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On Predictors and Estimators under a Constrained Partitioned Linear Model and its Reduced Models

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Abstract

In this study, we consider a partitioned linear model with linear partial parameter constrains, known as a constrained partitioned linear model (CPLM), and its reduced models. A group of formulas on best linear unbiased predictors (BLUPs) and best linear unbiased estimators (BLUEs) in CPLM is derived via some quadratic matrix optimization methods, and further many basic properties of the predictors and estimators are established under some general assumptions. Our main purpose is to derive various inequalities and equalities for the comparison of covariance matrices of BLUPs and BLUEs under CPLM and its reduced models.

1. Introduction and preliminary results

We first introduce the following notations. $\mathbf{A}'$, $r(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$, and $\mathbf{A}^{+}$ denote, respectively, the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$, where $\mathbb{R}^{m\times n}$ stands for the set of all $m\times n$ real matrices. $\mathbf{E}_{\mathbf{A}} = \mathbf{A}^{\perp} = \mathbf{I}_m - \mathbf{A}\mathbf{A}^{+}$ stands for the orthogonal projector, where $\mathbf{I}_m$ denotes the identity matrix of size $m\times m$. $i_{+}(\mathbf{A})$ and $i_{-}(\mathbf{A})$ denote the positive and the negative inertias of symmetric matrix $\mathbf{A}$, respectively, and for both $i_{\pm}(\mathbf{A})$ and $i_{\mp}(\mathbf{A})$ are used. The inequality $\mathbf{A}_1 - \mathbf{A}_2 \succcurlyeq \mathbf{0}$ or $\mathbf{A}_1 \succcurlyeq \mathbf{A}_2$ means that the difference $\mathbf{A}_1 - \mathbf{A}_2$ is positive semi-definite (psd) matrix in the Löwner partial ordering (LPO) for the symmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of same size, further, we use $\mathbf{A}_1 \prec \mathbf{A}_2$, $\mathbf{A}_1 \preccurlyeq \mathbf{A}_2$, and $\mathbf{A}_1 \succ \mathbf{A}_2$ in cases where the difference $\mathbf{A}_1 - \mathbf{A}_2$ is negative definite, negative semi-definite, and positive definite matrix, respectively.

As a linear model with its partitioned form, we consider

$$\mathscr{M} : \mathbf{y} = \mathbf{X}\alpha + \varepsilon = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix}\begin{bmatrix}\alpha_1', & \alpha_2'\end{bmatrix}' + \varepsilon = \mathbf{X}_1\alpha_1 + \mathbf{X}_2\alpha_2 + \varepsilon, \tag{1.1}$$

$$\mathrm{E}(\varepsilon) = \mathbf{0} \quad \text{and } \operatorname{cov}(\varepsilon,\varepsilon) = \mathrm{D}(\varepsilon) = \sigma^2\Sigma, \tag{1.2}$$

and its reduced model

$$\mathscr{M}_R : \mathbf{X}_2^{\perp}\mathbf{y} = \mathbf{X}_2^{\perp}\mathbf{X}_1\alpha_1 + \mathbf{X}_2^{\perp}\varepsilon, \tag{1.3}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix} \in \mathbb{R}^{n\times k}$ is a known matrix of arbitrary rank with $\mathbf{X}_i \in \mathbb{R}^{n\times k_i}$, $\alpha = \begin{bmatrix}\alpha_1', & \alpha_2'\end{bmatrix}' \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters with $\alpha_i \in \mathbb{R}^{k_i\times 1}$, $\varepsilon \in \mathbb{R}^{n\times 1}$ is an unobservable vector of random errors, σ^2 is a positive unknown parameter, and $\Sigma \in \mathbb{R}^{n\times n}$ is a known psd matrix of arbitrary rank, $i = 1,2$,

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$k_1+k_2=k$. The reduced linear model $\mathscr{M}_R$ in (1.3), also known as the correctly-reduced model, is obtained by pre-multiplying $\mathbf{X}_2^{\perp}$ on both sides of the partitioned linear model $\mathscr{M}$ in (1.1); see, e.g., [1] and [2]. The model in (1.3) is one of the different forms of the model in (1.1) and, especially, this model can be considered when estimation/prediction problems in general parametric functions of partial parameters are considered.

In statistical theory and its applications, there often exist certain restrictions on unknown parameters in linear regression models. These kinds of restrictions occur in many situations such as the linear hypothesis testing on parameters. Let us considered the partitioned linear model in (1.1) with a certain restriction on α_1, known as constrained partitioned linear model (CPLM), as follows:

$$\mathscr{N}: \mathbf{y}=\mathbf{X}\alpha+\varepsilon=\mathbf{X}_1\alpha_1+\mathbf{X}_2\alpha_2+\varepsilon, \quad \mathbf{A}_1\alpha_1=\mathbf{b}_1, \tag{1.4}$$

and its constrained reduced linear model (CRLM),

$$\mathscr{N}_R: \mathbf{X}_2^{\perp}\mathbf{y}=\mathbf{X}_2^{\perp}\mathbf{X}_1\alpha_1+\mathbf{X}_2^{\perp}\varepsilon, \quad \mathbf{A}_1\alpha_1=\mathbf{b}_1, \tag{1.5}$$

