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Research Article

Some structures on the coframe bundle with Cheeger-Gromoll metric

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Abstract

In this paper an almost paracomplex structures on the coframe bundle with Cheeger-Gromoll metric are defined and later we obtained the integrability conditions of these structures. Also we proved that para-Norden structures which exists on coframe bundle are non-Kahler-Norden.

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Keywords. coframe bundle, Cheeger-Gromoll metric, para-Norden structure, Nijenhuis tensor, integrability condition

1. Introduction

Almost paracomplex and para-Hermitian structures on a differentiable manifold were initially introduced by Rashevskii in 1948 [14] and later by Libermann in 1952 [10]. These type structures have been studied and used by many mathematicians and physicists, for example Kaneyuki-Kozai [9] and Gadea-Anilibia [6] (see [4] for a wide range of references). In [1] Bejan has extended these notions to an arbitrary vector bundles and call them para-complex and para-Hermitian vector bundles. The book by Vishnevskii, Shirokov and Shurygin [21] is a monography in which the authors study differential geometry on manifolds over general algebras. In particular, in this book the authors studied the paracomplex structures with additional properties, that is a para-Kahler manifolds. Many authors considered almost complex structures on the tangent, cotangent and tensor bundles (see, for example [5], [13], [15]),

The coframe bundle is widely used not only in mathematics, but also in theoretical physics in the sense that gravity can be mathematically defined as a coframe bundle $C_M = (GL(d, R), M)$, where M is a $d-$dimensional spacetime manifold ([2], [12], [20]). The present paper is devoted to the study of paracomplex structure on the coframe bundle with the Cheeger-Gromoll metric. In 2 we briefly describe the definitions and results that a needed later, after which the Cheeger-Gromoll metric ${}^{CG}g$ on coframe bundle $F^*(M)$ introduced in 3. In 4 we define an almost paracomplex structures ${}^{CG}F_\alpha, \alpha = 1, ..., n$, on $F^*(M)$. The integrability conditions of ${}^{CG}F_\alpha, \alpha = 1, ..., n$ are investigated in 5. In 6 we calculate the covariant derivatives of paracomplex structures ${}^{CG}F_\alpha, \alpha = 1, ..., n$.

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2. Peliminaries

Let $F^*(M)$ be the linear coframe bundle of $n-$dimensional smooth manifold M. We denote by π the natural projection of $F^*(M)$ to M defined by $\pi(x,u^*)=x$, where $x\in M$ and u^* is a basis (coframe) for the cotangent space T^*_xM of M at x (see, [16]). If $(U;x^1,x^2,...,x^n)$ is a system of local coordinates in M, then a coframe $u^*=(X^\alpha)=(X^1,X^2,...,X^n)$ for T^*_xM can be expressed uniquely in the form $X^\alpha=X^\alpha_i(dx^i)_x$. Therefore,

$$(\pi^{-1}(U);x^1,...,x^n,X^1_1,...,X^n_n)$$

is a system of local coordinates in $F^*(M)$, that is $F^*(M)$ is a $C^\infty-$manifold of dimension $n+n^2$. We note that indices $i,j,k,...,\alpha,\beta,\gamma,...$ have range in $\{1,2,...,n\}$, indices $A,B,C,...$ have range in $\{1,...,n,n+1,...,n+n^2\}$. We put $i_\alpha=\alpha\cdot n+i$. Obviously that indices $i_\alpha,j_\beta,k_\gamma,...$ have range in $\{n+1,n+2,...,n+n^2\}$. The set of all tensor fields of type (p,q) on M we denote by $\Im^p_q(M)$. Summation over repeated indices is always implied.

Let $X=X^i\frac{\partial}{\partial x^i}$ and $\omega=\omega_i dx^i$ be the local expressions in U of $X\in\Im^1_0(M)$ and $\omega\in\Im^0_1(M)$. Then the horizontal lift ${}^HX$ of X and the $\alpha-$th vertical lift ${}^{V_\alpha}\omega$ of ω to $F^*(M)$ are given, in the induced coordinates (x^j,X^β_j) by

$${}^HX=X^j\partial_j+X^\beta_m\Gamma^m_{lj}X^l\partial_{j_\beta}, \tag{2.1}$$

$${}^{V_\alpha}\omega=\delta^\alpha_\beta\omega_j\partial_{j_\beta}, \tag{2.2}$$

where Γ^k_{ij} are the coefficients of the Levi-Civita connections ∇ of g and $\alpha=1,2,...,n$ (for more details see [16]). In $U\subset M$, we put

$$X_{(i)}=\frac{\partial}{\partial x^i},\quad \theta^{(i)}=dx^i, i=1,2,...,n.$$

Taking into account of (2.1) and (2.2), we see that

$${}^HX_{(i)}=D_i=\begin{pmatrix}\delta^j_i\\ X^\beta_m\Gamma^m_{ij}\end{pmatrix}, \tag{2.3}$$

$${}^{V_\alpha}\theta^{(i)}=D_{i_\alpha}=\begin{pmatrix}0\\ \delta^\alpha_\beta\delta^i_j\end{pmatrix} \tag{2.4}$$

with respect to the natural frame $\left\{\partial_j,\partial_{j_\beta}\right\}$. This $n+n^2$ vector fields are linearly independent and generate, respectively the horizontal distribution of linear connection ∇ and the vertical distribution of coframe bundle $F^*(M)$. The set $\{D_I\}=\{D_i,D_{i_\alpha}\}$ is called the frame adapted to linear connection ∇. From (2.1)-(2.4) it follows that

$${}^HX=\begin{pmatrix}X^j\\ 0\end{pmatrix}, \tag{2.5}$$

$${}^{V_\alpha}\omega=\begin{pmatrix}0\\ \delta^\alpha_\beta\omega_j\end{pmatrix} \tag{2.6}$$

with respect to the adapted frame $\{D_J\}$. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{aligned}&[{}^{V_\alpha}\omega,{}^{V_\beta}\theta]=0,\\ &[{}^HX,{}^{V_\beta}\theta]={}^{V_\beta}(\nabla_X\theta),\\ &[{}^HX,{}^HY]={}^H[X,Y]+\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ R(X,Y))\end{aligned} \tag{2.7}$$

for all $X,Y\in\Im^1_0(M)$ and $\omega,\theta\in\Im^0_1(M)$, where R is the Riemanian curvature of g defined by

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}.$$

If f is a differentiable function on M, ${}^{V}f = f \circ \pi$ denotes its canonical vertical lift to the coframe bundle $F^*(M)$.

3. The Cheeger-Gromoll metric on the coframe bundle

Let (M, g) be a Riemannian manifold. A Riemannian metric $\tilde{g}$ on the coframe bundle $F^*(M)$ is said to be natural with respect to g on M if

$$\begin{aligned} &\tilde{g}({}^{H}X, {}^{H}Y) = g(X, Y),\\ &\tilde{g}({}^{H}X, {}^{V_\beta}\theta) = 0 \end{aligned}$$

for all vector fields $X, Y \in \Im^1_0(M)$ and 1-form $\theta \in \Im^0_1(M)$. A natural metric $\tilde{g}$ is constructed in such a way that the horizontal and vertical distributions are orthogonal. The well-known example of natural metric is Sasaki metric ${}^{S}g$ (or diagonal lift of g) introduced in [18]. The Sasaki metric ${}^{S}g$ in coframe bundle $F^*(M)$ is defined by

$$\begin{aligned} &{}^{S}g({}^{H}X, {}^{H}Y) = g(X, Y),\\ &{}^{S}g({}^{H}X, {}^{V_\beta}\theta) = 0,\\ &{}^{S}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = \delta^\alpha_\beta g^{-1}(\omega, \theta) \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Another well-known natural Riemannian metric ${}^{CG}g$ on tangent bundle $T(M)$ was considered by Musso and Tricerri [11] who inspired by the paper [3] of Cheeger and Gromoll called it the Cheeger-Gromoll metric. The Levi-Civita connection of ${}^{CG}g$ and its Riemannian curvature tensor were studied by Sekizawa [19]. The geometries of Cheeger-Gromoll type metrics on tangent and cotangent bundles has been intensively studied by many geometers (see, for example [7]).

