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Factorable Surfaces in Pseudo-Galilean Space with Prescribed Mean and Gaussian Curvatures

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Abstract — We study the so-called factorable surfaces in the pseudo-Galilean space, the graphs of the product of two functions of one variable. We then classify these surfaces when the mean and Gaussian curvatures are functions of one variable.

Keywords — *Pseudo-Galilean space; factorable surface; Gaussian curvature; mean curvature*

Mathematics Subject Classification (2020) — 53A35, 53C42

1. Introduction

A *Cayley-Klein space* is defined as a real projective space $P(\mathbb{R})$ with a certain absolute figure which is a subset of quadrics and planes. Let $(u_0 : u_1 : u_2 : u_3)$ denote the homogeneous coordinates in $P(\mathbb{R})$. The pseudo-Galilean 3-space G_3^1 that we are interested in is a Cayley-Klein space $P(\mathbb{R})$ with the absolute figure $\{\omega, f, I\}$ such that ω is the absolute plane $u_0 = 0$, f the line $u_0 = u_1 = 0$ and I the fixed hyperbolic involution of points of f. The hyperbolic involution is $(0 : 0 : u_2 : u_3) \mapsto (0 : 0 : u_3 : u_2)$ and $u_2^2 - u_3^2 = 0$ is the absolute conic, [1–5].

Consider the affine coordinates in G_3^1 defined by $(u_0 : u_1 : u_2 : u_3) = (1 : x : y : z)$. Then, a plane of the form $x = d$, $d \in \mathbb{R}$, in G_3^1 is said to be *Lorentzian* since its induced geometry is Lorentzian. We call other planes *isotropic.*

The main purpose of this study, in this special ambient space, is determining the surfaces with prescribed mean (H) and Gaussian (K) curvatures which is a common problem in differential geometry of surfaces. For this, we focus on a graphical surface. Because of the absolute figure of G_3^1, the geometric structure of the surface depends on if it is graph on an isotropic or a Lorentzian plane.

Without lose of generality we may consider the coordinate planes. Hence a graph on the isotropic $xy-$plane (resp. the Lorentzian $yz-$plane) is said to be of *type 1* (*type 2*). Let M be a non-degenerate graph of a smooth function $u = u(s,t)$, $s \in I \subset \mathbb{R}$, $t \in J \subset \mathbb{R}$. If M is of type 1 then it parametrizes $\mathbf{r}(s,t) = (s, t, u(s,t))$ and hence its mean and Gaussian curvatures are given by

$$u_{ss}u_{tt} - u_{st}^2 = -\epsilon K(s,t)\left|1 - u_t^2\right|^2 \tag{1}$$

$$u_{tt} = 2\epsilon H(s,t)\, 2\left|1 - u_t^2\right|^{3/2} \tag{2}$$

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where ϵ is 1 if $1-u_t^2>0$ and -1 otherwise. Here we notice $u_s=\partial u/\partial s$, $u_{st}=\partial^2 u/\partial s\partial t$, and so. If M is of type 2 then parameterizes $\mathbf{r}(s,t)=(u(s,t),s,t)$ and

$$u_{ss}u_{tt}-u_{st}^2=-\epsilon K(s,t)\left|u_s^2-u_t^2\right|^2 \tag{3}$$

$$u_s^2u_{tt}-2u_su_tu_{st}+u_t^2u_{ss}=-2\epsilon H(s,t)\left|u_s^2-u_t^2\right|^{3/2} \tag{4}$$

We point out that the PDEs (1) and (3) are of Monge-Ampère type and their importance is due to economics, meteorology, oceanography etc. [6–11].

In principle, we will concern with the PDEs (1)-(4). Finding their solutions is complicated and one way to reduce their complexity is to use the technique of separation of variables, namely

$$u(s,t)=f(s)+g(t),\quad u(s,t)=f(s)g(t)$$

for smooth functions f,g. Notice that the graphs $u(s,t)=f(s)+g(t)$ are known as *translation surfaces*. The name is because kinematic point of view, obtained by translating one curve along the other one. If the so-called *generating curves* are denoted by $\alpha(s)$ and $\beta(t)$ then

$$\begin{aligned}\mathbf{r}(s,t) &= \alpha(s)+\beta(t)=(s,t,f(s)+g(t))\\ \mathbf{r}(s,t) &= \alpha(s)+\beta(t)=(f(s)+g(t),s,t)\end{aligned}$$

Those surfaces were completely obtained in [12–16] when H and K are a constant function.