where the linear restriction equation $\mathbf{A}_1\alpha_1=\mathbf{b}_1$ is consistent for given $\mathbf{A}_1\in\mathbb{R}^{m\times k_1}$ of arbitrary rank and $\mathbf{b}_1\in\mathbb{R}^{m\times 1}$. The two given equation parts in (1.4) and (1.5) can merge into the following combined form of vectors

$$\widehat{\mathscr{N}}: \widehat{\mathbf{y}}=\widehat{\mathbf{X}}\alpha+\widehat{\varepsilon}=\widehat{\mathbf{X}}_1\alpha_1+\widehat{\mathbf{X}}_2\alpha_2+\widehat{\varepsilon}, \tag{1.6}$$

$$\widehat{\mathscr{N}_R}: \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}=\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\alpha_1+\widehat{\mathbf{X}}_2^{\perp}\widehat{\varepsilon}, \tag{1.7}$$

respectively, and according to the expectation and covariance matrix assumptions in (1.2),

$$\mathrm{E}(\widehat{\mathbf{y}})=\widehat{\mathbf{X}}\alpha, \quad \mathrm{E}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}})=\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\alpha_1, \quad \mathrm{D}(\widehat{\mathbf{y}})=\mathrm{D}(\widehat{\varepsilon})=\sigma^2\begin{bmatrix}\Sigma & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}:=\widehat{\Sigma}, \quad \mathrm{D}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}})=\mathrm{D}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\varepsilon})=\sigma^2\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} \tag{1.8}$$

are obtained, where

$$\widehat{\mathbf{y}}=\begin{bmatrix}\mathbf{y}\\ \mathbf{b}_1\end{bmatrix}, \quad \widehat{\mathbf{X}}=[\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2]=\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\\ \mathbf{A}_1 & \mathbf{0}\end{bmatrix}, \quad \widehat{\mathbf{X}}_1=\begin{bmatrix}\mathbf{X}_1\\ \mathbf{A}_1\end{bmatrix}, \quad \widehat{\mathbf{X}}_2=\begin{bmatrix}\mathbf{X}_2\\ \mathbf{0}\end{bmatrix}, \quad \widehat{\varepsilon}=\begin{bmatrix}\varepsilon\\ \mathbf{0}\end{bmatrix}, \quad \widehat{\mathbf{X}}_2^{\perp}=\begin{bmatrix}\mathbf{X}_2^{\perp}\\ \mathbf{0}\end{bmatrix}.$$

This merging operation in (1.6) and (1.7) is a well-known method of including equality restrictions in constrained linear regression models.

We make statistical inference of the models in (1.6) and (1.7) under the assumptions that the models are consistent, i.e., we assume that $\widehat{\mathbf{y}}\in\mathscr{C}\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right]$ holds with probability (wp) 1, corresponding the consistency of $\widehat{\mathscr{N}}$, in this case, the model $\widehat{\mathscr{N}_R}$ in (1.7) is consistent, i.e., $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}\in\mathscr{C}\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right]$ holds wp 1; see, e.g., [3].

To estimate the unknown parameter vector α_1 and to predict random error vector ε jointly in (1.4) and (1.5), we construct a general vector containing the both unknown vectors as follows

$$\phi_1=\mathbf{K}_1\alpha_1+\mathbf{H}\widehat{\varepsilon}=\left[\mathbf{K}_1,\mathbf{0}\right]\alpha+\mathbf{H}\widehat{\varepsilon}:=\widehat{\mathbf{K}}\alpha+\mathbf{H}\widehat{\varepsilon} \tag{1.9}$$

for given matrices $\widehat{\mathbf{K}}=\left[\mathbf{K}_1,\mathbf{0}\right]\in\mathbb{R}^{s\times k}$ with $\mathbf{K}_1\in\mathbb{R}^{s\times k_1}$ and $\mathbf{H}\in\mathbb{R}^{s\times(n+m)}$. It can be seen from the expectation and covariance matrix assumptions in (1.2) and (1.8),

$$\mathrm{E}(\phi_1)=\mathbf{K}_1\alpha_1, \quad \mathrm{D}(\phi_1)=\sigma^2\mathbf{H}\widehat{\Sigma}\mathbf{H}', \quad \mathrm{cov}(\phi_1,\widehat{\mathbf{y}})=\sigma^2\mathbf{H}\widehat{\Sigma}, \quad \mathrm{cov}(\phi_1,\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}})=\sigma^2\mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}. \tag{1.10}$$

In the present paper, we concern with the problems of constrained prediction/estimation under a CPLM and its CRLMs. We first review some of the results related to the subject that we consider in the study including the consistency of CPLMs, predictability/estimability of ϕ_1 in (1.9), the best linear unbiased predictors (BLUPs), and the best linear unbiased estimators (BLUEs). We show how to establish the BLUPs and the BLUEs of all unknown vectors in a CPLM and its CRLMs and present some fundamental properties of the BLUPs/BLUEs by solving certain constrained quadratic matrix-valued function optimization problems in LPO including ranks and inertias of block matrices. Our main purpose is to derive various inequalities and equalities for comparison of covariance matrices of the BLUPs/BLUEs of all unknown vectors in the CPLM and its CRLMs. Previous and recent work on the problems of the inference of CPLMs can be found in; see e.g., [4]-[18] among others.

