için matris temsili oluşturulmuştur. k mertebeden Fibonacci sayıları için tanımlanan Q_k matrisi, genelleştirilmiş k mertebeden Fibonacci hibrit kuaterniyonlar için verilmiştir. Bu matris ve tanımlanmış diğer matrisler yardımıyla bu kuaterniyonların bazı özdeşlikleri de elde edilmiştir.

Anahtar Kelimeler: Fibonacci dizisi, k-mertebeden Fibonacci dizisi, hibrit sayılar, kuaterniyonlar.

1. Introduction

Hybrid numbers are defined as a generalization of dual numbers, complex numbers and hyperbolic numbers [1]. The system of the hybrid numbers is a number system formed by three number systems together. With the help of these three sets of numbers, the set of hybrid numbers is defined as follow:

$$\mathbb{K} = \{z = a + b\mathrm{i} + c\varepsilon + d\mathrm{h} : a, b, c, d \in \mathbb{R}, \mathrm{i}^2 = -1, \varepsilon^2 = 0, \mathrm{h}^2 = 1, \mathrm{ih} = -\mathrm{hi} = \varepsilon + 1\} \qquad (1)$$

The multiplications of the units i, ε and h are given in the following table and the hybrid product is defined with the help of this table.

***Corresponding Author**: kubra.gul@kafkas.edu.tr

Kübra GÜL, https://orcid.org/0000-0002-8732-5718

·	1	i	ε	h
1	1	i	ε	h
i	i	-1	1-h	ε+i
ε	ε	h+1	0	-ε
h	h	-ε-i	ε	1

Table 1. Multiplication table of hybrid units

From the Table 1, it is seen that the multiplication operation in the hybrid numbers is associativity and not commutative. The conjugate of z is denoted by $\overline{z}$, and it is given as

$$\overline{z} = a - b\mathrm{i} - c\varepsilon - d\mathrm{h}.$$

The hybrid number character is defined by

$$C(z) = z\overline{z} = \overline{z}z = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2.$$

The norm of any hybrid number z is the root of $C(z)$, that is $\|z\| = \sqrt{C(z)}$. The readers can find more information about the hybrid number system in [1].

In recent years, special types of hybrid numbers have been studied by several authors. In [2], the authors introduced Fibonacci hybrid numbers and gave miscellaneous properties of these numbers. In [3, 4], the authors defined the generalizations of the Fibonacci and Lucas hybrid numbers and obtained some results for these numbers. They gave some number sequences for special values of k. In [5], the authors obtained the Euler's and De Moivre's formulas for the 4×4 matrix representation of hybrid numbers. Moreover, they gave the roots of the matrix representation of hybrid numbers and some results. In [6], the authors described Mersenne-Lucas numbers. Also, they presented the Binet formula, the generating function, several identities. For more information, we refer to [7, 8] and closely related references.

Quaternions were investigated by Hamilton [9] as an extension of the complex numbers. A quaternion is defined by

$$q = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \tag{2}$$

where a_0, a_1, a_2, a_3 are real numbers and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are quaternionic units that satisfy the following rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \text{ and } \boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{ji}, \boldsymbol{jk} = \boldsymbol{i} = -\boldsymbol{kj}, \boldsymbol{ki} = \boldsymbol{j} = -\boldsymbol{ik}. \tag{3}$$

Many authors studied different quaternions and their generalizations, some of which can be found in [10-18]. Dağdeviren and Kürüz defined a new class of quaternions as called hybrid quaternions in [19]. Also, the authors described the hybrid quaternions with Horadam numbers components and gave some properties. In [20], Uysal and Özkan defined Padovan hybrid quaternions.

Hybrid quaternions are a generalization of complex, dual and hyperbolic quaternions. The set of hybrid quaternions is defined by

$$\mathbb{H}_{\mathbb{K}} = \{Q = z_0 + z_1\boldsymbol{i} + z_2\boldsymbol{j} + z_3\boldsymbol{k}: z_0, z_1, z_2, z_3 \in \mathbb{K}\} \tag{4}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are quaternionic units satisfied the equations in (3). The hybrid quaternion Q can be written as

$$Q = q_0 + q_1\mathrm{i} + q_2\varepsilon + q_3\mathrm{h}$$

where q_0, q_1, q_2, q_3 are quaternions and i, ε, h are hybrid units.