Most recently, as a generalization, the present authors [17] classified translation surfaces when H and K are a non-constant function of one variable, that is, $K=K(s)$ and $K=K(t)$ (or $H=H(s)$ and $H=H(t)$). The authors found the motivation in Ruiz-Hernández's paper [18] where the translation hypersurfaces in the Euclidean $n-$space $\mathbb{R}^n$ were obtained when mean and Gauss-Kronocker curvatures depend on its first p (or second q) variables, $p+q=n$. This is indeed, in 3-dimensional setting, a well-known framework for surfaces of revolution or, more generally, helicoidal surfaces due to the fact the mean and Gaussian curvatures only depend on the parameter of the profile curve, see [19, 20].

Following Ruiz-Hernández's idea, we will consider the graphs $u(s,t)=f(s)g(t)$ called *factorable* (or *homothetical*) *surfaces* [21]. These surfaces were studied from various point of view in the (pseudo-) Galilean ambient space, see [22–24].

When $u(s,t)=f(s)g(t)$, the PDEs (1)-(4) that we will solve are now

$$fgf''g''-(f'g')^2=-\epsilon K\left|1-(fg')^2\right|^2 \tag{5}$$

$$fg''=2\epsilon H\left|1-(fg')^2\right|^{3/2} \tag{6}$$

and

$$fgf''g''-(f'g')^2=-\epsilon K\left|(f'g)^2-(fg')^2\right|^2 \tag{7}$$

$$(f'g)^2fg''-2fg(f'g')^2+(fg')^2f''g=-2\epsilon H\left|(f'g)^2-(fg')^2\right|^{3/2} \tag{8}$$

where H and K only depend on s or t and a prime denotes the derivative with respect to the related variable. The Equations (5)-(8) were solved in [13, 25, 26] when K and H are a constant.

In Section 3, we will solve (5) and (6), obtaining the graphs are a cylindrical ruled surface of type 3 from geometric point of view. The detailed properties of ruled surfaces may be found in [2, 27]. We remark that K has to be a function of s in (5) while H has to be a function of t in (6). Contrary to this, the solution of (7) is that, up to a change in the roles of the functions f,g, $f(s)=ae^{bs}$, $K(s)=cb^2f^{-2}(s)$ and $g(t)$ is the solution to the following autonomous differential equation

$$gg''-g'^2=c\left(g'^2-(cg)^2\right)^2,\ a,b\in\mathbb{R},\ a,b,c\neq 0$$

We also provide an example admits a solution when H depends on only one variable.

2. Preliminaries

The *pseudo-Galilean distance* between the points $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ is

$$d(\mathbf{p}, \mathbf{q}) = \begin{cases} |q_1 - p_1|, & p_1 \neq q_1 \\ \sqrt{\left|(q_2 - p_2)^2 - (q_3 - p_3)^2\right|}, & p_1 = q_1 \end{cases}$$

Let $a_1, .., a_5, \varphi$ be some constants. Then, the six-parameter group of motions of G_3^1 which leaves invariant the absolute figure and pseudo-Galilean distance is given in terms of affine coordinates by

$$\begin{aligned} \overline{x} &= a_1 + x \\ \overline{y} &= a_2 + a_3 x + y \cosh\varphi + z \sinh\varphi \\ \overline{z} &= a_4 + a_5 x + y \sinh\varphi + z \cosh\varphi \end{aligned}$$

A line in G_3^1 is said to be *isotropic* if its intersection with the absolute line f is non-empty and *non-isotropic* otherwise. A vector $\mathbf{v} = (v_1, v_2, v_3)$ is said to be *isotropic* (*non-isotropic*) if $v_1 = 0$ ($\neq 0$). Let $\mathbf{w} = (w_1, w_2, w_3)$ and $\langle \cdot, \cdot \rangle_G$ denote the *pseudo-Galilean dot product.* Then, $\langle \mathbf{v}, \mathbf{w} \rangle_G$ is the Lorentzian scalar product if both $\mathbf{v}$ and $\mathbf{w}$ are isotropic. Otherwise, $v_1^2 + w_1^2 \neq 0$, it is defined by $\langle \mathbf{v}, \mathbf{w} \rangle_G = v_1 w_1$. The pseudo-Galilean angle between $\mathbf{v}$ and $\mathbf{w}$ is defined as the Lorentzian angle if $\mathbf{v}$ and $\mathbf{w}$ are isotropic. Otherwise, it is given by the pseudo-Galilean distance. We call $\mathbf{v}$ and $\mathbf{w}$ *orthogonal* if $\langle \mathbf{v}, \mathbf{w} \rangle_G = 0$.