The results, in the present paper, are established by making use of formulas of ranks of block matrices and elementary matrix operations. We review well-known results, which we need later, related to block matrices as follows.

Lemma 1.1 ([19]). *Let $\mathbf{A}_1$, $\mathbf{A}_2\in\mathbb{R}^{m\times n}$, or, let $\mathbf{A}_1=\mathbf{A}_1'$, $\mathbf{A}_2=\mathbf{A}_2'\in\mathbb{R}^{m\times m}$. Then,*

1. $\mathbf{A}_1=\mathbf{A}_2 \Leftrightarrow r(\mathbf{A}_1-\mathbf{A}_2)=0$.
2. $\mathbf{A}_1\succ\mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1-\mathbf{A}_2)=m$ *and* $\mathbf{A}_1\prec\mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1-\mathbf{A}_2)=m$.
3. $\mathbf{A}_1\succcurlyeq\mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1-\mathbf{A}_2)=0$ *and* $\mathbf{A}_1\preccurlyeq\mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1-\mathbf{A}_2)=0$.

Lemma 1.2 ([19]). *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{n\times n}$, $\mathbf{P} \in \mathbb{R}^{m\times n}$, *and* $c \in \mathbb{R}$. *Then,*

$$r(\mathbf{A}_1) = i_+(\mathbf{A}_1) + i_-(\mathbf{A}_1).$$

$$i_\pm(c\mathbf{A}_1) = \begin{cases} i_\pm(\mathbf{A}_1) & if\ \ c > 0 \\ i_\mp(\mathbf{A}_1) & if\ \ c < 0 \end{cases}.$$

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{P} \\ \mathbf{P}' & \mathbf{A}_2 \end{bmatrix} = i_\pm \begin{bmatrix} \mathbf{A}_1 & -\mathbf{P} \\ -\mathbf{P}' & \mathbf{A}_2 \end{bmatrix} = i_\mp \begin{bmatrix} -\mathbf{A}_1 & \mathbf{P} \\ \mathbf{P}' & -\mathbf{A}_2 \end{bmatrix}.$$

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} = i_\pm(\mathbf{A}_1) + i_\pm(\mathbf{A}_2). \ \ i_+ \begin{bmatrix} \mathbf{0} & \mathbf{P} \\ \mathbf{P}' & \mathbf{0} \end{bmatrix} = i_- \begin{bmatrix} \mathbf{0} & \mathbf{P} \\ \mathbf{P}' & \mathbf{0} \end{bmatrix} = r(\mathbf{P}).$$

Lemma 1.3 ([19]). *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$, $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n\times n}$, *and* $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$. *Then,*

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) + i_\pm(\mathbf{E}_{\mathbf{A}_2}\mathbf{A}_1\mathbf{E}_{\mathbf{A}_2}). \tag{1.11}$$

$$i_+ \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{A}_1, & \mathbf{A}_2 \end{bmatrix} \ \ and \ \ i_- \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) \ \ if \ \ \mathbf{A}_1 \succcurlyeq \mathbf{0}. \tag{1.12}$$

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{B} \end{bmatrix} = i_\pm(\mathbf{A}_1) + i_\pm(\mathbf{B} - \mathbf{A}_2'\mathbf{A}_1^+\mathbf{A}_2) \ \ if \ \ \mathscr{C}(\mathbf{A}_2) \subseteq \mathscr{C}(\mathbf{A}_1). \tag{1.13}$$

Lemma 1.4 ([20]). *Let* $\mathbf{Q} \in \mathbb{R}^{n\times n}$ *be a symmetric psd matrix. Assume that there exists* $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *for given* $\mathbf{A} \in \mathbb{R}^{n\times p}$, $\mathbf{B} \in \mathbb{R}^{m\times p}$. *Then the maximal positive inertia of* $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{X}\mathbf{Q}\mathbf{X}'$ *subject to all solutions of* $\mathbf{X}\mathbf{A} = \mathbf{B}$ *is*

$$\max_{\mathbf{X}\mathbf{A}=\mathbf{B}} i_+(\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{X}\mathbf{Q}\mathbf{X}') = r\begin{bmatrix} \mathbf{X}_0\mathbf{Q} \\ \mathbf{A}' \end{bmatrix} - r(\mathbf{A}) = r(\mathbf{X}_0\mathbf{Q}\mathbf{A}^\perp). \tag{1.14}$$

Hence there exists solution $\mathbf{X}_0$ *of* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *such that holds for all solutions of* $\mathbf{X}\mathbf{A} = \mathbf{B} \Leftrightarrow \mathbf{X}_0$ *satisfies both* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *and* $\mathbf{X}_0\mathbf{Q}\mathbf{A}^\perp = \mathbf{0}$.