The Cheeger-Gromoll metric ${}^{CG}g$ is a positive defined metric on coframe bundle $F^*(M)$ which is described in terms of lifted vector fields as follows.

Definition 3.1. Let g be a Riemannian metric on a manifold M. Then a Cheeger-Gromoll metric is a Riemannian metric ${}^{CG}g$ on the coframe bundle $F^*(M)$ such that

$$\begin{aligned} &{}^{CG}g({}^{H}X, {}^{H}Y) = g(X, Y),\\ &{}^{CG}g({}^{H}X, {}^{V_\beta}\theta) = 0,\\ &{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = 0, \quad \alpha \neq \beta,\\ &{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta) = \tfrac{1}{1+r_\alpha^2}(g^{-1}(\omega, \theta) + g^{-1}(\omega, X^\alpha) g^{-1}(\theta, X^\alpha)) \end{aligned} \tag{3.1}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where $r_\alpha^2 = |X^\alpha|^2 = g^{-1}(X^\alpha, X^\alpha)$.

The Levi-Civita connection ${}^{CG}\nabla$ of Cheeger-Gromoll metric ${}^{CG}g$ satisfies the following relations

i) ${}^{CG}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) + \frac{1}{2}\sum\limits_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X, Y))$,

$$ii) \qquad {}^{CG}\nabla_{{}^{H}X}{}^{V_\beta}\theta = {}^{V_\beta}(\nabla_X \theta) + \frac{1}{2h_\beta}{}^{H}(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\theta})), \tag{3.2}$$

iii) ${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{H}Y = \frac{1}{2h_\alpha}{}^{H}(X^\alpha(g^{-1} \circ R(, Y)\tilde{\omega}))$,

iv) ${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = 0$ for $\alpha \neq \beta$,

$${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\theta = -\tfrac{1}{h_\alpha}({}^{CG}g({}^{V_\alpha}\omega, \gamma\delta){}^{V_\alpha}\theta + {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{V_\alpha}\omega)$$

$$+\tfrac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)\gamma\delta - \tfrac{1}{h_\alpha}{}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)\gamma\delta$$

for all $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$, where $\tilde{\omega} = g^{-1} \circ \omega$, $R(\ \ , X)\tilde{\omega} \in \Im^1_1(M)$, $h_\alpha = 1 + r_\alpha^2$, R and $\gamma\delta$ denotes respectively the Riemanniian curvature of g and the canonical vertical vector field on $F^*(M)$ with local expression $\gamma\delta = X^\sigma_i D_{i_\sigma}$.

Using (3.2) it is easy to prove that the components ${}^{CG}\Gamma_{IJ}^{K}$ of Levi-Civita connection ${}^{CG}\nabla$ for different indices are then found to be

$$
\begin{aligned}
{}^{CG}\Gamma_{ij}^{k} &= \Gamma_{ij}^{k},\ {}^{CG}\Gamma_{ij}^{k_\gamma} = \frac{1}{2}X_m^\gamma R_{ijk}^m,\\
{}^{CG}\Gamma_{ij_\beta}^{k} &= \frac{1}{2h_\beta}X_m^\beta R_{\cdot\, i\, \cdot}^{k\ jm},\ {}^{CG}\Gamma_{ij_\beta}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j,\\
{}^{CG}\Gamma_{i_\alpha j}^{k} &= \frac{1}{2h_\alpha}X_m^\alpha R_{\cdot\, j\, \cdot}^{k\ im},\ {}^{CG}\Gamma_{i_\alpha j}^{k_\gamma} = {}^{CG}\Gamma_{i_\alpha j_\beta}^{k} = 0,
\end{aligned}
\tag{3.3}
$$

${}^{CG}\Gamma_{i_\alpha j_\beta}^{k_\gamma} = 0$ for $\alpha \neq \beta$,

$$
{}^{CG}\Gamma_{i_\alpha j_\alpha}^{k_\gamma} = -\tfrac{1}{h_\alpha}(\tilde{X}^{\alpha i}\delta_\gamma^\alpha \delta_k^j + \tilde{X}^{\alpha j}\delta_\gamma^\alpha \delta_k^i)+
$$

$$
+\tfrac{1+h_\alpha}{h_\alpha^2} g^{ij} X_k^\gamma + \tfrac{1}{h_\alpha^2}\tilde{X}^{\alpha i}\tilde{X}^{\alpha j}X_k^\gamma,
$$

where $\tilde{X}^{\alpha i} = g^{is}X_s^\alpha$.

4. Para-Norden structures on the coframe bundle

Let (M, g) be an $n-$dimensional Riemannian manifold. An almost paracomplex manifold is an almost product manifold $(M, \varphi), \varphi^2 = id, \varphi \neq id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively. The dimension of almost paracomplex manifold is even. Let (M_{2k}, φ) be an almost paracomplex manifold. A Riemannian metric g is a para-Norden metric (or $B-$metric) if

$$
g(\varphi X, \varphi Y) = g(X, Y),
$$

or

$$
g(\varphi X, Y) = g(X, \varphi Y)
$$

for any $X, Y \in \Im_0^1(M_{2k})$. We say that the triple (M_{2k}, φ, g) is an almost paracomplex Norden manifold ([8], [17], [22]) if (M_{2k}, φ) is an almost paracomplex manifold with a para-Norden metric g. If φ is integrable, then (M_{2k}, φ, g) is called paracomplex Norden manifold.

Let $(F^*(M), {}^{CG}g)$ be the linear coframe bundle with the Cheeger-Gromoll metric ${}^{CG}g$. Define a tensor field ${}^{CG}F_\alpha$ of type $(1,1)$ on $F^*(M)$ for each $\alpha = 1, 2, ..., n$, by

$$
\begin{aligned}
{}^{CG}F_\alpha({}^{H}X) &= \sqrt{h_\alpha}\,{}^{V_\alpha}\tilde{X} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{V_\alpha}X^\alpha,\\
{}^{CG}F_\alpha({}^{V_\beta}\omega) &= 0, \quad \beta \neq \alpha,\\
{}^{CG}F_\alpha({}^{V_\alpha}\omega) &= \frac{1}{\sqrt{h_\alpha}}\left({}^{H}\tilde{\omega} + \frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \omega)^{H}\tilde{X}^\alpha\right)
\end{aligned}
\tag{4.1}
$$

for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, where $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$ and the horizontal lifts are considered with respect to the Levi-Civita connection of g. Each ${}^{CG}F_\alpha$ satisfies the condition

$$
{}^{CG}F_\alpha^2 = I.
$$

Indeed, by virtue of (4.1), we have

$$
\begin{aligned}
{}^{CG}F_\alpha^2({}^{H}X) &= {}^{CG}F_\alpha({}^{CG}F_\alpha({}^{H}X)) = {}^{CG}F_\alpha(\sqrt{h_\alpha}\,{}^{V_\alpha}\tilde{X}\\
&-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{V_\alpha}X^\alpha) = \sqrt{h_\alpha}\,{}^{CG}F_\alpha({}^{V_\alpha}\tilde{X})\\
&-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{CG}F_\alpha({}^{V_\alpha}X^\alpha) = \sqrt{h_\alpha}\left(\frac{1}{\sqrt{h_\alpha}}{}^{H}X\right.
\end{aligned}
$$