In [21], the definition of order-k Fibonacci numbers is given as follows:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \text{for } n > 0, 1 \le i \le k$$

with initial conditions

$$g_n^i = \begin{cases} 1 & \text{if} \quad i = 1 - n \\ 0 & \text{otherwise} \end{cases}, \text{ for} 1 - k \le n \le 0$$

where g_n^i is the nth term of the ith sequence. Many authors studied k-order Fibonacci numbers, see, for example, [22, 23, 24] . In [4], the generalized k-order Fibonacci numbers are defined as follows;

$$V_n^{(k)} = d_1V_{n-1}^{(k)} + d_2V_{n-2}^{(k)} + d_3V_{n-3}^{(k)} + \cdots + d_kV_{n-k}^{(k)}$$

for $n > k \ge 2$, where

$$V_1^{(k)} = V_2^{(k)} = V_3^{(k)} = \cdots = V_{k-2}^{(k)} = 0, V_{k-1}^{(k)} = q, V_k^{(k)} = d_1.$$

Also, they introduced hybrid numbers with generalized k-order Fibonacci numbers components and obtained several properties for some important number sequences.

2. Main Results

In this section, we introduce the generalized k-order Fibonacci hybrid quaternions, and present some results obtained from the definition. Then we give generating function, summation formula, matrix representation and Simson (Cassini) identity for these quaternions. Note that we present the following table to avoid confusion of notations.

Notation	Sequence
$HQ_n^{(k)}$	generalized k-order Fibonacci hybrid quaternion
$HV_n^{(k)}$	generalized k- order Fibonacci hybrid number
$Q_n^{(k)}$	generalized k-order Fibonacci quaternion
$V_n^{(k)}$	generalized k-order Fibonacci number

Table 2: Notation table

Definition 2.1 The generalized k-order Fibonacci hybrid quaternions $\left\{HQ_n^{(k)}\right\}$, $n \in \mathbb{R}$, is defined as

$$HQ_n^{(k)} = HV_n^{(k)} + \boldsymbol{i}HV_{n+1}^{(k)} + \boldsymbol{j}HV_{n+2}^{(k)} + \boldsymbol{k}HV_{n+3}^{(k)} \tag{5}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are quaternionic units and $HV_n^{(k)}$ is nth generalized k-order hybrid Fibonacci numbers.

Any generalized k-order Fibonacci hybrid quaternion can be given by

$$HQ_n^{(k)} = Q_n^{(k)} + \mathrm{i}Q_{n+1}^{(k)} + \varepsilon Q_{n+2}^{(k)} + \mathrm{h}Q_{n+3}^{(k)}, \tag{6}$$

where i, ε, h are hybrid units and $Q_n^{(k)}$ is nth generalized k-order Fibonacci quaternion.

If we take as $k = 2$ in the equation (5), we get generalized Fibonacci hybrid quaternion defined in [19]. Certain special cases are given as follows:

when $d_1 = d_2 = 1$, $q = 1$, it is the Fibonacci hybrid quaternion ,

when $d_1 = d_2 = 1$, $q = 2$, it is the Lucas hybrid quaternion,

when $d_1 = 2, d_2 = 1$, $q = 1$, it is the Pell hybrid quaternion,

when $d_1 = 2, d_2 = 1$, $q = 2$, it is the Pell-Lucas hybrid quaternion,

when $d_1 = 1, d_2 = 2$, $q = 1$, it is the Jacobsthal hybrid quaternion,

when $d_1 = 1, d_2 = 2$, $q = 2$, it is the Jacobsthal-Lucas hybrid quaternion.

If we take as $k = 3$ in the equation (5),

when $d_1 = 0$, $d_2 = d_3 = 1$, $q = 1$, it is the Padovan hybrid quaternion,

when $d_1 = 0$, $d_2 = d_3 = 1$, $q = 3$, it is the Perrin hybrid quaternion.

Note that, for $n \geq 0$, there is the following recurrence relation:

$$HQ_n^{(k)} = \sum_{m=1}^{k} d_m HQ_{n-m}^{(k)}, \qquad \text{for} \quad k \geq 2. \tag{7}$$

Definition 2.2 The hybrid quaternions defined by both hybrid number $HV_n^{(k)}$ and hybrid quaternion $Q_n^{(k)}$ get three different conjugations as follows:

quaternion conjugate:

$\overline{HQ_n^{(k)}} = HV_n^{(k)} - \boldsymbol{i}HV_{n+1}^{(k)} - \boldsymbol{j}HV_{n+2}^{(k)} - \boldsymbol{k}HV_{n+3}^{(k)}$

or

$\overline{HQ_n^{(k)}} = \overline{Q_n^{(k)}} + \mathrm{i}\overline{Q_{n+1}^{(k)}} + \varepsilon\overline{Q_{n+2}^{(k)}} + \mathrm{h}\overline{Q_{n+3}^{(k)}}$,

hybrid conjugate:

$\widehat{HQ_n^{(k)}} = Q_n^{(k)} - \mathrm{i}Q_{n+1}^{(k)} - \varepsilon Q_{n+2}^{(k)} - \mathrm{h}Q_{n+3}^{(k)}$,

or

$\widehat{HQ_n^{(k)}} = \widehat{HV_n^{(k)}} + \boldsymbol{i}\widehat{HV_{n+1}^{(k)}} + \boldsymbol{j}\widehat{HV_{n+2}^{(k)}} + \boldsymbol{k}\widehat{HV_{n+3}^{(k)}}$,

total conjugate:

$(HQ_n^{(k)})^\dagger = \overline{Q_n^{(k)}} - \mathrm{i}\overline{Q_{n+1}^{(k)}} - \varepsilon\overline{Q_{n+2}^{(k)}} - \mathrm{h}\overline{Q_{n+3}^{(k)}}$

or

$(HQ_n^{(k)})^\dagger = \widehat{HV_n^{(k)}} - \boldsymbol{i}\widehat{HV_{n+1}^{(k)}} - \boldsymbol{j}\widehat{HV_{n+2}^{(k)}} - \boldsymbol{k}\widehat{HV_{n+3}^{(k)}}$.

Proposition 2.3 Let $\overline{HQ_n^{(k)}}, \widehat{HQ_n^{(k)}}, (HQ_n^{(k)})^\dagger$ be the quaternion conjugate, hybrid conjugate and total conjugate of $HQ_n^{(k)}$, respectively. For $n \geq 0$ and $k \geq 2$, we give the following relations:

i. $HQ_n^{(k)} + \overline{HQ_n^{(k)}} = 2HV_n^{(k)}$,

ii. $HQ_n^{(k)} + \widehat{HQ_n^{(k)}} = 2Q_n^{(k)}$,

iii. $(HQ_n^{(k)})^\dagger - HQ_n^{(k)} = -2HV_n^{(k)} - 2Q_n^{(k)} + 4V_n^{(k)}$,

iv. $HQ_n^{(k)}\widehat{HQ_n^{(k)}} = (Q_n^{(k)})^2 + (Q_{n+1}^{(k)})^2 - \left(Q_{n+3}^{(k)}\right)^2 - Q_{n+1}^{(k)}Q_{n+2}^{(k)} - Q_{n+2}^{(k)}Q_{n+1}^{(k)}$.

Theorem 2.4 The generating function for $HQ_n^{(k)}$ hybrid quaternion is given by

$$g(t) = \frac{HQ_0^{(k)} + \left(HQ_1^{(k)} - d_1HQ_0^{(k)}\right)t + \left(HQ_2^{(k)} - d_1HQ_1^{(k)} - d_2HQ_0^{(k)}\right)t^2 + \cdots + (HQ_{k-1}^{(k)} - \sum_{m=1}^{k-1} d_m HQ_{k-m-1}^{(k)})t^{k-1}}{1 - \sum_{m=1}^{k} d_m t^m}.$$

Proof. Suppose that the generating function for $HQ_n^{(k)}$ is

$$g(t) = \sum_{n=0}^{\infty} HQ_n^{(k)} t^n = HQ_0^{(k)} + HQ_1^{(k)}t + HQ_2^{(k)}t^2 + \cdots + HQ_n^{(k)}t^n + \cdots.$$

Multiplying $g(t)$ with $-d_1t, -d_2t^2, -d_3t^3, \cdots, -d_kt^k$ and then summing obtained equations, we have the following equation;

$$g(t) = \frac{1}{(1 - d_1t - d_2t^2 - d_3t^3 - \cdots - d_kt^k)}(HQ_0^{(k)} + (HQ_1^{(k)} - d_1HQ_0^{(k)})t$$

$$+(HQ_2^{(k)} - d_1 HQ_1^{(k)} - d_2 HQ_0^{(k)})t^2$$

$$+(HQ_3^{(k)} - d_1 HQ_2^{(k)} - d_2 HQ_1^{(k)} - d_3 HQ_0^{(k)})t^3$$

$$+\cdots+(HQ_n^{(k)} - \sum_{m=1}^{n-1} d_m HQ_{n-m}^{(k)})t^n + \cdots).$$

From the recurrence relation (7), we obtain

$$g(t) = \frac{HQ_0^{(k)} + \left(HQ_1^{(k)} - d_1 HQ_0^{(k)}\right)t + \left(HQ_2^{(k)} - d_1 HQ_1^{(k)} - d_2 HQ_0^{(k)}\right)t^2 + \cdots + (HQ_{k-1}^{(k)} - \sum_{m=1}^{k-1} d_m HQ_{k-m-1}^{(k)})t^{k-1}}{1 - \sum_{m=1}^{k} d_m t^m}.$$

So, the proof is completed.

