



## The Theory of Ultra - Groups and GAP Applications

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*Ultra-group,  
normal ultra-group,  
GAP programming  
language*

**Abstract** — In this work, we introduce a system computational programming of the ultra-group notions with GAP, Groups, Algorithms and Programming. We have constructed some algorithms about ultra-groups and their substructures with GAP language. Also, we give the GAP algorithm about ultra-group homomorphism. So we have presented a GAP application for ultra-group theory.

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### 1. Introduction

The notion of ultra-groups over a group was introduced in [6] by Moghaddasi et al. In fact, an ultra-group is an algebraic structure whose underlying set is determined by a group and its subgroup. An ultra-group is a special subset of a group that has a binary operation that combines any two elements to generate a third element and a unary operation. Every group is an ultra-group, but the reverse is not always valid. The idea of an ultra-group was studied in [7],[12] and [13] with details.

By similarity with “computational group theory,” we provide new GAP functions in this study. As an application, we have constructed some algorithms to create ultra-groups and subultra-groups. Also, the algorithm for checking the ultra-group homomorphism conditions as well as controlling an ultra-group structure and its various substructures has been presented. In [2], [1], [9] and [10], one can find several algorithms for many concepts which are implemented in the GAP packages.

### 2. Theory of Ultra-Groups

The concept of transversals was first introduced by Kurosh in [5], which is the foundation of the ultra-group concept.

We will recall this notion and some fundamental notions such as ultra-groups and their substructures from [6].

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**Definition 2.1.** Let  $G$  be a finite group. Assume  $M$  and  $H$  are two any subsets of group  $G$ . Obviously,  $HM \neq MH$ .  $(A, B)$  is a pair of subsets of group  $G$  and is named **transversal** if the equality  $xy = x'y'$  implies  $x = x'$  and  $y = y'$  for  $x, x' \in A, y, y' \in B$ .

From this definition, it is easy to understand that a pair  $(H, M)$  of subgroups of  $G$  is transversal if and only if  $H \cap M = \{e\}$ .

Furthermore, if  $G$  is a group,  $H$  is a subgroup of  $G$ , and  $M$  is a subset of  $G$ , then we conclude that the pair  $(H, M)$  is a transversal if and only if  $M \cap Hg$  contains **at most one element**, for all  $g \in G$  [12].

We demonstrate the left and right quotient sets by

$$G/H = \{xH | x \in G\}$$

$$H/G = \{Hx | x \in G\}$$

respectively. These sets are the partitions of  $G$  where  $H$  is the subgroup of  $G$ .

A transversal of a partition is a set with just one member from each part of the partition. Throughout the work, we will assume that a right transversal contains the identity of the group. In other words,  $|M \cap Hg| = 1$  with all  $g \in G$ , subgroup  $H$  and a set  $M$ . Then  $G = HM$ . And so, we have  $MH \subseteq G = HM$ , [6].

**Definition 2.2.** Let  $G$  be a group with multiplication and  $H$  be a subgroup of  $G$ . A subset  $M$  of  $G$  is described as (right unitary) **complementary set** concerning the subgroup  $H$ , if for all elements  $h \in H$  and  $m \in M$  there exist the **unique** members  $m' \in M$  and  $h' \in H$  satisfying

$$h'm' = mh$$

and the identity of  $G$  in  $M$ . We denote  $m'$  and  $h'$  by  $m^h$  and  ${}^m h$ , respectively.

In a similar way, for all  $m_1, m_2 \in M$  there is a unique  $[m_1, m_2] \in M$  and  $(m_1, m_2) \in H$  satisfying

$$m_1 m_2 = (m_1, m_2)[m_1, m_2]$$

For all  $x \in M$ , there exists  $x^{-1} \in G$ .

In  $G = HM$ , there exists  $x^{[-1]} \in M$  and  $x^{(-1)} \in H$  satisfying

$$x^{-1} = x^{(-1)} x^{[-1]}$$

[6].