$$+\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\tilde{X})^H\tilde{X}^\alpha\Big)-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)\frac{1}{\sqrt{h_\alpha}}\Big({}^H\tilde{X}^\alpha$$
$$+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,X^\alpha)^H\tilde{X}^\alpha\Big)={}^HX+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\tilde{X})^H\tilde{X}^\alpha$$
$$-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)\frac{1}{\sqrt{h_\alpha}}{}^H\tilde{X}^\alpha-\frac{1}{(\sqrt{h_\alpha}+1)^2}\frac{1}{\sqrt{h_\alpha}}X^\alpha(X)(h_\alpha-1)^H\tilde{X}^\alpha$$
$$={}^HX+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\tilde{X})^H\tilde{X}^\alpha-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)\frac{1}{\sqrt{h_\alpha}}{}^H\tilde{X}^\alpha$$
$$-\frac{1}{\sqrt{h_\alpha}+1}\frac{\sqrt{h_\alpha}-1}{\sqrt{h_\alpha}}X^\alpha(X)^H\tilde{X}^\alpha={}^HX,$$

$${}^{CG}F_\alpha^2({}^{V_\alpha}\omega)={}^{CG}F_\alpha({}^{CG}F_\alpha({}^{V_\alpha}\omega))={}^{CG}F_\alpha\Big(\frac{1}{\sqrt{h_\alpha}}\,({}^H\tilde{\omega}$$
$$+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\omega)^H\tilde{X}^\alpha)=\frac{1}{\sqrt{h_\alpha}}{}^{CG}F_\alpha({}^H\tilde{\omega})$$
$$+\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega)^{CG}F_\alpha({}^H\tilde{X}^\alpha)=\frac{1}{\sqrt{h_\alpha}}\Big(\sqrt{h_\alpha}\,{}^{V_\alpha}\omega$$
$$-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(\tilde{\omega})^{V_\alpha}X^\alpha\Big)+\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega)\sqrt{h_\alpha}{}^{V_\alpha}X^\alpha$$
$$-\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega)\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,X^\alpha)^{V_\alpha}X^\alpha$$
$$={}^{V_\alpha}\omega-\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}X^\alpha(\tilde{\omega})^{V_\alpha}X^\alpha+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\omega)^{V_\alpha}X^\alpha$$
$$-\frac{\sqrt{h_\alpha}-1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega)^{V_\alpha}X^\alpha={}^{V_\alpha}\omega$$

for any $X\in\Im_0^1(M)$ and $\omega\in\Im_1^0(M)$, which implies that ${}^{CG}F_\alpha^2=I$ for each $\alpha=1,2,...,n$.
The following theorem holds.

Theorem 4.1. *Let (M,g) be a Riemannian manifold and $F^*(M)$ be its linear coframe bundle with Cheeger-Gromoll metric ${}^{CG}g$ and the almost paracomplex structures ${}^{CG}F_\alpha,\alpha=1,2,...,n$, defined by (11). Then the triple $(F^*(M),{}^{CG}g,{}^{CG}F_\alpha)$ for each $\alpha=1,2,...,n$, is an almost paracomplex Norden manifold.*

Proof. We put

$$A(\tilde{X},\tilde{Y})={}^{CG}g({}^{CG}F_\alpha\tilde{X},\tilde{Y})-{}^{CG}g(\tilde{X},{}^{CG}F_\alpha\tilde{Y})$$

for any $\tilde{X},\tilde{Y}\in\Im_0^1(F^*(M))$. Then direct calculations using (2.5), (2.6), (3.1) and (4.1) give

$$A({}^HX,{}^HY)={}^{CG}g({}^{CG}F_\alpha{}^HX,{}^HY)-{}^{CG}g({}^HX,{}^{CG}F_\alpha{}^HY)$$
$$={}^{CG}g(\sqrt{h_\alpha}{}^{V_\alpha}\tilde{X}-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{V_\alpha}X^\alpha,{}^HY)-{}^{CG}g({}^HX,\sqrt{h_\alpha}{}^{V_\alpha}\tilde{Y}$$
$$-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{V_\alpha}X^\alpha)=\sqrt{h_\alpha}{}^{CG}g({}^{V_\alpha}\tilde{X},{}^HY)$$
$$-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{CG}g({}^{V_\alpha}X^\alpha,{}^HY)-\sqrt{h_\alpha}{}^{CG}g({}^HX,{}^{V_\alpha}\tilde{Y})$$
$$+\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{CG}g({}^HX,{}^{V_\alpha}X^\alpha)=0,$$
$$A({}^{V_\alpha}\omega,{}^HY)={}^{CG}g({}^{CG}F_\alpha{}^{V_\alpha}\omega,{}^HY)-{}^{CG}g({}^{V_\alpha}\omega,{}^{CG}F_\alpha{}^HY)$$
$$={}^{CG}g\Big(\frac{1}{\sqrt{h_\alpha}}\,({}^H\tilde{\omega}+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\omega)^H\tilde{X}^\alpha),{}^HY\Big)$$

$$-{}^{CG}g({}^{V_\alpha}\omega, \sqrt{h_\alpha}{}^{V_\alpha}\tilde{Y} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y),\omega)^{V_\alpha}X^\alpha)$$
$$= \frac{1}{\sqrt{h_\alpha}}{}^{CG}g({}^H\tilde{\omega}, {}^HY) + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega){}^{CG}g({}^H\tilde{X}^\alpha, {}^HY)$$
$$+\sqrt{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\tilde{Y}) + \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y){}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}X^\alpha)$$
$$= \frac{1}{\sqrt{h_\alpha}}g(\tilde{\omega}, Y) + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega)g(\tilde{X}^\alpha, Y)$$
$$-\sqrt{h_\alpha}\frac{1}{h_\alpha}g^{-1}(\omega,\tilde{Y}) - \sqrt{h_\alpha}\frac{1}{h_\alpha}g^{-1}(\omega, X^\alpha)g^{-1}(X^\alpha,\tilde{Y})$$
$$+\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)\left(\frac{1}{h_\alpha}\,g^{-1}(\omega,X^\alpha) + \frac{1}{h_\alpha}g^{-1}(X^\alpha,\omega)(h_\alpha-1)\right)$$
$$= \left(\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)} - \frac{1}{\sqrt{h_\alpha}} + \frac{1}{h_\alpha(\sqrt{h_\alpha}+1)}\right.$$
$$\left.+\frac{h_\alpha - 1}{h_\alpha(\sqrt{h_\alpha}+1)}\right)X^\alpha(Y)g^{-1}(X^\alpha,\omega) = 0,$$
$$A({}^HX, {}^{V_\alpha}\theta) = {}^{CG}g({}^{CG}F_\alpha{}^HX, {}^{V_\alpha}\theta) - {}^{CG}g({}^HX, {}^{CG}F_\alpha{}^{V_\alpha}\theta)$$
$$= -({}^{CG}g({}^HX, {}^{CG}F_\alpha{}^{V_\alpha}\theta) - {}^{CG}g({}^{CG}F_\alpha{}^HX, {}^{V_\alpha}\theta)) = -({}^{CG}g({}^{CG}F_\alpha{}^{V_\alpha}\theta, {}^HX)$$
$$-{}^{CG}g({}^{V_\alpha}\theta{}^{CG}F_\alpha{}^HX)) = 0,$$
$$A({}^{V_\alpha}\omega, {}^{V_\alpha}\theta) = {}^{CG}g({}^{CG}F_\alpha{}^{V_\alpha}\omega, {}^{V_\alpha}\theta) - {}^{CG}g({}^{V_\alpha}\omega, {}^{CG}F_\alpha{}^{V_\alpha}\theta)$$
$$= {}^{CG}g\left(\frac{1}{\sqrt{h_\alpha}}\,({}^H\tilde{\omega} + \frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\omega){}^H\tilde{X}^\alpha), {}^{V_\alpha}\theta\right)$$
$$-{}^{CG}g({}^{V_\alpha}\omega, \frac{1}{\sqrt{h_\alpha}}({}^H\tilde{\theta} + \frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha,\theta){}^H\tilde{X}^\alpha))$$
$$= \frac{1}{\sqrt{h_\alpha}}{}^{CG}g({}^H\tilde{\omega}, {}^{V_\alpha}\theta) + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega){}^{CG}g({}^H\tilde{X}^\alpha, {}^{V_\alpha}\theta)$$
$$-\frac{1}{\sqrt{h_\alpha}}{}^{CG}g({}^{V_\alpha}\omega, {}^H\tilde{\theta}) - \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\theta){}^{CG}g({}^{V_\alpha}\omega, {}^H\tilde{X}^\alpha) = 0.$$