2. BLUPs/BLUEs' computations

A group of computational formulas on the BLUPs/BLUEs of all unknown vectors in CPLM and its CRLMs are given with many basic properties of BLUPs/BLUEs by using quadratic matrix optimization methods given as in Lemma 1.4. Under our considerations, firstly, we review the predictability/estimability requirement of ϕ_1 and its special cases under the models (1.6) and (1.7) before giving the definition of the BLUPs/BLUEs.

1. ϕ_1 in (1.9) is predictable by $\widehat{\mathbf{y}}$ under $\widehat{\mathscr{N}}$ in (1.6), i.e., $\mathrm{E}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \mathbf{0}$ holds for some $\mathbf{L} \Leftrightarrow \mathscr{C}(\widehat{\mathbf{K}}') \subseteq \mathscr{C}(\widehat{\mathbf{X}}') \Leftrightarrow \widehat{\mathbf{K}}\alpha$ is estimable under (1.6), i.e., $\mathbf{K}_1\alpha_1$ is estimable under (1.6),
2. $\widehat{\mathbf{X}}\alpha$ is always estimable and $\widehat{\varepsilon}$ is always predictable under (1.6),
3. ϕ_1 in (1.9) is predictable by $\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{y}}$ under $\widehat{\mathscr{N}_R}$ in (1.7), i.e., $\mathrm{E}(\mathbf{G}\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{y}} - \phi_1) = \mathbf{0}$ holds for some $\mathbf{G} \Leftrightarrow \mathscr{C}(\mathbf{K}_1') \subseteq \mathscr{C}(\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^\perp)$ $\Leftrightarrow \mathbf{K}_1\alpha_1$ is estimable under (1.7),
4. $\widehat{\mathbf{X}}_1\alpha_1$ is estimable under (1.7) $\Leftrightarrow \mathscr{C}(\widehat{\mathbf{X}}_1') \subseteq \mathscr{C}(\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^\perp)$,
5. $\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{X}}_1\alpha_1$ is always estimable and $\widehat{\mathbf{X}}_2^\perp\widehat{\varepsilon}$ is always predictable under (1.7),
6. α_1 is estimable under (1.7) $\Leftrightarrow r(\widehat{\mathbf{X}}_2^\perp\widehat{\mathbf{X}}_1) = k_1$ and $\widehat{\varepsilon}$ is always predictable under (1.7);

see, e.g., [21]. Further, ϕ_1 is predictable under $\widehat{\mathscr{N}}$ when it is predictable under $\widehat{\mathscr{N}_R}$.

Definition 2.1 ([22],[23]). *The* BLUP/BLUE *definitions for models in* (1.6) *and* (1.7) *are given as follows, respectively.*

1. *Let* ϕ_1 *be predictable by* $\widehat{\mathbf{y}}$ *in* (1.6). *If there exists* $\mathbf{L}\widehat{\mathbf{y}}$ *such that*

$$\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \ min\ s.t.\ \ \mathrm{E}(\mathbf{L}\widehat{\mathbf{y}} - \phi_1) = \mathbf{0} \tag{2.1}$$

holds in the LPO, *the linear statistic* $\mathbf{L}\widehat{\mathbf{y}}$ *is defined to be the* BLUP *of* ϕ_1 *and is denoted by* $\mathbf{L}\widehat{\mathbf{y}} = \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1) = \mathrm{BLUP}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{K}}\alpha + \mathbf{H}\widehat{\varepsilon})$. *If* $\mathbf{H} = \mathbf{0}$ *in* ϕ_1 *or* $\widehat{\mathbf{K}} = \mathbf{0}$ *in* ϕ_1, $\mathbf{L}\widehat{\mathbf{y}}$ *corresponds the* BLUE *of* $\widehat{\mathbf{K}}\alpha$, *denoted by* $\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{K}}\alpha)$ *and* BLUP *of* $\mathbf{H}\widehat{\varepsilon}$, *denoted by* $\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\mathbf{H}\widehat{\varepsilon})$, *under* (1.6).

2. *Let ϕ_1 be predictable by $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ in (1.7). If there exists $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ such that*

$$\mathrm{D}(\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}-\phi_1)=\ \textit{min s.t. } \mathrm{E}(\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}-\phi_1)=\mathbf{0}$$

holds in the LPO*, the linear statistic $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ is defined to be the* BLUP *of ϕ_1 and is denoted by $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}=\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)=\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1+\mathbf{H}\widehat{\varepsilon})$. If $\mathbf{H}=\mathbf{0}$ in ϕ_1 or $\mathbf{K}_1=\mathbf{0}$ in ϕ_1, $\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ corresponds the* BLUE *of $\mathbf{K}_1\alpha_1$, denoted by $\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)$ and* BLUP *of $\mathbf{H}\widehat{\varepsilon}$, denoted by $\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\mathbf{H}\widehat{\varepsilon})$, under (1.7).*

The fundamental results on BLUP of ϕ_1 under (1.6) and (1.7) are collected in the following theorems. The results given below are obtained from [24] by considering the models and notation used in this paper. For different approaches; see, e.g, [23], [25].