Corollary 2.5 If $k = 2$, the generating function for the Horadam hybrid quaternions is given by

$$g(t) = \frac{HQ_0 + (HQ_1 - d_1 HQ_0)t}{1 - d_1 t - d_2 t^2}.$$

The generating functions for special cases of (d_1, d_2, q) is given as follows:

when $d_1 = d_2 = 1$, $q = 1$, for the Fibonacci hybrid quaternion, it is

$$g(t) = \frac{\hat{F}_0 + (\hat{F}_1 - \hat{F}_0)t}{1 - t - t^2},$$

when $d_1 = d_2 = 1$, $q = 2$, for the Lucas hybrid quaternion, it is

$$g(t) = \frac{\hat{L}_0 + (\hat{L}_1 - \hat{L}_0)t}{1 - t - t^2},$$

when $d_1 = 2$, $d_2 = 1$, $q = 1$, for the Pell hybrid quaternion, it is

$$g(t) = \frac{\hat{P}_0 + (\hat{P}_1 - \hat{P}_0)t}{1 - 2t - t^2},$$

when $d_1 = 1$, $d_2 = 2$, $q = 1$, for the Jacobsthal hybrid quaternion, it is

$$g(t) = \frac{\hat{J}_0 + (\hat{J}_1 - \hat{J}_0)t}{1 - t - 2t^2}.$$

If we take as $k = 3$, we can derive the following generating functions:

when $d_1 = d_2 = d_3 = 1$, $q = 1$, for the Tribonacci hybrid quaternion, it is

$$g(t) = \frac{\hat{T}_0 + (\hat{T}_1 - \hat{T}_0)t + (\hat{T}_2 - \hat{T}_1 - \hat{T}_0)t^2}{1 - t - t^2 - t^3},$$

when $d_1 = 0$, $d_2 = d_3 = 1$, $q = 1$, for the Padovan hybrid quaternion, it is

$$g(t) = \frac{\widehat{HP}_0 + \widehat{HP}_1 t + (\widehat{HP}_2 - \widehat{HP}_0)t^2}{1 - t^2 - t^3},$$

when $d_1 = 0$, $d_2 = d_3 = 1$, $q = 3$, for the Perrin hybrid quaternion, it is

$$g(t) = \frac{\widehat{RH}_0 + \widehat{RH}_1 t + (\widehat{RH}_2 - \widehat{RH}_0)t^2}{1 - t^2 - t^3}.$$

Theorem 2.6 The summation formula for the generalized k-order Fibonacci quaternions is given as follows:

$$\sum_{m=1}^{n} HQ_m^{(k)} = \frac{1}{d_k}\Big(\sum_{m=1}^{l} HQ_{k+m}^{(k)} - \sum_{m=1}^{k-1}\sum_{s=1}^{m} d_s HQ_{k+l-m}^{(k)}$$

$$- \sum_{m=1}^{k-1}\sum_{s=1}^{m} d_{k-s} HQ_{m+1}^{(k)} - \sum_{m=1}^{l-k} (d_1 + d_2 + \cdots + d_{k-1}) HQ_{k+m}^{(k)}\Big).$$

Proof. By the recurrence relation in the equation (7), for $n = k+1, \ldots, k+l$, we have

$$HQ_{k+1}^{(k)} = d_1 HQ_k^{(k)} + d_2 HQ_{k-1}^{(k)} + \cdots + d_{k-1} HQ_2^{(k)} + d_k HQ_1^{(k)},$$

$$HQ_{k+2}^{(k)} = d_1 HQ_{k+1}^{(k)} + d_2 HQ_k^{(k)} + \cdots + d_{k-1} HQ_3^{(k)} + d_k HQ_2^{(k)},$$

$$HQ_{k+3}^{(k)} = d_1 HQ_{k+2}^{(k)} + d_2 HQ_{k+1}^{(k)} + \cdots + d_{k-1} HQ_4^{(k)} + d_k HQ_3^{(k)},$$

$$\vdots$$

$$HQ_{k+l-1}^{(k)} = d_1 HQ_{k+l-2}^{(k)} + d_2 HQ_{k+l-3}^{(k)} + \cdots + d_{k-1} HQ_l^{(k)} + d_k HQ_{l-1}^{(k)},$$

$$HQ_{k+l}^{(k)} = d_1 HQ_{k+l-1}^{(k)} + d_2 HQ_{k+l-2}^{(k)} + \cdots + d_{k-1} HQ_{l+1}^{(k)} + d_k HQ_l^{(k)}.$$