So ${}^{CG}g$ is pure with respect to ${}^{CG}F_\alpha$, for each $1, 2, ..., n$ and Theorem 2 is proved. □

5. Integrability conditions

Now we shall study the integrability of ${}^{CG}F_\alpha, \alpha = 1, 2, ..., n$. As we know, the integrability of ${}^{CG}F_\alpha$ for each $\alpha = 1, 2, ..., n$, is equivalent to the vanishing of the Nijenhuis tensor. The Nijenhuis tensor of ${}^{CG}F_\alpha$ is given by

$$N_{{}^{CG}F_\alpha}(\tilde{X},\tilde{Y}) = [{}^{CG}F_\alpha\tilde{X}, {}^{CG}F_\alpha\tilde{Y}] - {}^{CG}F_\alpha[{}^{CG}F_\alpha\tilde{X},\tilde{Y}] - {}^{CG}F_\alpha[\tilde{X}, {}^{CG}F_\alpha\tilde{Y}] + [\tilde{X},\tilde{Y}],$$

where $\tilde{X},\tilde{Y} \in \Im^1_0(F^*(M))$. It is easy to check that the values $N_{{}^{CG}F_\alpha}({}^HX, {}^{V_\gamma}\theta)$ and $N_{{}^{CG}F_\alpha}({}^{V_\beta}\omega, {}^{V_\gamma}\theta)$ of the Nijenhuis tensor $N_{{}^{CG}F_\alpha}$ can be expressed in terms of the values $N_{{}^{CG}F_\alpha}({}^HX, {}^HY)$ of this tensor, where $X, Y \in \Im^1_0(M), \omega, \theta \in \Im^0_1(M)$. Indeed, by using of (2.5), (2.6) and (4.1), we obtain

$$N_{{}^{CG}F_\alpha}({}^HX, {}^{V_\gamma}\theta) = [{}^{CG}F_\alpha{}^HX, {}^{CG}F_\alpha{}^{V_\gamma}\theta] - {}^{CG}F_\alpha[{}^{CG}F_\alpha{}^HX, {}^{V_\gamma}\theta]$$
$$-{}^{CG}F_\alpha[{}^HX, {}^{CG}F_\alpha{}^{V_\gamma}\theta] + [{}^HX, {}^{V_\gamma}\theta]$$
$$= [{}^{CG}F_\alpha{}^HX, {}^{CG}F_\alpha(\delta^\gamma_\alpha{}^{CG}F_\alpha{}^HW)]$$
$$-{}^{CG}F_\alpha[{}^{CG}F_\alpha{}^HX, {}^{CG}F_\alpha(\delta^\gamma_\alpha{}^{CG}F_\alpha{}^HW)$$
$$-{}^{CG}F_\alpha[{}^HX, {}^{CG}F_\alpha(\delta^\gamma_\alpha{}^{CG}F_\alpha{}^HW)]$$
$$+[{}^HX, \delta^\gamma_\alpha{}^{CG}F_\alpha{}^HW] = \delta^\gamma_\alpha[{}^{CG}F_\alpha{}^HX, {}^HW]$$

$$-\delta_\alpha^{\gamma CG}F_\alpha[^{CG}F_\alpha{}^H X, {}^H W] - \delta_\alpha^{\gamma CG}F_\alpha[^H X, {}^H W]$$
$$+\delta_\alpha^\gamma[^H X, {}^{CG}F_\alpha{}^H W] = -\delta_\alpha^\gamma N_{^{CG}F_\alpha}(^H X, {}^H W),$$

where

$$^{V_\gamma}\theta = \delta_\alpha^{\gamma CG}F_\alpha{}^H W = \delta_\alpha^\gamma(\sqrt{h_\alpha}{}^{V_\alpha}\tilde{W} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(W)^{V_\alpha}X^\alpha)$$
$$= \delta_\alpha^{\gamma V_\alpha}(\sqrt{h_\alpha}\tilde{W} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(W)X^\alpha), W \in \Im_0^1(M).$$

Similarly, we get

$$N_{^{CG}F_\alpha}(^{V_\beta}\omega, {}^{V_\gamma}\theta) = [^{CG}F_\alpha{}^{V_\beta}\omega., {}^{CG}F_\alpha{}^{V_\gamma}\theta] - {}^{CG}F_\alpha[^{CG}F_\alpha{}^{V_\beta}\omega., {}^{V_\gamma}\theta]$$
$$-{}^{CG}F_\alpha[^{V_\beta}\omega., {}^{CG}F_\alpha{}^{V_\gamma}\theta] + [^{V_\beta}\omega, {}^{V_\gamma}\theta]$$
$$= [^{CG}F_\alpha(\delta_\alpha^{\beta CG}F_\alpha{}^H Z)., {}^{CG}F_\alpha(\delta_\alpha^{\gamma CG}F_\alpha{}^H W)]$$
$$-{}^{CG}F_\alpha[^{CG}F_\alpha(\delta_\alpha^{\beta CG}F_\alpha{}^H Z)., \delta_\alpha^{\gamma CG}F_\alpha{}^H W]$$
$$-{}^{CG}F_\alpha[\delta_\alpha^{\beta CG}F_\alpha{}^H Z., {}^{CG}F_\alpha(\delta_\alpha^{\gamma CG}F_\alpha{}^H W)]$$
$$+[\delta_\alpha^{\beta CG}F_\alpha{}^H Z, \delta_\alpha^{\gamma CG}F_\alpha{}^H W] = \delta_\alpha^\beta\delta_\alpha^\gamma[^H Z, {}^H W]$$
$$-\delta_\alpha^\beta\delta_\alpha^{\gamma CG}F_\alpha[^H Z, {}^{CG}F_\alpha{}^H W]-$$
$$-\delta_\alpha^\beta\delta_\alpha^{\gamma CG}F_\alpha[^{CG}F_\alpha{}^H Z, {}^H W]$$
$$+\delta_\alpha^\beta\delta_\alpha^\gamma[^{CG}F_\alpha{}^H Z, {}^{CG}F_\alpha{}^H W]$$
$$= \delta_\alpha^\beta\delta_\alpha^\gamma N_{^{CG}F_\alpha}(^H Z, {}^H W),$$

where $^{V_\beta}\omega = \delta_\alpha^{\beta CG}F_\alpha{}^H Z, Z \in \Im_0^1(M)$.

Therefore, we have

Lemma 5.1. *An almost paracomplex structure* $^{CG}F_\alpha$ *on* $(F^*(M), {}^{CG}g)$ *for any* $\alpha = 1, 2, ..., n$ *is integrable if and only if* $N_{^{CG}F_\alpha}(^H X, {}^H Y) = 0$ *for all* $X, Y \in \Im_0^1(M)$.

Let us consider

$$N_{^{CG}F_\alpha}(^H X, {}^H Y) = [^{Cg}F_\alpha{}^H X, {}^{CG}F_\alpha{}^H Y] - {}^{CG}F_\alpha[^{CG}F_\alpha{}^H X, {}^H Y]$$
$$-{}^{CG}F_\alpha[^H X, {}^{CG}F_\alpha{}^H Y] + [^H X, {}^H Y].$$

Before calculating $N_{^{CG}F_\alpha}(^H X, {}^H Y)$ it is necessary to prove the following.