Theorem 2.2. *Let ϕ_1 be predictable by $\widehat{\mathbf{y}}$ in (1.6). Then,*

$$\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)=\mathbf{L}\widehat{\mathbf{y}}=\left(\left[\widehat{\mathbf{K}}\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]\mathbf{W}_1^{+}+\mathbf{P}_1\mathbf{W}_1^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.2}$$

where $\mathbf{P}_1\in\mathbb{R}^{s\times(n+m)}$ is an arbitrary matrix and $\mathbf{W}_1=\left[\widehat{\mathbf{X}},\quad \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]$. In particular,

1. *$\mathbf{L}$ is unique $\Leftrightarrow r(\mathbf{W}_1)=(n+m)$.*
2. *$\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)$ is unique wp 1 $\Leftrightarrow \widehat{\mathscr{N}}$ is consistent.*
3. *$r(\mathbf{W}_1)=r\left[\widehat{\mathbf{X}},\quad \widehat{\Sigma}\right]=r\left[\widehat{\mathbf{X}},\quad \widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\right]$.*
4. *Further, the following dispersion matrix equalities hold.*

$$\mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]=\sigma^2\left[\widehat{\mathbf{K}},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]\mathbf{W}_1^{+}\widehat{\Sigma}\left(\left[\widehat{\mathbf{K}},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]\mathbf{W}_1^{+}\right)', \tag{2.3}$$

$$\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]=\sigma^2\left(\left[\widehat{\mathbf{K}},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]\mathbf{W}_1^{+}-\mathbf{H}\right)\widehat{\Sigma}\left(\left[\widehat{\mathbf{K}},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]\mathbf{W}_1^{+}-\mathbf{H}\right)'. \tag{2.4}$$

5. *In particular,*

$$\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{K}}\alpha)=\left(\left[\widehat{\mathbf{K}},\quad \mathbf{0}\right]\mathbf{W}_1^{+}+\mathbf{P}_2\mathbf{W}_1^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.5}$$

$$\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\mathbf{H}\widehat{\varepsilon})=\left(\left[\mathbf{0},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]\mathbf{W}_1^{+}+\mathbf{P}_3\mathbf{W}_1^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.6}$$

where $\mathbf{P}_2$ and $\mathbf{P}_3\in\mathbb{R}^{s\times(n+m)}$ are arbitrary matrices.

Proof. Let $\mathbf{L}\widehat{\mathbf{y}}$ be an unbiased linear predictor for ϕ_1 under the model in (1.6). Then,

$$\mathrm{E}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)=\mathbf{0}\Leftrightarrow \mathbf{L}\widehat{\mathbf{X}}=\widehat{\mathbf{K}},\ \text{ i.e., }\ \left[\mathbf{L},\quad -\mathbf{I}_s\right]\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}=\mathbf{0}, \tag{2.7}$$

$$\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)=\sigma^2(\mathbf{L}-\mathbf{H})\widehat{\Sigma}(\mathbf{L}-\mathbf{H})'=\sigma^2\left[\mathbf{L},\quad -\mathbf{I}_s\right]\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\left[\mathbf{L},\quad -\mathbf{I}_s\right]' \tag{2.8}$$

for unbiased linear predictor $\mathbf{L}\widehat{\mathbf{y}}$. The similar expressions can be written for the other unbiased linear predictor $\mathbf{T}\widehat{\mathbf{y}}$ for ϕ_1 under the model in (1.6) by writing $\mathbf{T}$ instead of $\mathbf{L}$ in (2.7) and (2.8). Then the expression in (2.1) can be expressed as to find solution $\mathbf{L}$ of the consistent linear matrix equation $\mathbf{L}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}$ such that $\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)\preccurlyeq\mathrm{D}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)$ s.t. $\mathbf{T}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}$, i.e.,

$$\left[\mathbf{L},\quad -\mathbf{I}_s\right]\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\left[\mathbf{L},\quad -\mathbf{I}_s\right]'\preccurlyeq\left[\mathbf{T},\quad -\mathbf{I}_s\right]\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\left[\mathbf{T},\quad -\mathbf{I}_s\right]'\quad \text{s.t.}\quad \mathbf{T}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}. \tag{2.9}$$

Applying (1.14) to (2.9), the maximal positive inertia of $\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)-\mathrm{D}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)$ subject to $\mathbf{T}\widehat{\mathbf{X}}=\widehat{\mathbf{K}}$ is obtained as follows:

$$\begin{aligned}\max_{\mathrm{E}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1)=\mathbf{0}} i_{+}(\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)-\mathrm{D}(\mathbf{T}\widehat{\mathbf{y}}-\phi_1))&=r\begin{bmatrix}\left[\mathbf{L},\quad -\mathbf{I}_s\right]\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\\ \begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}'\end{bmatrix}-r\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}\\ &=r\left(\left[\mathbf{L},\quad -\mathbf{I}_s\right]\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}^{\perp}\right).\end{aligned} \tag{2.10}$$