By adding the last terms of the above equations, we can write as follows:

$$\sum_{m=1}^{l} HQ_m^{(k)} = \frac{1}{d_k}\Big(HQ_{k+l}^{(k)} + (1-d_1) HQ_{k+l-1}^{(k)} + (1-d_1-d_2) HQ_{k+l-2}^{(k)}$$

$$+ (1-d_1-d_2-d_3) HQ_{k+l-3}^{(k)} + \cdots + (1-d_1-d_2-\cdots-d_{k-3}) HQ_{l+3}^{(k)}$$

$$+ (1-d_1-d_2-\cdots-d_{k-2}) HQ_{l+2}^{(k)} + (1-d_1-d_2-\cdots-d_{k-1}) HQ_{l+1}^{(k)}$$

$$+ (1-d_1-d_2-\cdots-d_{k-1}) HQ_l^{(k)} + \cdots + (1-d_1-d_2-\cdots-d_{k-1}) HQ_{k+1}^{(k)}$$

$$+ (-d_1-d_2-\cdots-d_{k-1}) HQ_k^{(k)} + (-d_2-d_3-\cdots-d_{k-1}) HQ_{k-1}^{(k)} + \cdots$$

$$+ (-d_{k-3}-d_{k-2}-d_{k-1}) HQ_4^{(k)} + (-d_{k-2}-d_{k-1}) HQ_3^{(k)} - d_{k-1} HQ_2^{(k)}$$

$$= \frac{1}{d_k}\Big(\sum_{m=1}^{l} HQ_{k+m}^{(k)} - \sum_{m=1}^{k-1}\sum_{s=1}^{m} d_s HQ_{k+l-m}^{(k)}$$

$$- \sum_{m=1}^{k-1}\sum_{s=1}^{m} d_{k-s} HQ_{m+1}^{(k)} - \sum_{m=1}^{l-k} (d_1 + d_2 + \cdots + d_{k-1}) HQ_{k+m}^{(k)}\Big).$$

Corollary 2.7 For $k = 2$ and $d_1 = d_2 = q = 1$, the sum for the Fibonacci hybrid quaternions

$$\sum_{m=1}^{l} \hat{F}_m = \hat{F}_{l+2} - \hat{F}_2.$$

Corollary 2.8 For $k = 3$ and $d_1 = d_2 = d_3 = q = 1$, the sum for the Tribonacci hybrid quaternions

$$\sum_{m=1}^{l} \hat{T}_m = \frac{1}{2}(\hat{T}_{l+3} - \hat{T}_{l+1} - \hat{T}_3 + \hat{T}_1).$$

for $d_1 = 0$, $d_2 = d_3 = 1$ and $q = 1$, the sum for the Padovan hybrid quaternions

$$\sum_{m=1}^{n} \widehat{HP}_m = \widehat{HP}_{l+3} + \widehat{HP}_{l+2} - \widehat{HP}_2 - \widehat{HP}_3.$$

Based on the work of Asci and Aydinyuz [4], we now present the matrix representation for the generalized order k Fibonacci hybrid quaternions. The Q-matrix Q_k introduced in [4] is given as follow:

$$Q_k = \begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_{k-1} & d_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For $n \geq 1$, the elements of the quaternion sequence $\{ HQ_n^{(k)}\}$ can be derived by the following matrix relation:

$$\begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_{k-1} & d_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} HQ_{k-1}^{(k)} \\ HQ_{k-2}^{(k)} \\ HQ_{k-3}^{(k)} \\ \vdots \\ HQ_1^{(k)} \\ HQ_0^{(k)} \end{bmatrix} = \begin{bmatrix} HQ_{n+k-1}^{(k)} \\ HQ_{n+k-2}^{(k)} \\ HQ_{n+k-3}^{(k)} \\ \vdots \\ HQ_{n+1}^{(k)} \\ HQ_n^{(k)} \end{bmatrix}.$$

Lemma 2.9 Let

$$HQ_{k,n} = \begin{bmatrix} HQ_{n+k-1}^{(k)} & HQ_{n+k-2}^{(k)} & HQ_{n+k-3}^{(k)} & \cdots & HQ_{n+1}^{(k)} & HQ_n^{(k)} \\ HQ_{n+k-2}^{(k)} & HQ_{n+k-3}^{(k)} & HQ_{n+k-4}^{(k)} & \cdots & HQ_n^{(k)} & HQ_{n-1}^{(k)} \\ HQ_{n+k-3}^{(k)} & HQ_{n+k-4}^{(k)} & HQ_{n+k-5}^{(k)} & \cdots & HQ_{n-1}^{(k)} & HQ_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ HQ_{n+1}^{(k)} & HQ_n^{(k)} & HQ_{n-1}^{(k)} & \cdots & HQ_{n+3-k}^{(k)} & HQ_{n+2-k}^{(k)} \\ HQ_n^{(k)} & HQ_{n-1}^{(k)} & HQ_{n-2}^{(k)} & \cdots & HQ_{n+2-k}^{(k)} & HQ_{n+1-k}^{(k)} \end{bmatrix}$$