**Definition 2.3.** A complementary set of  $H$  over group  $G$  namely  ${}_H M$  is called a (right) ultra-group with a binary operation

$$\alpha : {}_H M \times {}_H M \longrightarrow {}_H M$$

and unary operation

$$\beta: {}_H M \longrightarrow {}_H M$$

which is defined as  $\alpha(m_1, m_2) := [m_1, m_2]$ ,  $\beta_h(m) := m^h$ , for all  $h \in H$ , respectively.

**Example 2.4.** Let  $G = S_3$  then  $H = \{(1), (13)\}$  is a subgroup of  $G$ . With the right quotient set  $H/G = \{(1), (13)\}, \{(12), (132)\}, \{(23), (123)\}$ , we have four alternatives for  $M$ . For instance, if we choose the set  $M = \{(1), (12), (23)\}$ , we get the following multiplication tables:

$\alpha$	(1)	(23)	(12)	$\beta$	(1)	(13)
(1)	(1)	(23)	(12)	(1)	(1)	(1)
(23)	(23)	(1)	(23)	(23)	(23)	(12)
(12)	(12)	(12)	(1)	(12)	(12)	(23)

$(M, \alpha, \beta)$  is an ultra-group of  $H$  over group  $G$ .

**Example 2.5.** Let  $G = S_3$  then  $H = \{(1), (13)\}$  is a subgroup of  $G$ . With the quotient set  $H/G = \{(1), (13)\}, \{(12), (132)\}, \{(23), (123)\}$ , we have four candidates for  $M$ . For instance if we choose the set  $M = \{(1), (123), (132)\}$ , we get the following multiplication tables:

$\alpha$	(1)	(123)	(132)	$\beta$	(1)	(13)
(1)	(1)	(123)	(132)	(1)	(1)	(1)
(123)	(123)	(132)	(1)	(123)	(123)	(123)
(132)	(132)	(1)	(123)	(132)	(132)	(123)

$(M, \alpha, \beta)$  is an ultra-group of  $H$  over group  $G$ .

**Example 2.6.** Let  $D_9$  be a dihedral group of order 18 and  $H = \langle a^3 \rangle$  its subgroup. We get 243 ultra-groups and two of them are the following:

$$M = \{e, a, b, a^2, ab, a^2b\}$$

$\alpha$	$e$	$a$	$b$	$a^2$	$ab$	$a^2b$	$\beta$	$e$	$a^3$	$a^6$
$e$	$e$	$a$	$b$	$a^2$	$ab$	$a^2b$	$e$	$e$	$e$	$e$
$a$	$a$	$a^2$	$ab$	$e$	$a^2b$	$b$	$a$	$a$	$a$	$a$
$b$	$b$	$a^2b$	$e$	$ab$	$a^2$	$a$	$b$	$b$	$b$	$b$
$a^2$	$a^2$	$e$	$a^2b$	$a$	$ab$	$ab$	$a^2$	$a^2$	$a^2$	$a^2$
$ab$	$ab$	$b$	$a$	$a^2b$	$b$	$a^2$	$ab$	$ab$	$ab$	$ab$
$a^2b$	$a^2b$	$ab$	$a^2$	$b$	$a$	$e$	$a^2b$	$a^2b$	$a^2b$	$a^2b$

and

$$M = \{e, b, ab, a^2b, a^5, a^4\}$$

$\alpha$	$e$	$b$	$ab$	$a^2b$	$a^5$	$a^4$	$\beta$	$e$	$a^3$	$a^6$
$e$	$e$	$b$	$ab$	$a^2b$	$a^5$	$a^4$	$e$	$e$	$e$	$e$
$b$	$b$	$e$	$a^5$	$a^4$	$ab$	$a^2b$	$b$	$b$	$b$	$b$
$ab$	$ab$	$a^4$	$b$	$a^5$	$a^2b$	$b$	$ab$	$ab$	$ab$	$ab$
$a^2b$	$a^2b$	$a^5$	$a^4$	$e$	$b$	$ab$	$a^2b$	$a^2b$	$a^2b$	$a^2b$
$a^5$	$a^5$	$a^2b$	$b$	$ab$	$a^4$	$e$	$a^5$	$a^5$	$a^5$	$a^5$
$a^4$	$a^4$	$ab$	$a^2b$	$b$	$e$	$a^5$	$a^4$	$a^4$	$a^4$	$a^4$

**Definition 2.7.** Let  $M$  be an ultra-group of  $H$  over  $G$ . A subset  $U \subseteq M$  that includes the identity is defined as **subultra-group** of  $H$  over  $G$ , if  $U$  is closed under the operations  $\alpha$  and  $\beta_h$  in the Definition 3.