Lemma 5.2. *Let* $^{CG}\nabla$ *be the Levi-Civita connection of the Cheeger-Gromoll metric* ^{CG}g *and* $f : R \to R$ *any smooth function. Then*

$$^H X(f(r_\alpha^2)) = 0, \tag{5.1}$$
$$^{V_\beta}\omega(f(r_\alpha^2)) = 2\delta_\alpha^\beta f'(r_\alpha^2)g^{-1}(\omega, X^\alpha), \tag{5.2}$$
$$^H X(g^{-1}(X^\alpha, \theta)) = g^{-1}(X^\alpha, \nabla_X\theta), \tag{5.3}$$
$$^{V_\alpha}\omega(g^{-1}(\theta, X^\beta)) = \delta_\beta^\alpha g^{-1}(\omega, \theta), \tag{5.4}$$
$$^{CG}\nabla_{^H X}{}^{V_\alpha}X^\alpha = \frac{1}{2h_\alpha}{}^H(X^\alpha(g^{-1} \circ R(\quad, X)\tilde{X}^\alpha)), \tag{5.5}$$
$$^{CG}\nabla_{^{V_\alpha}\omega}{}^{V_\alpha}X^\alpha = \frac{1}{h_\alpha}{}^{V_\alpha}\omega + \frac{1}{h_\alpha}g^{-1}(\omega, X^\alpha)\gamma\delta \tag{5.6}$$

for all $X \in \Im_0^1(M)$ *and* $\omega, \theta \in \Im_0^1(M)$, *where* $r_\alpha^2 = g^{-1}(X^\alpha, X^\alpha) = h_\alpha - 1$.

Proof. (i) Direct calculations using (2.1) give

$$\begin{aligned}
{}^{H}X(f(r_{\alpha}^{2})) &= {}^{H}X(f(g^{-1}(X^{\alpha},X^{\alpha}))) = (X^{i}D_{i})(f(g^{-1}(X^{\alpha},X^{\alpha})))\\
&= X^{i}(\partial_{i}+X_{r}^{\sigma}\Gamma_{ip}^{r}\partial_{p_{\sigma}})(f(g^{-1}(X^{\alpha},X^{\alpha})))\\
&= X^{i}\partial_{i}(f(g^{-1}(X^{\alpha},X^{\alpha})))\\
&+X^{i}X_{r}^{\sigma}\Gamma_{ip}^{r}\partial_{p_{\sigma}}(f(g^{-1}(X^{\alpha},X^{\alpha})))\\
&= X^{i}f'(r_{\alpha}^{2})(\partial_{i}g^{ms})(X_{m}^{\alpha}X_{s}^{\alpha})\\
&+f'(r_{\alpha}^{2})X^{i}X_{r}^{\sigma}\Gamma_{ip}^{r}\partial_{p\sigma}(g^{ms}X_{m}^{\alpha}X_{s}^{\alpha})\\
&= X^{i}f'(r_{\alpha}^{2})(-\Gamma_{il}^{m}g^{ls}-\Gamma_{il}^{s}g^{ml})X_{m}^{\alpha}X_{s}^{\alpha}\\
&+f'(r_{\alpha}^{2})X^{i}X_{r}^{\sigma}\Gamma_{ip}^{r}g^{ms}(\delta_{\sigma}^{\alpha}\delta_{m}^{p}X_{s}^{\alpha}+\delta_{\sigma}^{\alpha}\delta_{s}^{p}X_{m}^{\alpha})\\
&= f'(r_{\alpha}^{2})X^{i}X_{r}^{\sigma}X_{s}^{\alpha}(-\Gamma_{il}^{r}g^{ls}-\Gamma_{il}^{s}g^{rl})\\
&+f'(r_{\alpha}^{2})X^{i}X_{m}^{\sigma}X_{s}^{\alpha}(\Gamma_{il}^{m}g^{ls}-\Gamma_{il}^{s}g^{ml})=0.
\end{aligned}$$

(ii) Calculations like above using (2.2) give

$$\begin{aligned}
{}^{V_{\beta}}\omega(f(r_{\alpha}^{2})) &= \omega_{H}\delta_{\sigma}^{\beta}f'(r_{\alpha}^{2})\partial_{i_{\sigma}}(g^{rs}X_{r}^{\alpha}X_{s}^{\alpha})\\
&= \omega_{H}\delta_{\sigma}^{\beta}f'(r_{\alpha}^{2})g^{rs}(\delta_{\alpha}^{\sigma}\delta_{r}^{i}X_{s}^{\alpha}+\delta_{\alpha}^{\sigma}\delta_{s}^{i}X_{r}^{\alpha})X\\
&= 2\omega_{i}\delta_{\alpha}^{\beta}f'(r_{\alpha}^{2})g^{is}X_{s}^{\alpha}=2\delta_{\alpha}^{\beta}f'(r_{\alpha}^{2})g^{-1}(\omega,X^{\alpha}).
\end{aligned}$$

(iii) Using (2.5) we obtain

$$\begin{aligned}
{}^{H}X(g^{-1}(X^{\alpha},\theta)) &= (X^{i}D_{i})(g^{-1}(X^{\alpha},\theta))=X^{i}(\partial_{i}\\
&+X_{r}^{\sigma}\Gamma_{ip}^{r}\partial_{p_{\sigma}})(g^{-1}(X^{\alpha},\theta))=X^{i}\partial_{i}(g^{rs}X_{r}^{\alpha}\theta_{s})\\
&+X^{i}X_{r}^{\sigma}\Gamma_{ip}^{r}\partial_{p_{\sigma}}(g^{ms}X_{m}^{\alpha}\theta_{s})=X^{i}(\partial_{i}g^{rs})X_{r}^{\alpha}\theta_{s}\\
&+X^{i}g^{rs}X_{r}^{\alpha}\partial_{i}\theta_{s}+X^{i}X_{r}^{\sigma}\Gamma_{ip}^{r}g^{ms}\delta_{\sigma}^{\alpha}\delta_{m}^{p}\theta_{s}\\
&= X^{i}(-\Gamma_{il}^{r}g^{ls}-\Gamma_{il}^{s}g^{rl})X_{r}^{\alpha}\theta_{s}+X^{i}g^{rs}X_{r}^{\alpha}\partial_{i}\theta_{s}\\
&+X^{i}X_{r}^{\alpha}\Gamma_{im}^{r}g^{ms}\theta_{s}=-X^{i}\Gamma_{il}^{r}g^{ls}X_{r}^{\alpha}\theta_{s}\\
&-X^{i}\Gamma_{il}^{s}g^{rl}X_{r}^{\alpha}\theta_{s}+X^{i}g^{rs}X_{r}^{\alpha}\partial_{i}\theta_{s}+X^{i}\Gamma_{im}^{r}g^{ms}\theta_{s}X_{r}^{\alpha}\\
&= X^{i}g^{rs}X_{r}^{\alpha}\partial_{i}\theta_{s}--X^{i}\Gamma_{il}^{s}g^{rl}X_{r}^{\alpha}\theta_{s}=X_{r}^{\alpha}X^{i}(\partial_{i}\theta_{s}\\
&-\Gamma_{is}^{l}\theta_{l})g^{rs}=X_{r}^{\alpha}(\nabla_{X}\theta)_{s}g^{rs}=g^{-1}(X^{\alpha},\nabla_{X}\theta).
\end{aligned}$$

(iv) Direct calculations using (2.6) give

$$\begin{aligned}
{}^{V_{\alpha}}\omega(g^{-1}(\theta,X^{\beta})) &= \omega_{i}\delta_{\sigma}^{\alpha}\partial_{i_{\sigma}}(g^{rs}\theta_{r}X_{s}^{\beta})=\omega_{i}\delta_{\sigma}^{\alpha}g^{rs}\theta_{r}\delta_{\beta}^{\sigma}\delta_{s}^{i}\\
&= \omega_{s}\delta_{\beta}^{\alpha}g^{rs}\theta_{r}=\delta_{\beta}^{\alpha}g^{-1}(\omega,\theta).
\end{aligned}$$