Combining (2.7) with (2.10), we conclude that $\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}}-\phi_1)=\min \Leftrightarrow$ there exists $\mathbf{L}$ satisfying both

$$\mathbf{L}\widehat{\mathbf{X}}=\widehat{\mathbf{K}} \text{ and } \begin{bmatrix}\mathbf{L}, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}\widehat{\Sigma}\begin{bmatrix}\mathbf{I}_{n+m}\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\widehat{\mathbf{X}}\\ \widehat{\mathbf{K}}\end{bmatrix}^{\perp}=\mathbf{0},$$

i.e., $\mathbf{L}\widehat{\mathbf{y}}=\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)\Leftrightarrow\mathbf{L}\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}=\begin{bmatrix}\widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}$. This matrix equation is consistent and the general solution of the equation can be written as in (2.2); see, e.g., [26]. Results in items 1 and 2 follow from (2.2). For the result in item 3, we refer [27, Lemma 2.1(a)]. (2.3) is seen from (2.2) and the assumptions in (1.2). Further,

$$\mathrm{cov}\{\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1),\phi_1\}=\begin{bmatrix}\widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\end{bmatrix}^{+}\widehat{\Sigma}\mathbf{H}' \tag{2.11}$$

by using (1.8) and (1.10). (2.4) is seen from (2.3) and (2.11). (2.5) and (2.6) follow directly from (2.2). □

Theorem 2.3. *Let ϕ_1 be predictable by $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}$ in (1.7). Then,*

$$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)=\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}=\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}+\mathbf{P}_4\mathbf{W}_2^{\perp}\right)\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}, \tag{2.12}$$

where $\mathbf{P}_4\in\mathbb{R}^{s\times(n+m)}$ is an arbitrary matrix and $\mathbf{W}_2=\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}$. In particular,

1. *$\mathbf{G}$ is unique $\Leftrightarrow r(\mathbf{W}_2)=(n+m)$.*
2. *$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)$ is unique wp 1 $\Leftrightarrow\widehat{\mathscr{N}_R}$ is consistent.*
3. *$r(\mathbf{W}_2)=r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix}=r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix}$.*
4. *The following dispersion matrix equalities hold.*

$$\mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]=\sigma^2\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\right)',$$

$$\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]=\sigma^2\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{H}\right)\widehat{\Sigma}\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{H}\right)'. \tag{2.13}$$

5. *In particular,*

$$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)=\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}=\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{0}\end{bmatrix}\mathbf{W}_2^{+}+\mathbf{P}_5\mathbf{W}_2^{\perp}\right)\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}},$$

$$\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)=\mathbf{G}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}}=\left(\begin{bmatrix}\mathbf{0}, & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_2^{+}+\mathbf{P}_6\mathbf{W}_2^{\perp}\right)\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{y}},$$

where $\mathbf{P}_5$ and $\mathbf{P}_6\in\mathbb{R}^{s\times(n+m)}$ are arbitrary matrices.

Proof. The proof of the theorem is obtained in a similar way to the proof of the Theorem 2.3. □

3. Main results

Theorem 3.1. *Let consider models $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ in (1.6) and (1.7), respectively, and assume that ϕ_1 is predictable under these models. Let $\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)$ and $\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)$ be as given in (2.2) and (2.12), and*

$$\mathbf{A}=\begin{bmatrix}\widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} & \widehat{\mathbf{X}}\\ \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0}\\ \mathbf{H}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1 & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} & \mathbf{0}\\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}.$$

Then

$$i_+(\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]-\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)])=i_+(\mathbf{A})-r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix}-r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1), \tag{3.1}$$

$$i_-(\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]-\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)])=i_-(\mathbf{A})-r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix}-r(\widehat{\mathbf{X}}), \tag{3.2}$$

$$r(\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]-\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)])=r(\mathbf{A})-r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix}-r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)-r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix}-r(\widehat{\mathbf{X}}). \tag{3.3}$$

Further,

1. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \succ \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_+(\mathbf{A}) = r\left[\widehat{\mathbf{X}},\quad \widehat{\Sigma}\right] + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)+s.$
2. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \prec \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_-(\mathbf{A}) = r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + r(\widehat{\mathbf{X}})+s.$
3. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \succcurlyeq \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_-(\mathbf{A}) = r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + r(\widehat{\mathbf{X}}).$
4. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] \preccurlyeq \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow i_+(\mathbf{A}) = r\left[\widehat{\mathbf{X}},\quad \widehat{\Sigma}\right] + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1).$
5. $\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] = \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)] \Leftrightarrow r(\mathbf{A}) = r\left[\widehat{\mathbf{X}},\quad \widehat{\Sigma}\right] + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) + r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + r(\widehat{\mathbf{X}}).$