It is clear that  $\{e\}$  is a trivial subultra-group for all ultra-groups  ${}_H M$  where  $e$  is the identity element of  $H$ .

**Definition 2.8.** Let  ${}_H M$  be a ultra-group and  $A, B$  are the subsets of  ${}_H M$ .  $[A, B]$  is defined as the set

$$\{[x, y] | x \in A, y \in B\}$$

Furthermore, if  $A$  is a subultra-group  ${}_H M$  and  $y \in {}_H M$  so the subset  $[A, y]$  is named a **right coset** of  $A$  in  ${}_H M$  [6].

**Definition 2.9.** A  $N$  subultra-group of  ${}_H M$  is named **normal** if

$$[N, [x, y]] = [x, [N, y]]$$

for all  $x, y \in {}_H M$ .

So we get the following features for normal ultra-groups:

- $[x, N] = [N, x]$ , for every  $x \in {}_H M$
- $[[N, x], [N, y]] = [N, [x, y]]$ , for all  $x, y \in {}_H M$ .
- If  $[N, y] = N$ , then  $y \in N$

[6].

**Definition 2.10.** Let  ${}_{H_i} M_i$  be the ultra-group of  $H_i$  with  $G_i$  for  $i = 1, 2$ . A mapping from  ${}_{H_1} M_1$  to  ${}_{H_2} M_2$  is an ultra-group homomorphism with

- a)  $f([m_1, m_2]) = [f(m_1), f(m_2)]$
- b)  $(f(m))^{\phi(h)} = f(m^h)$

for  $\phi$  is an group homomorphism from  $H_1$  to  $H_2$  and  $m, m_1, m_2 \in M_1, h \in H_1$ .

**Example 2.11.** Let  $M_1 = \{e, a, b, a^2, ab, a^2b\}$  and  $M_2 = \{e, b, ab, a^2b, a^5, a^4\}$  be as in example 2.6. Then there is an ultra-group isomorphism  $f : M_1 \rightarrow M_2$

$$f(x) = \begin{cases} a^4 & , \quad x = a \\ a^5 & , \quad x = a^2 \\ x & , \quad \text{otherwise.} \end{cases}$$

### 3. GAP Application

For computational discrete algebra, GAP is the most often used system. It is used to address problems involving symbolic computation. This type of computation illuminates number theory, combinatorics, and coding theory calculations in Mathematics and Computer Science.

GAP programming language;

- Free
- Open source
- Expandable separate package library

It is published with GNU Public License. The kernel of the system is written in C programming language. The library of functions and additional packages are in a special language, also called GAP [11].

In the GAP system and extension packages, there are now 900K lines of GAP code and 360K lines of C code exist. The Small Groups library in these packages has been given in landmark computations with the **Millennium Project** in [4] which classifies all finite groups of order smaller than 2000.

Besche and Eick [2] used group isomorphism to classify small group libraries up to rank of 1000, with the exception of 512 and 768. With technological advancements over the years, some higher computers now calculate a rank of 2000 [3, 8].

The following are the language characteristics:

- Streams,
- Garbage collection,
- Flexible list and record data types,
- Built-in data types for key algebraic objects,
- Control structures akin to those found in Pascal.

GAP is an **interactive environment** that includes features such as online help, break loops for debugging and profiling GAP programs, completion of tab, a graphical user interface for GAP, and other GAP interface programs created by users.

`SmallGroup(n, k)`

function makes a small group with a rank of  $n$  and group isomorphism class of  $k$ .

```
StructureDescription(G)
```

function gives the isomorphism class of group  $G$ .

In GAP [3] a polycyclic group(PC) is formed. It means that a group uses the polycyclic presentation for element arithmetic.

The Cayley Theorem is used in GAP to illustrate groups with permutation groups that are isomorphic to groups.