be the matrix form of the generalized order-k Fibonacci quaternion. For $n \geq 1$, we have

$$HQ_{k,n+1} = Q_k . HQ_{k,n}. \tag{8}$$

Theorem 2.10 Let $HQ_{k,n}$ be the matrix form of $HQ_n^{(k)}$. For $n \geq 1$, then we have

$$HQ_{k,n} = Q_k^n . A_k \tag{9}$$

where A_k is defined as a $k \times k$ matrix by

$$A_k = \begin{bmatrix} HQ_{k-1}^{(k)} & HQ_{k-2}^{(k)} & HQ_{k-3}^{(k)} & \cdots & HQ_1^{(k)} & HQ_0^{(k)} \\ HQ_{k-2}^{(k)} & HQ_{k-3}^{(k)} & HQ_{k-4}^{(k)} & \cdots & HQ_0^{(k)} & HQ_{-1}^{(k)} \\ HQ_{k-3}^{(k)} & HQ_{k-4}^{(k)} & HQ_{k-5}^{(k)} & \cdots & HQ_{-1}^{(k)} & HQ_{-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ HQ_1^{(k)} & HQ_0^{(k)} & HQ_{-1}^{(k)} & \cdots & HQ_{-(k-3)}^{(k)} & HQ_{-(k-2)}^{(k)} \\ HQ_0^{(k)} & HQ_{-1}^{(k)} & HQ_{-1}^{(k)} & \cdots & HQ_{-(k-2)}^{(k)} & HQ_{-(k-1)}^{(k)} \end{bmatrix}.$$

Proof. The proof is seen by the principle of mathematical induction on n. For $n = 1$, it is easy to see that

$$Q_k A_k = HQ_{k,1}.$$

Now, we assume that the formula (9) is true for n, that is

$$HQ_{k,n} = Q_k^n . A_k.$$

Then by induction, using the equality (8), it is shown that it is valid for $n+1$,

$$\begin{aligned} Q_k^{n+1} A_k &= Q_k Q_k^n . A_k \\ &= Q_k HQ_{k,n} \\ &= HQ_{k,n+1}. \end{aligned}$$

So, the proof is completed.

Corollary 2.11 If $k = 2$, the matrix representation of the Horadam hybrid quaternions is given by

$$\begin{aligned} Q_2^n A_2 &= \begin{bmatrix} d_1 & d_2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} HQ_1 & HQ_0 \\ HQ_0 & HQ_{-1} \end{bmatrix} \\ &= \begin{bmatrix} HQ_{n+1} & HQ_n \\ HQ_n & HQ_{n-1} \end{bmatrix} \\ &= HQ_{2,n}. \end{aligned}$$

The matrix representations for special cases of (d_1, d_2, q) is obtained similarly. For example; for $d_1 = d_2 = 1$, $q = 1$, the matrix representation of the Fibonacci hybrid quaternion is

$$\begin{aligned} Q_2^n A_2 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \hat{F}_1 & \hat{F}_0 \\ \hat{F}_0 & \hat{F}_{-1} \end{bmatrix} \\ &= \begin{bmatrix} \hat{F}_{n+1} & \hat{F}_n \\ \hat{F}_n & \hat{F}_{n-1} \end{bmatrix}. \end{aligned}$$

If we take as $k = 3$,
for $d_1 = d_2 = d_3 = 1$, $q = 1$, the matrix representation of the Tribonacci hybrid quaternion is given by

$$Q_3^n A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} \hat{T}_2 & \hat{T}_1 & \hat{T}_0 \\ \hat{T}_1 & \hat{T}_0 & \hat{T}_{-1} \\ \hat{T}_0 & \hat{T}_{-1} & \hat{T}_{-2} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{T}_{n+2} & \hat{T}_{n+1} & \hat{T}_n \\ \hat{T}_{n+1} & \hat{T}_n & \hat{T}_{n-1} \\ \hat{T}_n & \hat{T}_{n-1} & \hat{T}_{n-2} \end{bmatrix}.$$

Theorem 2.12 For all integers m, n such that $0 < m < n$, we have the following relations :

$$HQ_n^{(k)} = V_{m+1}^{(k)} HQ_{n-m}^{(k)} + (V_m^{(k)} + V_{m-1}^{(k)} + \cdots + V_{m+2-k}^{(k)}) HQ_{n-m-1}^{(k)} + \cdots$$
$$+ (V_m^{(k)} + V_{m-1}^{(k)} + V_{m-2}^{(k)}) HQ_{n-m-k+3}^{(k)} + (V_m^{(k)} + V_{m-1}^{(k)}) HQ_{n-m-k+2}^{(k)} + V_m^{(k)} HQ_{n-m-k+1}^{(k)}.$$