An isotropic vector $\mathbf{v}$ is called *spacelike* if $\langle \mathbf{v}, \mathbf{v} \rangle_L > 0$; *timelike* if $\langle \mathbf{v}, \mathbf{v} \rangle_L < 0$ and *lighlike* if $\langle \mathbf{v}, \mathbf{v} \rangle_L = 0$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be standard basis vectors and $\mathbf{v}$ and $\mathbf{w}$ no both isotropic vectors. Then, the *pseudo-Galilean cross-product* is

$$\mathbf{v} \times_G \mathbf{w} = \begin{vmatrix} 0 & -\mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Then, $\langle \mathbf{v} \times_G \mathbf{w}, \mathbf{z} \rangle_G = -\det(\mathbf{v}, \mathbf{w}, \tilde{\mathbf{z}})$, where $\tilde{\mathbf{z}}$ is the projection of $\mathbf{z}$ onto the $yz-$plane. Note that the vector $\mathbf{v} \times_G \mathbf{w}$ is orthogonal to the vectors $\mathbf{v}$ and $\mathbf{w}$.

Let S be a surface in G_3^1 locally given by a regular map

$$(u_1, u_2) \mapsto \mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)), \quad (u_1, u_2) \in D \subset \mathbb{R}^2$$

Denote $x_{,i} = \frac{\partial x}{\partial u_i}$ and $x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}$ and etc., $1 \leq i, j \leq 2$. Then, S is said to be *admissible* if $x_{,i} \neq 0$ for some $i = 1, 2$. For such an admissible surface S, the *first fundamental form* is

$$\langle d\mathbf{x}, d\mathbf{x} \rangle_G = E du_1^2 + 2F du_1 du_2 + G du_2^2$$

where $E = (x_{,1})^2$, $F = x_{,1} x_{,2}$, $G = (x_{,2})^2$. Since nowhere an admissible surface has Lorentzian tangent plane, up to the absolute figure, the isotropic vector $\mathbf{x}_{,1} \times_G \mathbf{x}_{,2}$ is *normal* to S. Let

$$W = \langle \mathbf{x}_{,1} \times_G \mathbf{x}_{,2}, \mathbf{x}_{,1} \times_G \mathbf{x}_{,2} \rangle_L$$

Then, the surface S is called *spacelike* if $W < 0$; *timelike* if $W > 0$; and *lightlike* if $W = 0$. The spacelike and timelike surfaces are so-called *non-degenerate* and, throughout this study, we deal with the only non-degenerate admissible surfaces. The *unit normal vector* to the non-degenerate surface S is

$$\mathbf{N} = \frac{\mathbf{x}_{,1} \times_G \mathbf{x}_{,2}}{\sqrt{|W|}}$$

Let $\epsilon = \langle \mathbf{N}, \mathbf{N} \rangle_L = \pm 1$ and

$$L_{ij} = \epsilon \frac{1}{x_{,1}} \left\langle x_{,1} \tilde{\mathbf{x}}_{,ij} - (x_{,i})_{,j} \tilde{\mathbf{x}}_{,1}, \mathbf{N} \right\rangle_L = \epsilon \frac{1}{x_{,2}} \left\langle x_{,2}\, \tilde{\mathbf{x}}_{,ij} - (x_{,i})_{,j} \tilde{\mathbf{x}}_{,2}, \mathbf{N} \right\rangle_L$$

in which one of $x_{,1}$ and $x_{,2}$ is always nonzero due to the admissibility. Then, the *second fundamental form* of S is

$$II = Ldu_1^2 + 2Mdu_1du_2 + Ndu_2^2$$

where $L = L_{11}$, $M = L_{12}$, $N = L_{22}$. Thereby, *the Gaussian and mean curvatures* are defined by

$$K = -\epsilon\frac{LN - M^2}{W} \text{ and } H = -\epsilon\frac{GL - 2FM + EN}{2W}$$

A surface is said to be *minimal* if H vanishes identically.

Let S be a ruled surface in G_3^1 locally given by

$$\mathbf{x}(u_1, u_2) = \gamma(u_1) + u_2 w(u_1)$$

where $\gamma(u_1)$ is a regular curve and $w(u_1)$ is a nonvanishing vector field along $\gamma(u_1)$. There are three types of such surfaces depending on the positions of $\gamma(u_1)$, $w(u_1)$ and the absolute figure:

- **Type 1:** $w(u_1)$ is non-isotropic and $\gamma(u_1)$ does not lie in a pseudo-Euclidean plane.
- **Type 2:** $w(u_1)$ is non-isotropic and $\gamma(u_1)$ lies in a pseudo-Euclidean plane.
- **Type 3:** $w(u_1)$ is isotropic and $\gamma(u_1)$ is an arbitrary curve lying a plane orthogonal to $w(u_1)$.