We define the functions that give ultra group structure and control some substructures in a short amount of time in this paper.

- The **ugroup** function is used to create an ultra-group object.
- The **isUltra** function determines whether or not an ultra-group exists.
- The subultra structure is controlled by the **iSubUltra** function.
- The **rightCoset** function creates the ultra-right group's coset.
- The **isNormalSubgroup** function determines whether or not an ultra-group is normal.
- With ultra-group, **bracket** function is employed for the alpha function.

The theoretical definition of the ultra-group structure is defined by the **Ugroup** function. With the subgroup  $H$  and the quotient group, it generates a suitable ultra-group. The alpha and beta function computations are listed in this function.

Take the symmetric group  $G := S_3$  as an example.  $G$  has a subgroup called  $H$ . As a result, we get the possibilities to satisfy the ultra-group conditions. The first input demonstrated the group  $G$ , the second is subgroup  $H$ .

```
gap>G:=SymmetricGroup(3);;eG:=Elements(G);;
gap>Ugroup(G,Subgroup(G,[eG[6]]));
[[(),(2,3),(1,2)],[(),(2,3),(1,3,2)],[(),(1,2),(1,2,3)],[(),(1,2,3),(1,3,2)]]
```

Let take the main group  $S_4$  symmetric group of order 4. We get 2048 ultra-group. Some are

```
gap>T:=SymmetricGroup(4);;eT:=Elements(T);;
gap>Ugroup(T,Subgroup(T,[eT[6]]));
[[(),(3,4),(2,3),(1,2),(1,2)(3,4),(1,2,3),(1,3,2),(1,3),(1,3,4),
(1,4,3),(1,4),(1,4)(2,3)],
[(),(3,4),(2,3),(1,2),(1,2)(3,4),(1,2,3),(1,3,2),(1,3,4),(1,3)(2,4),
```

```
(1,4,3), (1,4), (1,4)(2,3) ],
[ (), (3,4), (2,3), (1,2), (1,2)(3,4), (1,2,3), (1,3,4,2), (1,3), (1,3,2,4)
, (1,4,3), (1,4), (1,4)(2,3) ],
...
```

Another example of this function is the Dihedral group of order 18 also known as  $D_9$ . We get 243 ultra-groups and the computation tables arise from the  $H = \langle a^3, b \rangle$  subgroup which is the second parameter:

```
gap>G:=SmallGroup(18,1);;eG:=Elements(G);;
gap>Ugroup(G,Subgroup(G,[eG[4]]));
[[<identity> of ..., f1, f2, f3, f1*f2, f2^2, f2*f3, f1*f2^2, f2^2*f3 ],
[<identity> of ...,f1*f2,f2^2,f2*f3,f3^2,f1*f2^2,f1*f2*f3,f1*f3^2,f2^2*f3],
[<identity>of...,f2*f3,f3^2,f1*f2^2,f1*f2*f3,f1*f3^2,f2^2*f3,f2*f3^2,f1*f2^2*f3],
[<identity> of ...,f1*f2,f2^2,f2*f3,f3^2,f1*f2*f3,f1*f3^2,f2^2*f3,f2^2*f3^2],
[<identity> of..., f3^2, f1*f2*f3, f1*f3^2, f2*f3^2, f1*f2^2*f3, f1*f2*f3^2,
f2^2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f1*f2, f2^2, f3^2, f1*f2*f3, f1*f3^2, f1*f2*f3^2,
f2^2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f3^2, f1*f2^2, f1*f2*f3, f1*f3^2, f2*f3^2,
f1*f2^2*f3, f1*f2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f3, f1*f2, f3^2, f1*f2*f3, f1*f2^2*f3,
f1*f2*f3^2, f2^2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f1, f2, f3, f2^2, f2*f3, f1*f2^2, f2^2*f3, f2*f3^2],
...
```

Each list corresponds to the ultra-group  $M$ .