Proof. In [21], it is seen that

$$Q_k^n = \begin{bmatrix} V_{n+1}^{(k)} & \cdots & V_n^{(k)} + V_{n-1}^{(k)} + V_{n-2}^{(k)} & V_n^{(k)} + V_{n-1}^{(k)} & V_n^{(k)} \\ V_n^{(k)} & \cdots & V_{n-1}^{(k)} + V_{n-2}^{(k)} + V_{n-3}^{(k)} & V_{n+1}^{(k)} + V_n^{(k)} & V_{n-1}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ V_{n-k+3}^{(k)} & \cdots & V_{n-k+2}^{(k)} + V_{n-k+1}^{(k)} + V_{n-k}^{(k)} & V_{n-k+2}^{(k)} + V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} \\ V_{n-k+2}^{(k)} & \cdots & V_{n-k+1}^{(k)} + V_{n-k}^{(k)} + V_{n-k-1}^{(k)} & V_{n-k+3}^{(k)} + V_{n-k+2}^{(k)} & V_{n-k+1}^{(k)} \end{bmatrix}.$$

From definitions of the Q_k-matrix and the equality (9), we have

$$Q_k^n = Q_k^m Q_k^{n-m}$$

$$Q_k^n A_k = Q_k^m (Q_k^{n-m} A_k)$$

$$= Q_k^m HQ_{k,n-m}.$$

Considering the matrix equality and the product of matrices, the result is obtained as follows:

$$HQ_n^{(k)} = V_{m+1}^{(k)} HQ_{n-m}^{(k)} + (V_m^{(k)} + V_{m-1}^{(k)} + \cdots + V_{m+2-k}^{(k)}) HQ_{n-m-1}^{(k)} + \cdots$$
$$+ (V_m^{(k)} + V_{m-1}^{(k)} + V_{m-2}^{(k)}) HQ_{n-m-k+3}^{(k)} + (V_m^{(k)} + V_{m-1}^{(k)}) HQ_{n-m-k+2}^{(k)} + V_m^{(k)} HQ_{n-m-k+1}^{(k)}.$$

Theorem 2.13 Let m and n be positive. Then we have

$$HQ_{m+n}^{(k)} = V_{m+1}^{(k)} HQ_n^{(k)} + (V_m^{(k)} + V_{m-1}^{(k)} + \cdots + V_{m+2-k}^{(k)}) HQ_{n-1}^{(k)} + \cdots$$
$$+ (V_m^{(k)} + V_{m-1}^{(k)} + V_{m-2}^{(k)}) HQ_{n-k+3}^{(k)} + (V_m^{(k)} + V_{m-1}^{(k)}) HQ_{n-k+2}^{(k)} + V_m^{(k)} HQ_{n-k+1}^{(k)}.$$

Simson identity (formula) called also as Cassini identity (formula) was found by R. Simson in

1753. This formula is determined by

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which is also denoted by the following form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

In [25], the author gave a generalization of Simson identity and presented Simson's identities of some number sequences by certain special cases. Now, we derive Simson identity for the generalized order-k Fibonacci hybrid quaternions and also obtain Simson identity for well-known hybrid quaternions in some special cases.

Theorem 2.14 For $k \geq 2$,

$$\left|HQ_{k,n}\right| = (-1)^{(k-1)n}(d_k)^n|A_k|. \tag{10}$$

Proof. We prove by the principle of mathematical induction on n. For $n = 0$, it is clear that the formula is true. Now, we assume that the formula (10) is true for n. Therefore, we have to show that it is true for $n + 1$.

Taking into account the recurrence relation $HQ_n^{(k)} = \sum_{m=1}^{k} d_m HQ_{n-m}^{(k)}$, we write the elements of the first column as follows:

$$HQ_{n+k}^{(k)} = d_1 HQ_{n+k-1}^{(k)} + d_2 HQ_{n+k-2}^{(k)} + \cdots + d_k HQ_n^{(k)},$$

$$HQ_{n+k-1}^{(k)} = d_1 HQ_{n+k-2}^{(k)} + d_2 HQ_{n+k-3}^{(k)} + \cdots + d_k HQ_{n-1}^{(k)},$$

$$HQ_{n+k-2}^{(k)} = d_1 HQ_{n+k-3}^{(k)} + d_2 HQ_{n+k-4}^{(k)} + \cdots + d_k HQ_{n-2}^{(k)},$$

$$\vdots$$

$$HQ_{n+1}^{(k)} = d_1 HQ_n^{(k)} + d_2 HQ_{n-1}^{(k)} + \cdots + d_k HQ_{n-k+1}^{(k)}.$$