3. The Graphs of Type 1

Let the graph $z(x,y) = f(x)g(y)$ be of type 1, then

$$\mathbf{r}(x,y) = (x, y, f(x)g(y)), \ \ (s,t) \in I \times J \subset \mathbb{R}^2$$

The Gaussian curvature is

$$K = \frac{fgf''g'' - (f'g')^2}{\left[1 - (fg')^2\right]^2} \tag{9}$$

where a prime denotes the derivative with respect to the related variable. We study the case that K is a non-constant function; that is, at least a partial derivative of K with respect to x and y is non-vanishing. It is equivalent to the statement that the first derivatives of both f and g are nonzero on $I \times J$.

In the following we obtain the graphs with $K(x,y) = k(x)$ where $k(x)$ is some smooth function of x.

Theorem 3.1. If the Gaussian curvature $K(x,y)$ of the graph $z(x,y) = f(x)g(y)$ is a non-constant function depending one variable then it is of the form $K(x,y) = k(x)$, where $k(x)$ is some negative smooth function of x. Furthermore, the graph is a cylindrical ruled surface of type 3 such that, up to a translation of y,

$$z(x,y) = \pm y \tanh\left(\pm\int^x \sqrt{-k(s)}ds\right)$$

Proof. Since there is not a symmetry in Equation (9) up to the functions f and g, we distinguish two cases:

1. **Case** $(\partial K/\partial y)(x,y) = 0$. We set $K(x,y) = k(x)$, for some smooth function $k(x)$. Then,

$$k(x) = \frac{fgf''g'' - (f'g')^2}{\left[1 - (fg')^2\right]^2} \tag{10}$$

Here, if there is a point $y_0 \in J$ such that $g''(y_0) = 0$ then we may assume $g'' = 0$ on a neighborhood of y_0 in J. Letting $g' = c$, $c \neq 0$, (10) is now

$$k(x) = \frac{-c^2 f'^2}{(1 - c^2 f^2)^2}$$

which implies that $k(x)$ is a negative function. Notice that, up to a translation of y, the graph is written by

$$\mathbf{r}(x, y) = (x, 0, 0) + y(0, 1, cf(x))$$

which is a parametrization of a ruled surface of type 3. Here $f(x)$ is the solution to

$$\frac{cf'}{1 - c^2 f^2} = \pm\sqrt{-k(x)}$$

Integrating gives

$$f(x) = |c|^{-1} \tanh\left(\pm \int^x \sqrt{-k(s)}ds\right)$$

which proves the result. Next, we will show that Equation (10) has no a solution provided $g''(y) \neq 0$ on J. In order to overcome difficulties in our calculations we introduce

$$\begin{aligned} \alpha_1 &= ff'' \\ \alpha_2 &= -f'^2 \\ \alpha_3 &= -f^2 \end{aligned}$$

Then, Equation (10) turns into

$$k(x) = \frac{\alpha_1 gg'' + \alpha_2 g'^2}{(1 + \alpha_3 g'^2)^2} \tag{11}$$

We observe two subcases;

(a) **Subcase** $\alpha_1 \neq 0$. Then,

$$\frac{k(x)}{\alpha_1}(1 + \alpha_3 g'^2)^2 - \left(\frac{\alpha_2}{\alpha_1}\right) g'^2 = gg'' \tag{12}$$

By taking derivative of Equation (12) with respect to x we obtain

$$\sum_{n=0}^{4} P_n (g')^n = 0$$

where

$$\begin{aligned} P_0 &= (k(x)/\alpha_1)' \\ P_1 &= 0 \\ P_2 &= 2(k(x)/\alpha_1)'\alpha_3 + 2(k(x)/\alpha_1)\alpha_3' - (\alpha_2/\alpha_1)' \\ P_3 &= 0 \\ P_4 &= (k(x)/\alpha_1)'\alpha_3^2 + 2(k(x)/\alpha_1)\alpha_3\alpha_3' \end{aligned}$$

Because of the fact that g' is a non-constant function of y, $P_0, ..., P_4$ are zero all. From $P_0 = 0$ we get $(k(x)/\alpha_1)' = 0$, implying the existence of some nonzero constant c such that $k(x) = c\alpha_1$. Then, using this in $P_4 = 0$ we obtain $4cf^3 f' = 0$ which allows $k(x)$ to be zero. This is not possible by our assumption.