The **isUltra** control function is a control function. The boolean data type is what it responds to. Three parameters are required for this function. The first parameter  $M$  is the set over which we have control, whether or not this set is an ultra-group. The main group  $G$  last parameter and the subgroup  $H$ , the second list, are required.

```
gap>isUltra([(),(1,2),(2,3)],[(),(1,3)],SymmetricGroup(3));
true
```

A control function is the **iSubultra** function. It responds to the boolean data type. In this function, we need four parameters.  $S$ , the second parameter, is the set that we control whether this set is the subultra-group or not of the first parameter, the list  $M$ . We need the main group the last input  $G$  and the subgroup  $H$ , the third parameter.

```
gap>G:=SymmetricGroup(3);;eG:=Elements(G);;
gap>iSubultra([(),(1,2),(2,3)],[()],Subgroup(G,[eG[6]]),G);
true
```

The **Subultra** function defines subultra-groups of the  $M$  ultra-group with  $H$  over the  $G$  group. It generates the subultra-groups with three parameter,  $M, H, G$ .

```
gap>G:=SymmetricGroup(3);;eG:=Elements(G);;
gap>Subultra([(),(1,2),(2,3)],Subgroup(G,[eG[6]]),G);
[[ [ () ], [ () , (2,3) ], [ () , (2,3) , (1,2) ], [ () , (1,2) ] ],
[[ [ () ], [ () , (2,3) ] ],[[ [ () ], [ () , (1,2) ] ],
[[ [ () ], [ () , (1,2,3) , (1,3,2) ] ]]
```

**rightCoset** function composes the right coset for an ultra-group like group version. It takes four parameters. The main group and ultra-group must be given. For  $A$  and  $B$  subultra-groups, we find  $[A, B]$  set with the GAP function.

```
gap>rightCoset([(),(1,2),(2,3)],[],[],SymmetricGroup(3));
[(),(2,3)]
```

**isNormalSubGrp** function checks whether the second parameter is normal for the first parameter or not. For example in  $G := S_3$  symmetric group

```
gap>isNormalSubGrp([(),(1,2),(2,3)],[(1,2),(2,3)],SymmetricGroup(3));
True
```

An ultra-group structure is a triple notation. A set  $M$ , an alpha function and a beta function. The alpha function is called with bracket notation. So, we use this function occasionally. In GAP,

```
gap>bracket((1,2),(2,3),[(1,2),(2,3)],[(1,3)]);
(1,2)
```

As we mentioned in the example 2.6 we build an ultra-group homomorphism. We have some outputs about ultra-group homomorphism control function.

```
gap>G:=SmallGroup(18,1);
<pc group of size 18 with 3 generators>
gap>eG:=Elements(G);
[<identity>of ...,f1,f2,f3, f1*f2, f1*f3, f2^2, f2*f3, f3^2, f1*f2^2,f1*f2*f3,
f1*f3^2, f2^2*f3, f2*f3^2, f1*f2^2*f3, f1*f2*f3^2, f2^2*f3^2, f1*f2^2*f3^2 ]
gap>H:=Subgroup(G,[eG[9]]);
Group([ f3^2 ])
gap>M:=Ugroup(G,H);;
```



```

gap>f:=GeneralMappingByElements(Domain(M[12]),Domain(M[76]),
gap>[Tuple([M[12][1],M[76][1]],Tuple([M[12][2],M[76][2]],Tuple([M[12][3],
gap>M[76][4])),Tuple([M[12][4],M[76][6]],Tuple([M[12][5],M[76][3]))
gap>,Tuple([M[12][6],M[76][5]]));;
gap>IsUltraHom(f,G,G,H,H);
True

```

and the algorithm of ultra-group homomorphism is below.

An algorithm of the ultra-group homomorphism is located at the end of this.