(v) By using of (3.2) we get

$$\begin{aligned}
{}^{CG}\nabla_{{}^{H}X}{}^{V_{\alpha}}X^{\alpha} &= {}^{CG}\nabla_{X^{i}D_{i}}(\delta_{\alpha}^{\beta}X_{j}^{\alpha}\partial_{j_{\beta}})=X^{i}\delta_{\beta}^{\alpha}D_{i}(X_{j}^{\alpha})\partial_{j_{\beta}}\\
&+X^{i}\delta_{\beta}^{\alpha}X_{j}^{\alpha CG}\nabla_{D_{i}}D_{j_{\beta}}=X^{i}\delta_{\beta}^{\alpha}(\partial_{i}\\
&+X_{r}^{\sigma}\Gamma_{ip}^{r}\partial_{p_{\sigma}})(X_{j}^{\alpha})\partial_{j_{\beta}}+X^{i}\delta_{\beta}^{\alpha CG}X_{j}^{\alpha}\Gamma_{ij_{\beta}}^{K}D_{K}\\
&= X^{i}\delta_{\beta}^{\alpha}X_{r}^{\sigma}\Gamma_{ip}^{r}\delta_{\sigma}^{\alpha}\delta_{j}^{p}\partial_{j_{\beta}}+X^{i}\delta_{\beta}^{\alpha}X_{j}^{\alpha CG}\Gamma_{ij_{\beta}}^{k}D_{k}\\
&+X^{i}\delta_{\beta}^{\alpha}X_{j}^{\alpha CG}\Gamma_{ij_{\beta}}^{k_{\gamma}}D_{k_{\gamma}}=\delta_{\beta}^{\alpha}X^{i}\Gamma_{ij}^{r}X_{r}^{\alpha}\partial_{j_{\beta}}\\
&+\frac{1}{2h_{\beta}}\delta_{\beta}^{\alpha}X^{i}X_{j}^{\alpha}X_{m}^{\alpha}R_{\cdot\, i\,\cdot}^{k\ \ jm}D_{k}\\
&-X^{i}\delta_{\beta}^{\alpha}\delta_{\beta}^{\gamma}X_{j}^{\alpha}\Gamma_{ik}^{j}X_{r}^{\alpha}D_{k_{\gamma}}\\
&= \frac{1}{2h_{\alpha}}{}^{H}(X^{\alpha}(g^{-1}\circ R(\quad,X)\tilde{X}^{\alpha})).
\end{aligned}$$

(vi) Direct calculations using (3.2) and (3.3) give

$$
\begin{aligned}
{}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}X^\alpha &= {}^{CG}\nabla_{\delta^\alpha_\beta \omega_i D_{i_\beta}}(\delta^\alpha_\sigma X^\alpha_j D_{j_\sigma})
\end{aligned}
$$

‘

$$
\begin{aligned}
&= \delta^\alpha_\beta \omega_i {}^{CG}\nabla_{D_{i_\beta}}(\delta^\alpha_\sigma X^\alpha_j D_{j_\sigma}) \\
&= \delta^\alpha_\beta \omega_i \delta^\alpha_\sigma \partial_{i_\beta}(X^\alpha_j) D_{j_\sigma} + \delta^\alpha_\beta \omega_i \delta^\alpha_\sigma X^{\alpha CG}_j \nabla_{D_{i_\beta}} D_{j_\sigma} \\
&= \delta^\alpha_\beta \omega_i \delta^\alpha_\sigma \delta^\alpha_\beta \delta^i_j D_{j_\sigma} + \delta^\alpha_\beta \omega_i \delta^\alpha_\sigma X^{\alpha CG}_j \Gamma^k_{i_\beta j_\sigma} D_k \\
&+ \delta^\alpha_\beta \omega_i \delta^\alpha_\sigma X^{\alpha CG}_j \Gamma^{k_\gamma}_{i_\beta j_\sigma} D_{k_\gamma} = \delta^\alpha_\beta \omega_i D_{j_\beta} \\
&+ \omega_i X^{\alpha CG}_j \Gamma^{k_\gamma}_{i_\alpha j_\alpha} D_{k_\gamma} = {}^{V_\alpha}\omega \\
&+ \omega_i X^\alpha_j (-\frac{1}{h_\alpha}(\tilde{X}^{\alpha i}\delta^\alpha_\gamma \delta^j_k + \tilde{X}^{\alpha j}\delta^\alpha_\gamma \delta^i_k) \\
&+ \frac{1+h_\alpha}{h_\alpha^2} g^{ij} X^\gamma_k + \frac{1}{h_\alpha^2}\tilde{X}^{\alpha i}\tilde{X}^{\alpha j} X^\gamma_k) D_{k_\gamma} \\
&= {}^{V_\alpha}\omega - \frac{1}{h_\alpha}\delta^\alpha_\gamma X^\alpha_k g^{-1}(X^\alpha, \omega) D_{k_\gamma} \\
&- \frac{1}{h_\alpha}\delta^\alpha_\gamma \omega_k g^{-1}(X^\alpha, X^\alpha) D_{k_\gamma} + \frac{1+h_\alpha}{h_\alpha^2} g^{-1}(\omega, X^\alpha) X^\gamma_k D_{k_\gamma} \\
&+ \frac{1}{h_\alpha^2} g^{-1}(\omega, X^\alpha) g^{-1}(X^\alpha, X^\alpha) X^\gamma_k D_{k_\gamma} = {}^{V_\alpha}\omega \\
&- \frac{1}{h_\alpha} g^{-1}(X^\alpha, \omega)^{V_\alpha}X^\alpha - \frac{h_\alpha - 1}{h_\alpha}{}^{V_\alpha}\omega \\
&+ \frac{h_\alpha - 1}{h_\alpha} g^{-1}(\omega, X^\alpha)^{V_\gamma}X^\gamma \\
&+ \frac{1}{h_\alpha^2} g^{-1}(\omega, X^\alpha)(h_\alpha - 1)^{V_\gamma}X^\gamma \\
&= \frac{1}{h_\alpha} g^{-1}(\omega, X^\alpha)\gamma\delta + \frac{1}{h_\alpha}{}^{V_\alpha}\omega + \frac{h_\alpha - 1}{h_\alpha} g^{-1}(\omega, X^\alpha)\gamma\delta \\
&+ \frac{1}{h_\alpha^2} g^{-1}(\omega, X^\alpha)(h_\alpha - 1)\gamma\delta = \frac{1}{h_\alpha}{}^{V_\alpha}\omega + \frac{1}{h_\alpha} g^{-1}(\omega, X^\alpha)\gamma\delta.
\end{aligned}
$$

This completes the proof of the lemma.

Direct calculations using (2.7), (3.1) and (4.1) give

$$
[{}^H X, {}^H Y] = {}^H[X,Y] + \sum_{\sigma=1}^{n} {}^\sigma(X^\sigma \circ R(X,Y)),
$$

$$
\begin{aligned}
{}^{CG}F_\alpha[{}^{CG}F_\alpha{}^H X, {}^H Y] &= {}^{CG}F_\alpha[\sqrt{h_\alpha}{}^{V_\alpha}\tilde{X} - \frac{1}{\sqrt{h_\alpha}+1} X^\alpha(X)^{V_\alpha}X^\alpha, {}^H Y] \\
&= {}^{CG}F_\alpha(\sqrt{h_\alpha}[{}^{V_\alpha}\tilde{X}, {}^H Y] - \frac{1}{\sqrt{h_\alpha}+1} g(\tilde{X}^\alpha, X)[{}^{V_\alpha}X^\alpha, {}^H Y] \\
&+ \frac{1}{\sqrt{h_\alpha}+1}{}^H Y(g(\tilde{X}^\alpha, X))^{V_\alpha}X^\alpha = {}^{CG}F_\alpha(-\sqrt{h_\alpha}{}^{V_\alpha}(\nabla_Y \tilde{X}) \\
&+ \frac{1}{\sqrt{h_\alpha}+1} g(\tilde{X}^\alpha, X)^{V_\alpha}(\nabla_Y X^\alpha) + \frac{1}{\sqrt{h_\alpha}+1}{}^H Y(g(\tilde{X}^\alpha, X))^{V_\alpha}X^\alpha) \\
&= {}^{CG}F_\alpha(-\sqrt{h_\alpha}{}^{V_\alpha}(\nabla_Y \tilde{X}) + \frac{1}{\sqrt{h_\alpha}+1} g^{-1}(\nabla_Y \tilde{X}, X^\alpha)^{V_\alpha}X^\alpha) \\
&= {}^{CG}F_\alpha(-{}^{CG}F_\alpha{}^H(\nabla_Y X)) = -{}^{CG}F^{2H}_\alpha(\nabla_Y X) = -{}^H(\nabla_Y X), \\
{}^{CG}F_\alpha[{}^H X, {}^{CG}F_\alpha{}^H Y] &= -{}^{CG}F_\alpha[{}^{CG}F_\alpha{}^H Y, {}^H X] = {}^H(\nabla_X Y)
\end{aligned}
$$