(b) **Subcase** $\alpha_1 = 0$. Then, because $\alpha_1 = ff''$, concludes $\alpha_2 = c$, $c \neq 0$. Therefore considering this in Equation (11) and next taking derivative with respect to y we obtain $g'' = 0$, which is a contradiction.

2. **Case** $(\partial K/\partial x)(x, y) = 0$. We set $K(x, y) = k(y)$, for some smooth function $k(y)$. Then, follows

$$k\,(y) = \frac{\alpha_1 g g'' + \alpha_2 g'^2}{\left[1 + \alpha_3 g'^2\right]^2} \tag{13}$$

where if g is a linear function then the left hand side of (13) is a function of y and the other side is a function of x. Thus, g cannot be a linear function by our assumption. Next, let $g'' \neq 0$. Then, by replacing $k\,(x)$ with $k\,(y)$ in Equation (12) we may easily show that no (13) has a solution.

□

Example 3.2. Take $z(x, y) = f(x)g(y)$ with $K(x, y) = -4x^2$. By Theorem 3.1, we have $z(x, y) = y \tanh(x^2)$ up to a translation of x. One can be drawn as in Fig. 1.

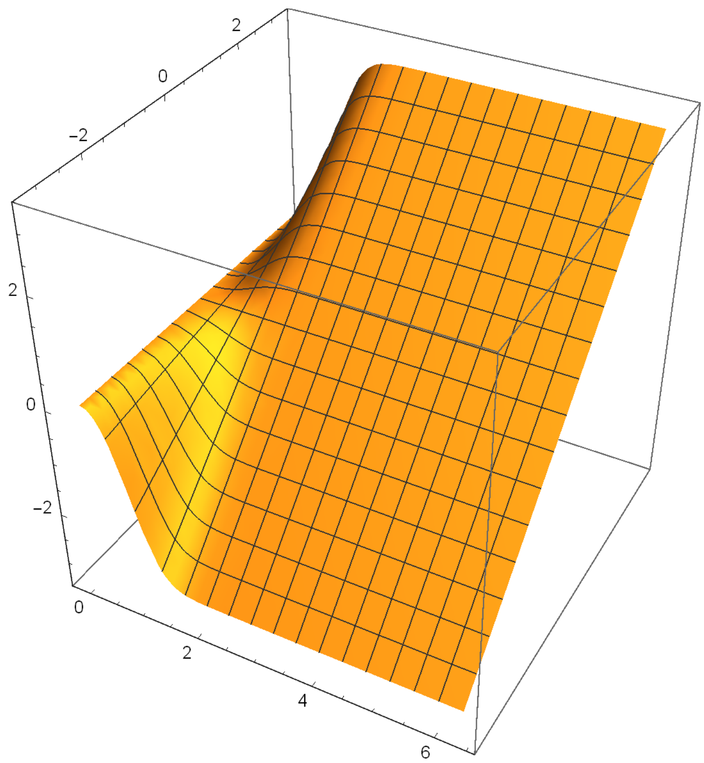

Fig. 1. Graph of $z(x, y) = y \tanh(x^2)$ with $0 \leq x \leq 2\pi$ and $-\pi \leq y \leq \pi$. The Gaussian curvature is $K(x, y) = x^2$.

We are also interested in the graphs $z\,(x, y) = f\,(x)\,g\,(y)$ whose mean curvature is non-vanishing function of one variable and present the following results:

Theorem 3.3. If the mean curvature H of the graph $z\,(x, y) = f\,(x)\,g\,(y)$ is a non-constant function depending one variable then it is of the form $H(x, y) = h(y)$, where $h(y)$ is some smooth function of y, and the graph is a cylindrical ruled surface of type 3 such that

$$z\,(x, y) = cg\,(y) = 2\int^{y} q(s)\,(1 + 4q(s))^{-1/2}\,ds \tag{14}$$

where $q(y) = \int^{y} h\,(s)\,ds$ and c is a non-zero constant.

Proof. We distinguish two cases:

1. $(\partial H/\partial x)(x,y)=0$. We set $H(x,y)=h(y)$, for some smooth function $h(y)$. Then, we have

$$h\left(y\right)=\frac{fg''}{2\left[1-\left(fg'\right)^{2}\right]^{3/2}} \tag{15}$$

   Suppose that f is non-constant function. By squaring we write

$$4h^{2}\left(y\right)\left[1-\left(fg'\right)^{2}\right]^{3}-f^{2}g''^{2}=0$$

   which is a polynomial equation of degree 6 on f whose leading coefficient is $-4h^{2}\left(y\right)g'^{6}$. Since $h\left(y\right)$ is nowhere vanish, we obtain a contradiction. Therefore we have $f=c$, where c is a nonzero constant. So the graph parameterizes as

$$x\left(1,0,0\right)+\left(0,y,cg\left(y\right)\right)$$

   which is locally a cylindrical ruled surface of type 3. The result follows by integrating (15).