---

**Input:**  $f, G_1, G_2, H_1, H_2$ , Homomorphism, Domain, Codomain

**Output:** true iff  $f$  is an ultra-group homomorphism

**begin**

$M_1 \leftarrow$  the ultra-group of  $H_1$  over  $G_1$

$M_2 \leftarrow$  the ultra-group of  $H_2$  over  $G_2$

$l \leftarrow$  the parameter of  $H_2$

$\alpha_1 \leftarrow$  the equality of  $l * f(m) * f(z)$

$\alpha_2 \leftarrow$  the equality of  $f(h * m * z)$

$\beta_1 \leftarrow$  the equality of  $l * f(m) * p(t)$

$\beta_2 \leftarrow$  the equality of  $f(h * m * t)$

**if**  $\alpha_1$  and  $\alpha_2$  are in ultra-groups **then**

**if**  $\alpha_1 = \alpha_2$  **then**

        | **return** true

**else**

        | **return** false

**end**

**else**

    | **return** false

**end**

**if**  $\beta_1$  and  $\beta_2$  are in ultra-groups **then**

**if**  $\beta_1 = \beta_2$  **then**

        | **return** true

**else**

        | **return** false

**end**

**else**

    | **return** false

**end**

**end**

---

A screen of the ultra-group homomorphism process is as follows.

```

f1---f3^2
Not in range of homomorphism
f1---f3^2
f3^2*f1*f3^2=f1SECOND CONDITION SUCCESS

second cond. ended-----
f2---f3
Not in range of homomorphism
f2---f3
Not in range of homomorphism
f2---f3
f3^2*f2*f3^2*f3=f2*f3^2SECOND CONDITION SUCCESS

second cond. ended-----
f2^2---f3
Not in range of homomorphism
f2^2---f3
Not in range of homomorphism
f2^2---f3
f3^2*f2^2*f3^2*f3=f2^2*f3^2SECOND CONDITION SUCCESS

second cond. ended-----
f1*f2*f3---f3^2
Not in range of homomorphism
f1*f2*f3---f3^2
Not in range of homomorphism
f1*f2*f3---f3^2
f3^2*f1*f2*f3*f3^2=f1*f2*f3SECOND CONDITION SUCCESS

second cond. ended-----
f1*f2^2*f3---f3^2
Not in range of homomorphism
f1*f2^2*f3---f3^2
Not in range of homomorphism
f1*f2^2*f3---f3^2
f3^2*f1*f2^2*f3*f3^2=f1*f2^2*f3SECOND CONDITION SUCCESS

second cond. ended-----
1
gap>

```

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## References

- [1] Alp, M., Wensley, C.D. (2015) Crossed Modules and cat1-groups in GAP Version 2.43, Manual for the XMod share package for GAP4, <http://www.gap-system.org/Packages/xmod.html>.
- [2] Besche, H. U., Eick, B. (1999) The groups of order at most 1000 except 512 and 768, *Journal of Symbolic Computation*, 27 (4): 405-413.
- [3] Eick, B. and Nickel, W. (2000) Polycyclic, A GAP 4.
- [4] Eick, B., Besche, H. U., O'Brien, E. A. (2002) A Millennium Project : Constructing Small Groups, *International Journal of Algebra and Computation*, 12 (05): 623-644.
- [5] Kurosh, A. (1960) *The theory of groups*, American Mathematical Society.
- [6] Moghaddasi, G.H., Tolue, B., Zolfaghari, P. (2016) On the structure of ultra-groups over a finite group,

U.P.B. Sci. Bull. Series A, 78(2): 173-184.

- [7] Moghaddasi, G.H., Zolfaghari, P. (2016) The quotient ultra-groups, Italian journal of pure and applied mathematics, 36: 211-218.
- [8] O'Brien, E. A. (1991) The groups of order 256, Journal of algebra, 143 (1): 219-235.
- [9] Odabas, A., Soylu Yilmaz, E. (2021) 2-Dimensional Groups with Action : The Category of Crossed Module of Groups with Action, Journal of Algebra and its Applications, Under review.
- [10] Odabas, A. (2020) An Algorithm for computing isoclinism families of groups, Comptes Rendus De L Academie Bulgare Des Sciences, 73 (1): 3-12.
- [11] The GAP Group, GAP - Groups, Algorithms, and Programming, (2021) version 4.11.1 (<https://www.gap-system.org>).
- [12] Tolue, B., Moghaddasi, G.H., Zolfaghari, P. (2017) On the category of ultra-groups, Hacettepe Journal of Mathematics and Statistics, 46(3) :437 - 447.
- [13] Tolue, B., Zolfaghari, P. and Moghaddasi, G.H. Free ultra-groups, generators and relations, *Algebra and Discrete Mathematics*, 28 (2) (2019) 308-316.