$$[{}^{CG}F_\alpha{}^HX, {}^{CG}F_\alpha{}^HY] = [\sqrt{h_\alpha}^{V_\alpha}\tilde{X} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{V_\alpha}X^\alpha, \sqrt{h_\alpha}^{V_\alpha}\tilde{Y}$$

$$-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{V_\alpha}X^\alpha] = [\sqrt{h_\alpha}^{V_\alpha}\tilde{X}, \sqrt{h_\alpha}^{V_\alpha}\tilde{Y}]$$

$$+[\sqrt{h_\alpha}^{V_\alpha}\tilde{X}, -\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{V_\alpha}X^\alpha] + [-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{V_\alpha}X^\alpha, \sqrt{h_\alpha}^{V_\alpha}\tilde{Y}]$$

$$+[-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X)^{V_\alpha}X^\alpha, -\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{V_\alpha}X^\alpha]$$

$$= \sqrt{h_\alpha}^{V_\alpha}\tilde{X}(\sqrt{h_\alpha})^{V_\alpha}\tilde{Y} - \sqrt{h_\alpha}^{V_\alpha}\tilde{Y}(\sqrt{h_\alpha})^{V_\alpha}\tilde{X}$$

$$-[-\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{Y})^{V_\alpha}X^\alpha, \sqrt{h_\alpha}^{V_\alpha}\tilde{X}]$$

$$+[-\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{X})^{V_\alpha}X^\alpha, \sqrt{h_\alpha}^{V_\alpha}\tilde{Y}]$$

$$= \sqrt{h_\alpha}\cdot\frac{1}{2\sqrt{h_\alpha}}\cdot 2g^{-1}(X^\alpha, \tilde{X})^{V_\alpha}\tilde{Y} - \sqrt{h_\alpha}\cdot\frac{1}{2\sqrt{h_\alpha}}\cdot 2g^{-1}(X^\alpha, \tilde{Y})^{V_\alpha}\tilde{X}$$

$$+\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{Y})^{V_\alpha}X^\alpha(\sqrt{h_\alpha})^{V_\alpha}\tilde{X}$$

$$+\frac{\sqrt{h_\alpha}}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{Y})[{}^{V_\alpha}X^\alpha, {}^{V_\alpha}\tilde{X}]$$

$$-\frac{1}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{X})^{V_\alpha}X^\alpha(\sqrt{h_\alpha})^{V_\alpha}\tilde{Y}$$

$$-\frac{\sqrt{h_\alpha}}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{X})[{}^{V_\alpha}X^\alpha, {}^{V_\alpha}\tilde{Y}] = g^{-1}(X^\alpha, \tilde{X})^{V_\alpha}\tilde{Y}$$

$$-g^{-1}(X^\alpha, \tilde{Y})^{V_\alpha}\tilde{X} + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha, \tilde{Y})g^{-1}(X^\alpha, X^\alpha)^{V_\alpha}\tilde{X}$$

$$-\frac{\sqrt{h_\alpha}}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{Y})^{V_\alpha}\tilde{X} - \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha, \tilde{X})g^{-1}(X^\alpha, X^\alpha)^{V_\alpha}\tilde{Y}$$

$$+\frac{\sqrt{h_\alpha}}{\sqrt{h_\alpha}+1}g^{-1}(X^\alpha, \tilde{X})^{V_\alpha}\tilde{Y} = \Big(g^{-1}\ (X^\alpha, \tilde{X})^{V_\alpha}\tilde{Y}$$

$$-g^{-1}(X^\alpha, \tilde{Y})^{V_\alpha}\tilde{X}\Big)\left(1 - \frac{r_\alpha^2}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+`1)} + \frac{\sqrt{h_\alpha}}{\sqrt{h_\alpha}+1}\right).$$

Therefore, we have

$$N_{{}^{CG}F_\alpha}({}^HX, {}^HY) = {}^{V_\alpha}\Big(g^{-1}\ (X^\alpha, \tilde{X})\tilde{Y} - g^{-1}(X^\alpha, \tilde{Y})\tilde{X}\Big)(1 - \frac{r_\alpha^2}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+`1)}$$

$$+\frac{\sqrt{h_\alpha}}{\sqrt{h_\alpha}+1}\Bigg) + {}^H(\nabla_Y X - \nabla_X Y) + {}^H[X, Y]$$

$$+\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(X, Y)) = \sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(X, Y))$$

$$+\frac{1+\sqrt{h_\alpha}+h_\alpha}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}{}^{V_\alpha}\Big(g^{-1}\ (X^\alpha, \tilde{X})\tilde{Y} - g^{-1}(X^\alpha, \tilde{Y})\tilde{X}\Big).$$

Thus the following theorem holds. □

Theorem 5.3. *An almost paracomplex structure* ${}^{CG}F_\alpha$ *on* $(F^*(M), {}^{CG}g)$ *for each* $\alpha = 1, 2, ..., n$ *is integrable if and only if*

$$\gamma R(X,Y) = \sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X,Y))$$

$$= \frac{1+\sqrt{h_\alpha}+h_\alpha}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)} {}^{V_\alpha}\left(g^{-1}\,(X^\alpha,\tilde{Y})\tilde{X} - g^{-1}(X^\alpha,\tilde{X})\tilde{Y}\right)$$

for all $X, Y \in \Im^1_0(M)$.

6. Non-existence of Kahler type structures

Let (M_{2k}, φ, g) be an almost paracomplex Norden manifold. If $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g, then we say that (M_{2k}, φ, g) is a para-Kahler-Norden manifold [16].

We now calculate the covariant derivative of para-Norden structures ${}^{CG}F_\alpha, \alpha = 1, 2, ..., n$. Direct calculations using (3.2) and (4.1)-(5.6) give

i)

$$({}^{CG}\nabla_{{}^H X}{}^{CG}F_\alpha)({}^H Y) = {}^{CG}\nabla_{{}^H X}({}^{CG}F_\alpha {}^H Y) - {}^{CG}F_\alpha({}^{CG}\nabla_{{}^H X}{}^H Y)$$

$$= {}^{CG}\nabla_{{}^H X}(\sqrt{h_\alpha}{}^{V_\alpha}\tilde{Y} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y){}^{V_\alpha}X^\alpha)$$

$$-{}^{CG}F_\alpha({}^H(\nabla_X Y) + \frac{1}{2}\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(X,Y))$$

$$= {}^H X(\sqrt{h_\varepsilon}){}^{V_\alpha}\tilde{Y} + \sqrt{h_\alpha}{}^{CG}\nabla_{{}^H X}{}^{V_\alpha}\tilde{Y}$$

$$-{}^H X(\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)){}^{V_\alpha}X^\alpha - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y){}^{CG}\nabla_{{}^H X}{}^{V_\alpha}X^\alpha$$

$$-{}^{CG}F_\alpha({}^H(\nabla_X Y)) - \frac{1}{2}\sum_{\sigma=1}^{n}{}^{CG}F_\alpha{}^{V_\sigma}(X^\sigma \circ R(X,Y))$$

$$= \frac{1}{2\sqrt{h_\alpha}}{}^H(X^\alpha(g^{-1}\circ[R(\quad,X),Y) - R(X,Y)])$$

$$-\frac{1}{2\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}[{}^H(X^\alpha(g^{-1}\circ R(\quad,X)\tilde{X}^\alpha)$$

$$+g^{-1}(X^\alpha, X^\alpha \circ R(X,Y))\tilde{X}^\alpha];$$

ii)