Proof. Let $\mathbf{D} = \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]$. By using (2.13) and (1.13),

$$
\begin{aligned}
& i_{\pm}\left(\mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)] - \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\phi_1)]\right) \\
&= i_{\pm}\left(\mathbf{D} - \left(\left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp} - \mathbf{H}\right)\widehat{\Sigma}\left(\left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp} - \mathbf{H}\right)'\right) \\
&= i_{\pm}\begin{bmatrix} \widehat{\Sigma} & \widehat{\Sigma}\left(\left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp} - \mathbf{H}\right)' \\ \left(\left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_2^{+}\widehat{\mathbf{X}}_2^{\perp} - \mathbf{H}\right)\widehat{\Sigma} & \mathbf{D} \end{bmatrix} - i_{\pm}(\widehat{\Sigma}) \\
&= i_{\pm}\left(\begin{bmatrix} \widehat{\Sigma} & -\widehat{\Sigma}\mathbf{H}' \\ -\mathbf{H}\widehat{\Sigma} & \mathbf{D} \end{bmatrix} + \begin{bmatrix} \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{W}_2 \\ \mathbf{W}_2' & \mathbf{0} \end{bmatrix}^{+}\begin{bmatrix} \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]' \end{bmatrix}\right) \\
&\quad - i_{\pm}(\widehat{\Sigma})
\end{aligned} \tag{3.4}
$$

is obtained, where $\mathbf{W}_2 = \left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]$. We can apply (1.13) to (3.4) since the column space inclusions $\mathscr{C}(\widehat{\Sigma}) \subseteq \mathscr{C}(\mathbf{W}_2)$ and $\mathscr{C}\left(\left[\mathbf{K}_1,\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]'\right) \subseteq \mathscr{C}(\mathbf{W}_2')$ hold. Then (3.4) is equivalently written as follows

$$
\begin{aligned}
& i_{\pm}\begin{bmatrix} \mathbf{0} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} \\ -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1' \\ -(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\Sigma} & -\widehat{\Sigma}\mathbf{H}' \\ \mathbf{0} & \mathbf{K}_1 & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & -\mathbf{H}\widehat{\Sigma} & \mathbf{D} \end{bmatrix} \\
&\quad - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] - i_{\pm}(\widehat{\Sigma}) \\
&= i_{\pm}\begin{bmatrix} -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1' \\ -(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1 & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp} & \mathbf{D}-\mathbf{H}\widehat{\Sigma}\mathbf{H}' \end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] \\
&= i_{\pm}\begin{bmatrix} -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' \\ -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{K}_1' \\ \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1 & \mathbf{D}-\mathbf{H}\widehat{\Sigma}\mathbf{H}' \end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] \\
&= i_{\mp}\left(\begin{bmatrix} \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\ \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{H}\widehat{\Sigma}\mathbf{H}' & \mathbf{K}_1 \\ \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \\ \mathbf{0} \end{bmatrix}\mathbf{D}\begin{bmatrix} \mathbf{0} & \mathbf{I}_s & \mathbf{0} \end{bmatrix}\right) + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] \\
&\quad - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right].
\end{aligned} \tag{3.5}
$$

We can reapply (1.13) to (3.5) after writing $\mathbf{D} = \mathrm{D}[\phi_1-\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\phi_1)]$ in (2.4). Then, (3.5) is equivalently written as follows by using the similar way to obtaining (3.4),

$$
\begin{aligned}
& i_{\mp}\left(\begin{bmatrix} \widehat{\Sigma} & \mathbf{0} & -\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\ -\mathbf{H}\widehat{\Sigma} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{H}\widehat{\Sigma}\mathbf{H}' & \mathbf{K}_1 \\ \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \widehat{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left[\widehat{\mathbf{K}},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{W}_1 \\ \mathbf{W}_1' & \mathbf{0} \end{bmatrix}^{+}\begin{bmatrix} \widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[\widehat{\mathbf{K}},\quad \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]' & \mathbf{0} \end{bmatrix}\right) \\
&\quad - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1,\quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] - i_{\mp}(\widehat{\Sigma}).
\end{aligned} \tag{3.6}
$$

We can apply (1.13) to (3.6) since $\mathscr{C}(\widehat{\Sigma}) \subseteq \mathscr{C}(\mathbf{W}_1)$, where $\mathbf{W}_1 = \left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]$. From Lemma 1.2 and 1.3, (3.6) is equivalent to

$$
\begin{aligned}
& i_{\mp}\begin{bmatrix}
\mathbf{0} & -\widehat{\mathbf{X}} & -\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & \widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
-\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\Sigma} & \mathbf{0} & -\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\mathbf{0} & \widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & -\mathbf{H}\widehat{\Sigma} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{H}}' & \mathbf{K}_1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - i_{\mp}(\widehat{\Sigma}) - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
& = i_{\mp}\begin{bmatrix}
-\widehat{\Sigma} & -\widehat{\mathbf{X}} & -\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
-\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
-\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\mathbf{H}\widehat{\Sigma} & \widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{K}_1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
& = i_{\mp}\begin{bmatrix}
-\widehat{\Sigma} & -\widehat{\mathbf{X}} & \mathbf{0} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
-\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\mathbf{H}\widehat{\Sigma} & \widehat{\mathbf{K}} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{K}_1 \\
\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
& = i_{\pm}\begin{bmatrix}
\widehat{\Sigma} & \mathbf{0} & \widehat{\mathbf{X}} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} \\
\mathbf{0} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\mathbf{H}' & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 \\
\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' & \mathbf{0} \\
\mathbf{H}\widehat{\Sigma} & \mathbf{H}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{K}} & \mathbf{0} & -\mathbf{K}_1 \\
\mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & -\mathbf{K}_1' & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right] \\
& \quad + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]
\end{aligned}
$$