When the first column of the determinant is subtracted with all terms except the last term in the sum on the right-hand side of the above equations, and the determinant is rearranged, using proportionality, switching and sum properties of determinant, we have

$$\left|HQ_{k,n+1}\right| = d_k \begin{vmatrix} HQ_n^{(k)} & HQ_{n+k-1}^{(k)} & HQ_{n+k-2}^{(k)} & \cdots & HQ_{n+2}^{(k)} & HQ_{n+1}^{(k)} \\ HQ_{n-1}^{(k)} & HQ_{n+k-2}^{(k)} & HQ_{n+k-3}^{(k)} & \cdots & HQ_{n+1}^{(k)} & HQ_n^{(k)} \\ HQ_{n-2}^{(k)} & HQ_{n+k-3}^{(k)} & HQ_{n+k-4}^{(k)} & \cdots & HQ_n^{(k)} & HQ_{n-1}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ HQ_{n-k+2}^{(k)} & HQ_{n+1}^{(k)} & HQ_n^{(k)} & \cdots & HQ_{n+4-k}^{(k)} & HQ_{n+3-k}^{(k)} \\ HQ_{n-k+1}^{(k)} & HQ_n^{(k)} & HQ_{n-1}^{(k)} & \cdots & HQ_{n+3-k}^{(k)} & HQ_{n+2-k}^{(k)} \end{vmatrix}$$

$$= (-1)^{k-1} d_k \begin{vmatrix} HQ_{n+k-1}^{(k)} & HQ_{n+k-2}^{(k)} & HQ_{n+k-3}^{(k)} & \cdots & HQ_{n+1}^{(k)} & HQ_{n}^{(k)} \\ HQ_{n+k-2}^{(k)} & HQ_{n+k-3}^{(k)} & HQ_{n+k-4}^{(k)} & \cdots & HQ_{n}^{(k)} & HQ_{n-1}^{(k)} \\ HQ_{n+k-3}^{(k)} & HQ_{n+k-4}^{(k)} & HQ_{n+k-5}^{(k)} & \cdots & HQ_{n-1}^{(k)} & HQ_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ HQ_{n+1}^{(k)} & HQ_{n}^{(k)} & HQ_{n-1}^{(k)} & \cdots & HQ_{n+3-k}^{(k)} & HQ_{n+2-k}^{(k)} \\ HQ_{n}^{(k)} & HQ_{n-1}^{(k)} & HQ_{n-2}^{(k)} & \cdots & HQ_{n+2-k}^{(k)} & HQ_{n+1-k}^{(k)} \end{vmatrix}$$

$$= (-1)^{(k-1)n+1} (d_k)^{n+1} |A_k|.$$

So, the proof is completed.

Corollary 2.15 Simson Formula of Horadam hybrid quaternions is

$$\begin{vmatrix} HQ_{n+1} & HQ_n \\ HQ_n & HQ_{n-1} \end{vmatrix} = (-1)^n (d_2)^n \begin{vmatrix} HQ_1 & HQ_0 \\ HQ_0 & HQ_{-1} \end{vmatrix}.$$

Corollary 2.16 Simson Formula of Jacobsthal hybrid quaternions is

$$\begin{vmatrix} \hat{J}_{\mathrm{n}+1} & \hat{J}_{\mathrm{n}} \\ \hat{J}_{\mathrm{n}} & \hat{J}_{\mathrm{n}-1} \end{vmatrix} = (-1)^n (2)^n \begin{vmatrix} \hat{J}_1 & \hat{J}_0 \\ \hat{J}_0 & \hat{J}_{-1} \end{vmatrix}.$$

Corollary 2.17 Simson Formula of Tribonacci hybrid quaternions is

$$\begin{vmatrix} \hat{T}_{n+1} & \hat{T}_n & \hat{T}_{n-1} \\ \hat{T}_n & \hat{T}_{n-1} & \hat{T}_{n-2} \\ \hat{T}_{n-1} & \hat{T}_{n-2} & \hat{T}_{n-3} \end{vmatrix} = \begin{vmatrix} \hat{T}_1 & \hat{T}_0 & \hat{T}_{-1} \\ \hat{T}_0 & \hat{T}_{-1} & \hat{T}_{-2} \\ \hat{T}_{-1} & \hat{T}_{-2} & \hat{T}_{-3} \end{vmatrix}.$$

3. Conclusion

In this paper, we extend the Fibonacci hybrid quaternions to the generalized k-order Fibonacci hybrid quaternions. We investigated the recurrence relation, generating function, the summation formula for these quaternions. Then, we gave the matrix representation for the generalized k-order Fibonacci hybrid quaternions. With the help of the Q_k matrix defined for the generalized k-order Fibonacci hybrid quaternions and other defined matrices, we also obtain some identities of these quaternions.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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