$$({}^{CG}\nabla_{{}^H X}{}^{CG}F_\alpha)({}^{V_\alpha}\omega) = {}^{CG}\nabla_{{}^H X}({}^{CG}F_\alpha{}^{V_\alpha}\omega) - {}^{CG}F_\alpha({}^{CG}\nabla_{{}^H X}{}^{V_\alpha}\omega)$$

$$= {}^{CG}\nabla_{{}^H X}(-\frac{1}{\sqrt{h_\alpha}}{}^H\tilde{\omega} - \frac{1}{\sqrt{h_\partial}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega){}^H\tilde{X}^\alpha)$$

$$-{}^{CG}F_\alpha({}^{V_\alpha}(\nabla_X\omega) + \frac{1}{2h_\alpha}(X^\alpha(g^{-1}\circ R(\quad,X)\tilde{\omega}))$$

$$= {}^H X(-\frac{1}{\sqrt{h_\alpha}}){}^H\tilde{\omega} - \frac{1}{\sqrt{h_\alpha}}{}^{CG}\nabla_{{}^H X}{}^H\tilde{\omega}$$

$$+{}^H X(-\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)})g^{-1}(X^\alpha,\omega){}^H\tilde{X}^\alpha$$

$$-\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}{}^H X(g^{-1}(X^\alpha,\omega)){}^H\tilde{X}^\alpha$$

$$-\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\omega){}^{CG}\nabla_{{}^H X}{}^H\tilde{X}^\alpha$$

$$+\frac{1}{\sqrt{h_\alpha}}{}^H(g^{-1}(\nabla_X\omega)) + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha, \nabla_X\omega)^H\tilde{X}^\alpha$$

$$-\frac{1}{2h_\alpha}\sqrt{h_\alpha}{}^{V_\alpha}(g(X^\alpha(g^{-1}\circ R(\quad, X)\tilde{\omega})))$$

$$+\frac{1}{2h_\alpha(\sqrt{h_\alpha}+1)}X^\alpha(X^\alpha(g^{-1}\circ R(\quad, X)\tilde{\omega}))^{V_\alpha}X^\alpha$$

$$= -\frac{1}{2\sqrt{h_\alpha}}(\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ R(X,\tilde{\omega})) + {}^{V_\alpha}(g(R(\tilde{X}^\alpha, X)\tilde{\omega}))$$

$$-\frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}(g^{-1}(X^\alpha,\omega)\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ R(X,\tilde{X}^\alpha)$$

$$+\frac{1}{\sqrt{h_\alpha}}g^{-1}(X^\alpha, R(\tilde{X}^\alpha, X)\tilde{\omega})^{V_\alpha}X^\alpha;$$

iii) $({}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{CG}F_\alpha)({}^HY) = {}^{CG}\nabla_{{}^{V_\alpha}\omega}({}^{CG}F_\alpha{}^HY) - {}^{CG}F_\alpha({}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^HY)$

$$= {}^{CG}\nabla_{{}^{V_\alpha}\omega}(\sqrt{h_\alpha}{}^{V_\alpha}\tilde{Y} - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{V_\alpha}X^\alpha)$$

$$-{}^{CG}F_\alpha(\frac{1}{2h_\alpha}{}^H(X^\alpha(g^{-1}\circ R(\quad, Y)\tilde{\omega})))$$

$$= {}^{V_\alpha}\omega(\sqrt{h_\alpha})^{V_\alpha}\tilde{Y} + \sqrt{h_\alpha}{}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\tilde{Y} - {}^{V_\alpha}\omega(\frac{1}{\sqrt{h_\alpha}+1})X^\alpha(Y)^{V_\alpha}X^\alpha$$

$$-{}^{V_\alpha}\omega(X^\alpha(Y))\frac{1}{\sqrt{h_\alpha}+1}{}^{V_\alpha}X^\alpha - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(Y)^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}X^\alpha$$

$$-\frac{1}{2h_\alpha}(\sqrt{h_\alpha}{}^{V_\alpha}(g(X^\alpha(g^{-1}\circ R(\quad, Y)\tilde{\omega})))$$

$$-\frac{1}{\sqrt{h_\alpha}+1}X^\alpha(X^\alpha(g^{-1}\circ R(\quad, Y)\tilde{\omega}))$$

$$= \left(-\frac{1}{\sqrt{h_\alpha}} - \frac{1}{2h_\alpha^2(\sqrt{h_\alpha}+1)}\right)g^{-1}(\tilde{Y}, X^\alpha)^{V_\alpha}\omega$$

$$+\left(\frac{1+\sqrt{h_\alpha}}{h_\alpha\sqrt{h_\alpha}} - \frac{1}{\sqrt{h_\alpha}+1}\right)g^{-1}(\omega,\tilde{Y})^{V_\alpha}X^\alpha$$

$$+\left(\frac{1+\sqrt{h_\alpha}}{h_\alpha\sqrt{h_\alpha}} - \frac{1}{\sqrt{h_\alpha}} + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}\right)g^{-1}(\tilde{Y}, X^\alpha)g^{-1}(\omega, X^\alpha)^{V_\alpha}X^\alpha$$

$$-\frac{1}{2\sqrt{h_\alpha}}{}^{V_\alpha}(g(R(\tilde{X}^\alpha, Y)\omega)) - \frac{1}{\sqrt{h_\alpha}+1}X^\alpha(R(\tilde{X}^\alpha, Y)\tilde{\omega});$$

iv)

$$({}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{CG}F_\alpha)({}^{V_\alpha}\theta) = {}^{CG}\nabla_{{}^{V_\alpha}\omega}({}^{CG}F_\alpha{}^{V_\alpha}\theta) - {}^{CG}F_\alpha({}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\theta)$$

$$= {}^{CG}\nabla_{{}^{V_\alpha}\omega}(\frac{1}{\sqrt{h_\alpha}}{}^H\tilde{\theta} + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(X^\alpha,\theta)^H\tilde{X}^\alpha)$$

$$-{}^{CG}F_\alpha(-\frac{1}{h_\alpha}({}^{CG}g({}^{V_\alpha}\omega,\gamma\delta)^{V_\alpha}\theta + {}^{CG}g({}^{V_\alpha}\theta,\gamma\delta)^{V_\alpha}\omega)$$

$$+\frac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)\gamma\delta - \frac{1}{h_\alpha}{}^{CG}g({}^{V_\alpha}\theta,\gamma\delta)^{CG}g({}^{V_\alpha}\omega,\gamma\delta)\gamma\delta)$$

$$= \frac{1}{h_\alpha\sqrt{h_\alpha}}g^{-1}(\theta, X^\alpha)^H\tilde{\omega} + \frac{1}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(\omega,\theta)^H\tilde{X}^\alpha$$

$$+\frac{h_\alpha - \sqrt{h_\alpha}+1}{h_\alpha\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)}g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha)^H\tilde{X}^\alpha$$

$$-\frac{1+h_\alpha}{h_\alpha^2}(g^{-1}(\omega,\theta)+g^{-1}(\omega,X^\alpha)g^{-1}(\theta,X^\alpha))^{V_\alpha}X^\alpha$$

$$+\frac{1}{2h_\alpha\sqrt{h_\alpha}}{}^H(R(\tilde{X}^\alpha,\tilde{\theta})\tilde{\omega}).$$

From *iii*) and *iv*) it follows that ${}^{CG}\nabla^{CG}F_\alpha \neq 0$ even for the locally flat manifold M. Thus we have

Theorem 6.1. *Let (M,g) be a Riemannian manifold and let $F^*(M)$ be its coframe bundle equipped with the Cheeger-Gromoll metric ${}^{CG}g$ and the paracomplex structures ${}^{CG}F_\alpha,\ \alpha = 1,2,...,n$ defined by (11). Then the triple $(F^*(M), {}^{CG}F_\alpha, {}^{CG}g)$ for each $\alpha = 1,2,...,n$ is never a para-Kahler-Norden manifold.*

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