2. **Case** $(\partial H/\partial y)(x,y)=0$. We set $H(x,y)=h(x)$, for some smooth function $h(x)$. Then, follows

$$h\left(x\right)=\frac{fg''}{2\left[1-\left(fg'\right)^{2}\right]^{3/2}} \tag{16}$$

   in which f cannot be constant function because otherwise the left hand side is a function of x and the other side is a function of y. In such a case, divide (16) with $f/2$ and derivative with respect to y, obtaining

$$\left(3g'g''^{2}-g'^{2}g'''\right)f^{2}+g'''=0$$

   Here is clear that the coefficients are not zero all, which is not possible. This completes the proof.

□

4. The Graphs of Type 2

Consider the graph $x(y,z)=f(y)g(z)$, $(y,z)\in I\times J$. Then,

$$\mathbf{r}\left(y,z\right)=\left(f\left(y\right)g\left(z\right),y,z\right),\ \ (s,t)\in I\times J\subset\mathbb{R}^{2}$$

The Gaussian curvature is

$$K(y,z)=\frac{fgf''g''-\left(f'g'\right)^{2}}{\left[\left(fg'\right)^{2}-\left(f'g\right)^{2}\right]^{2}} \tag{17}$$

Here we point out $f',g'\neq 0$ on $I\times J$ because K is a non-constant function in our case.

Thus we present the following result:

Theorem 4.1. If the Gaussian curvature K of the graph $x(y,z)=f(y)g(z)$ is a non-constant function of one variable then, up to a change in the roles of the functions f,g, $f\left(y\right)=ae^{by}$ and the following autonomous differential equation holds

$$gg''-g'^{2}=c\left(g'^{2}-(cg)^{2}\right)^{2},\ \ a,b\in\mathbb{R},\ \ a,b,c\neq 0$$

Furthermore, $K=c\left(ab^{-1}e^{by}\right)^{-2}$.

PROOF. Assume K is a non-constant function of one variable. Notice that the roles of f and g in Equation (17) are symmetric and therefore we only focus on the case $K = k(y)$. We previously observe the following cases:

1. **Case** $f(y) = cy + d,\ c, d \in \mathbb{R},\ c \neq 0$. Then, Equation (17) is now

$$\pm\sqrt{-k(y)} = \frac{cg'}{(fg')^2 - (cg)^2} \tag{18}$$

The partial derivative of (18) with respect to z is

$$-cg'^2 g'' f^2 - c^3 g^2 g'' + 2c^3 g g'^2 = 0$$

which is a polynomial equation of degree 2 on f. Because k is nonzero function, the coefficients are not zero all, which is a contradiction. Then, f cannot be a linear function. Similarly, if g is a linear function then we easily arrive the contradiction $f' = 0$. This discussion gives us that f and g must be non-linear functions.

2. **Case** $f' = cf^d,\ c, d \in \mathbb{R},\ c, d \neq 0.$. Then, the following two sub-cases are provided:

(a) **Subcase** $d = 1$. Then, $f(y) = ae^{cy},\ a \in \mathbb{R},\ a \neq 0$. Equation (17) is

$$c^{-2} f^2 k(y) = \frac{gg'' - g'^2}{[g'^2 - (cg)^2]^2} \tag{19}$$

where the left hand side is a function of y and the other hand side is a function of z. Then, there exists a nonzero constant λ such that $c^{-2} f^2 k(y) = \lambda$ and

$$gg'' - g'^2 = \lambda \left(g'^2 - (cg)^2\right)^2$$

which gives the result.