$$
= i_{\pm}\begin{bmatrix}
\widehat{\Sigma} & -\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}} & \mathbf{0} & \widehat{\Sigma}\mathbf{H}' \\
-\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & -\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\
\widehat{\mathbf{X}}' & -\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}' \\
\mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & -\mathbf{K}_1' \\
\mathbf{H}\widehat{\Sigma} & \mathbf{0} & \widehat{\mathbf{K}} & -\mathbf{K}_1 & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]
$$

$$
\begin{aligned}
& = i_{\pm}\begin{bmatrix}
\widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\Sigma}\mathbf{H}' & \mathbf{0} & \widehat{\mathbf{X}} \\
\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\
\mathbf{H}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\
\mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} & \mathbf{0} \\
\widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} - r\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\right] + i_{\pm}\left[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right] - r\left[\widehat{\mathbf{X}}, \quad \widehat{\Sigma}\right] \\
& \quad + i_{\mp}\left[\widehat{\mathbf{X}}^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}^{\perp}\right]. \qquad (3.7)
\end{aligned}
$$

In consequence, by using (1.11) and (1.12), we obtain (3.1) and (3.2) from (3.7). (3.3) is obtained by adding the equalities in (3.1) and (3.2). (a)-(e) is seen from (3.1)-(3.3) by using Lemma 1.1. □

As an immediate consequence of Theorem 3.1, the following results are obtained by setting $\mathbf{H} = \mathbf{0}$ and $\mathbf{K}_1 = \widehat{\mathbf{X}}_1$, respectively, in this theorem.

Corollary 3.2. *Let $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ be as given in (1.6) and (1.7), respectively, and assume that $\mathbf{K}_1\alpha_1$ is estimable under these models. Denote*

$$\mathbf{B} = \begin{bmatrix} \widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}} \\ \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2^{\perp} & \mathbf{K}_1' & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$i_+(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)]) = i_+(\mathbf{B}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1),$$

$$i_-(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)]) = i_-(\mathbf{B}) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}),$$

$$r(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)]) = r(\mathbf{B}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}).$$

Further,

1. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \succ \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) + s.$
2. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \prec \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}) + s.$
3. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \succcurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}).$
4. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] \preccurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1).$
5. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\mathbf{K}_1\alpha_1)] = \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\mathbf{K}_1\alpha_1)] \Leftrightarrow r(\mathbf{B}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1) + r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}).$

Corollary 3.3. *Let $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ be as given in (1.6) and (1.7), respectively, and assume that $\widehat{\mathbf{X}}_1\alpha_1$ is estimable under these models. Denote*

$$\mathbf{C} = \begin{bmatrix} \widehat{\Sigma} & \widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp} & \widehat{\mathbf{X}} \\ \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$i_+(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)]) = i_+(\mathbf{C}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix},$$

$$i_-(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)]) = i_-(\mathbf{C}) - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}_2),$$

$$r(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)]) = r(\mathbf{C}) - r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} - r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} - r(\widehat{\mathbf{X}}_2).$$

Further,

1. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \succ \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + m + n.$
2. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \prec \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}_2) + m + n.$
3. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \succcurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_-(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}_2).$
4. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] \preccurlyeq \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow i_+(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix}.$
5. $\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\widehat{\mathbf{X}}_1\alpha_1)] = \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\widehat{\mathbf{X}}_1\alpha_1)] \Leftrightarrow r(\mathbf{C}) = r\begin{bmatrix}\widehat{\mathbf{X}}, & \widehat{\Sigma}\end{bmatrix} + r\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\Sigma}\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} + r(\widehat{\mathbf{X}}_2).$

Corollary 3.4. *Let $\widehat{\mathscr{N}}$ and $\widehat{\mathscr{N}_R}$ be as given in (1.6) and (1.7), respectively, and assume that α_1 is estimable under these models. Then*

1. $i_{\pm}(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\alpha_1)]) = r(\mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}}}(\alpha_1)] - \mathrm{D}[\mathrm{BLUE}_{\widehat{\mathscr{N}_R}}(\alpha_1)]) = 0.$
2. $i_{\pm}(\widehat{\varepsilon} - \mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\widehat{\varepsilon})] - \mathrm{D}[\widehat{\varepsilon} - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\widehat{\varepsilon})]) = r(\widehat{\varepsilon} - \mathrm{D}[\mathrm{BLUP}_{\widehat{\mathscr{N}}}(\widehat{\varepsilon})] - \mathrm{D}[\widehat{\varepsilon} - \mathrm{BLUP}_{\widehat{\mathscr{N}_R}}(\widehat{\varepsilon})]) = 0.$

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Author's contributions

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