(b) **Subcase** $d \neq 1$. Equation (17) imply that

$$(c^2 f^{2d-4})^{-1} k(y) = \frac{dgg'' - g'^2}{\left(g'^2 - c^2 f^{2d-2} g^2\right)^2} \tag{20}$$

Letting $dgg'' - g'^2 = T$ and next derivating with respect to z of Equation (20) we obtain

$$T'g'^2 - 4Tg'g'' - c^2\left(T'g^2 - 4Tgg'\right) f^{2d-2} = 0$$

which is polynomial equation on f. Using the equality of coefficients we derive following equations

$$T'g'^2 - 4Tg'g'' = 0$$
$$T'g^2 - 4Tgg' = 0$$

From these equations, we have the contradiction $g' = bg,\ b = const.$

3. **Case** that f is neither a linear function nor of the form $f' = cf^d$. Then, derivating of (17) with respect to z

$$ff''\left[\left(g'g'' + gg'''\right)\left(g'^2 - g^2\right) - 4gg''\left(g'g'' - gg'\right)\right] + 4f'^2 g'^2\left(g'g'' - gg'\right) - 2f'^2 g'g''\left(g'^2 - g^2\right) = 0$$

Since $f' \neq 0$, we obtain

$$\frac{ff''}{f'^2} = \frac{-4g'^2\left(g'g'' - gg'\right) + 2g'g''(g'^2 - g^2)}{\left(g'g'' + gg'''\right)\left(g'^2 - g^2\right) - 4gg''\left(g'g'' - gg'\right)}$$

The left hand side is a function of y while the other side is a function of z. Then, both hand sides are a constant, which contradicts with our assumption.

□

The mean curvature of the graph $x(y,z) = f(y)g(z)$ is given by

$$H = \frac{(fg')^2 f''g - 2fg(f'g')^2 + (f'g)^2 fg''}{2\left[(fg')^2 - (f'g)^2\right]^{3/2}} \tag{21}$$

Apart from the previous results, we could not completely solve our problem when $H(y,z)$ is nonconstant and

$$\frac{\partial H(y,z)}{\partial y} = 0 \text{ or } \frac{\partial H(y,z)}{\partial z} = 0 \tag{22}$$

But we have an example indicating the existence of the graphs $x(y,z) = f(y)g(z)$ when Equation (22) holds.

Example 4.2. Let $a, b, c \in \mathbb{R} - \{0\}$, up to a change in the roles of the functions f, g, $g(z) = ae^{bz}$ and

$$f(y) = c\exp\left(\pm \int^y \left(\left(2a\int^y h(s)ds\right)^{-2} + a^{-2}\right)^{-1/2} ds\right) \tag{23}$$

Then, the mean curvature of the graph $x(y,z) = f(y)g(z)$ only depends on the variable y.

For the solution of Example, we set $\alpha = \frac{f}{f'}$ and $\beta = \frac{g}{g'}$. Then, Equation (21) is now

$$H(x,y) = -\frac{\alpha\beta(\alpha' + \beta')}{2(\alpha^2 - \beta^2)^{3/2}} \tag{24}$$

Up to a change in the roles of the functions f, g, suppose that β is constant, or equivalently, $g(z) = ae^{bz}$, for some nonzero constants a, b. Equation (24) is

$$-2aH(x,y) = \frac{\alpha\alpha'}{(\alpha^2 - a^{-2})^{3/2}} \tag{25}$$

which means that $H(x,y)$ only depends on the variable y. Put $H(x,y) = h(y)$, for some smooth function $h(y)$. Integrating Equation (25) gives the result of Example.

5. Conclusion

In this paper factorable surfaces are classified when the mean and Gaussian curvatures are functions of one variable in the pseudo-Galilean space. We have obtained that if the Gaussian curvature K of the graph $z(x,y) = f(x)g(y)$ is a non-constant function of one variable then it is a negative function of the form $K = k(x)$. Furthermore, the graph is a cylindrical ruled surface of type 3 and if the mean curvature H of the graph $z(x,y) = f(x)g(y)$ is a non-constant function of one variable then it is of the form $H = h(y)$ and the graph is a cylindrical ruled surface of type 3. Also we have shown that there does not exist a graph $x(y,z) = f(y)g(z)$ in G_3^1 when its mean curvature depends on one variable.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] O. Giering, Vorlesungen über höhere Geometrie, Friedr Vieweg & Sohn, Braunschweig, Germany, 1982.

[2] B. Divjak and Z. Milin-Sipus, *Special Curves on Ruled Surfaces in Galilean and Pseudo-Galilean Spaces*, Acta Mathematica Hungarica 98 (1) (2003) 203–215.

[3] E. Mólnar, *The Projective Interpretation of the Eight 3-Dimensional Homogeneous Geometries*, Beitrage zur Algebra und Geometrie 38 (2) (1997) 261–288.

[4] A. Onishchick and R. Sulanke, Projective and Cayley-Klein Geometries, Springer, 2006.

[5] I. M. Yaglom, A Simple Non-Euclidean Geometry and Its Physical Basis, Springer-Verlag, New York, 1979.

[6] B. Y. Chen, G. E. Vîlcu, *Geometric Classifications of Homogeneous Production Functions*, Applied Mathematics and Computation 225 (2013) 345–351.

[7] B. Y. Chen, *A Note on Homogeneous Production Models*, Kragujevac Journal of Mathematics 36 (1) (2012) 41–43.

[8] B. Y. Chen, *Solutions to Homogeneous Monge-Ampère Equations of Homothetic Functions and Their Applications to Production Models in Economics*, Journal of Mathematical Analysis and Applications 411 (2014) 223–229.

[9] M. J. P. Cullen, R. J. Douglas, *Applications of the Monge-Ampè re equation and Monge transport problem to meterology and oceanography*, In: L. A. Caffarelli, M. Milman (eds.), NSF-CBMS Conference on the Monge Amp‘ere Equation, Applications to Geometry and Optimization, July 9-13, Florida Atlantic University, 1997, pp. 33–54.

[10] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Berlin, Springer-Verlag, 1983.

[11] V. Ushakov, *The Explicit General Solution of Trivial Monge-Ampère Equation*, Commentarii Mathematici Helvetici 75 (2000) 125–133.

[12] M. E. Aydın, M. Alyamac Külahcı, A.O. Öğrenmis, *Constant Curvature Translation Surfaces in Galilean 3-Space*, International Electronic Journal of Geometry 12 (1) (2019) 9–19.

[13] A. Kelleci, *Translation-Factorable Surfaces with Vanishing Curvatures in Galilean 3-Spaces,* International Journal of Maps in Mathematics 4 (1) (2021) 14–26.

[14] Z. Milin-Sipus, B. Divjak, *Translation Surface in the Galilean Space*, Glasnik Matematicki 46 (66) (2011) 455–469.

[15] Z. Milin-Sipus, *On a Certain Class of Translation Surfaces in a Pseudo-Galilean Space*, International Mathematical Forum 6 (23) (2011) 1113–1125.

[16] D. W. Yoon, *Some Classification of Translation Surfaces in Galilean 3-Space*, International Journal of Mathematical Analysis 6 (28) (2012) 1355–1361.

[17] M.E. Aydın, S. Aykurt Sepet, H. Gün Bozok, *Translation Surfaces in Pseudo-Galilean Space with Prescribed Mean and Gaussian Curvatures*, Honam Mathematical Journal 44 (1) (2022) 36–51.

[18] G. Ruiz-Hernández, *Translation Hypersurfaces whose Curvature Depends Partially on Its Variables*, Journal of Mathematical Analysis and Applications 497 (2) (2021) 124913.

[19] C. Baikoussis, T. Koufogioros, *Helicoidal Surface with Prescribed Mean or Gauss Curvature*, Journal of Geometry 63 (1998) 25–29.

[20] K. Kenmotsu, *Surface of Revolution with Prescribed Mean Curvature*, Tohoku Mathematical Journal 32 (1980) 147–153.

[21] I. Van de Woestyne, *Minimal Homothetical Hypersurfaces of a Semi-Euclidean Space*, Results in Mathematics 27 (1995) 333–342.

[22] H. S. Abdel-Aziz, M. Khalifa Saad, A. Ali Haytham, *Affine Factorable Surfaces in Pseudo-Galilean Space*, arXiv:1812.00765v1[math.GM].

[23] P. Bansal, M. H. Shahid, *On Classification of Factorable Surfaces in Galilean Space* G_3, Jordan Journal of Mathematics and Statistics 12 (3) (2019) 289–306.

[24] M. S. Lone, *Homothetical Surfaces in Three Dimensional Pseudo-Galilean Spaces Satisfying* $\triangle^{II} \mathbf{x}_i = \lambda_i \mathbf{x}_i$, Advances in Applied Clifford Algebras 29 (92) (2019).

[25] M. E. Aydın, A. O. Öğrenmis, M. Ergüt, *Classification of Factorable Surfaces in the Pseudo-Galilean Space*, Glasnik Matematicki 70 (50) (2015) 441–451.

[26] M. E. Aydın, M. Alyamac Külahcı, A. O. Öğrenmis, *Non-Zero Constant Curvature Factorable Surfaces in Pseudo-Galilean Space*, Communications of the Korean Mathematical Society 33 (1) (2018) 247–259.

[27] B. Divjak, Z. Milin-Sipus, *Minding Isometries of Ruled Surfaces in Pseudo-Galilean Space*, Journal of Geometry 77 (2003) 35–47.