Turk. J. Math. Comput. Sci.
15(2)(2023) 258–269
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DOI : 10.47000/tjmcs.1138229

Spectral Properties of Discontinuous Boundary Value Problem with Transmission Conditions and Manypoint Singularities

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Received: 03-07-2022 • Accepted: 29-09-2022

Abstract. In this study, we investigate a discontinuous Sturm-Liouville boundary value problem on three intervals with manypoint-transmission conditions in direct sum of Sobolev space. We establish such spectral properties as Fredholmness and coreciveness with respect to the eigenvalue parameter.

2010 AMS Classification: 34A36, 34B08, 34B10, 34B24, 34L10

Keywords: Boundary-value problem, multipoint-transmission conditions, isomorphism, Fredholm, solvability.

1. Introduction

In recent years, many researchers have been interested in the discontinuous boundary value problems for their application in physics as regards theoretical investigations. The discontinuity of the coefficients of the equations in the boundary value problems relates to the fact that the nonhomogeneous media consists of two or more different materials. On the other hand, transmission boundary value problems together with additional transmission conditions appear frequently in various fields such as in electrostatics, magnetostatics and in solid mechanic for discontinuous problems (in these regard see, [8, 28]). Solvability and some spectral properties of nonlocal Sturm-Liouville problems have been investigated by many authors, see for example, [5, 6, 18, 20]. Various generalizations of classical boundary value problems for ordinary linear differential equations have attracted a lot of attention in recent years because of the appearance of new important applications in physical sciences and applied mathematics. An important special case of the generalized boundary value problems are so-called multipoint boundary value problems. Such problems have been extensively studied by many authors, see for example, [7, 16, 18]. Some of the mathematical problems encountered in the study of boundary value transmission problems or nonclassical problems cannot be treated with the usual techniques within the standard framework of boundary value problems. In classical theory, boundary-value problems for ordinary differential equations are usually considered for equations with continuous coefficients and for boundary conditions which contain only endpoints of the considered interval. However, this paper deals with one nonclassical boundary-value problem for a second order ordinary differential equation with discontinuous coefficients at two points and boundary conditions including not only endpoints of the considered interval, but also two point of discontinuity and finite number of internal points.

In this paper, we consider boundary value transmission problem for Sturm-Liouville equation given by

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$$L(\lambda)u := t(x)u''(x) + (k(x) - \lambda^2)u(x) = f(x), x \in [-1, d_1) \cup (d_1, d_2) \cup (d_2, 1], \tag{1.1}$$

with many point boundary conditions on intervals $[-1, d_1)$, (d_1, d_2) and $(d_2, 1]$ given by

$$L_1 u := \alpha_1 u^{(m_1)}(-1) + \beta_1 u^{(m_1)}(d_1 - 0) + \sum_{s=1}^{n_1} \gamma_{1s} u^{(m_1)}(x_{1s}) = h_1, \tag{1.2}$$

$$L_2 u := \alpha_2 u^{(m_2)}(d_2 + 0) + \beta_2 u^{(m_2)}(1) + \sum_{s=1}^{n_2} \gamma_{2s} u^{(m_2)}(x_{2s}) = h_2, \tag{1.3}$$

respectively, and additional transmission conditions at the interior points of interaction d_1, d_2 given by

$$\begin{aligned} L_j u \quad &: \quad = \alpha_j u^{(m_j)}(d_k - 0) + \beta_j u^{(m_j)}(d_k + 0) + \sum_{s=1}^{n_j} \gamma_{js} u^{(m_j)}(x_{js}) = h_j, \\ k \quad &= \quad 1 \text{ for } j = 3, 4 \text{ and } k = 2 \text{ for } j = 5, 6, \end{aligned} \tag{1.4}$$

where $-1 < d_1 < 0 < d_2 < 1, \alpha_j, \beta_j, \gamma_{js}, h_j$ are given complex numbers; $f(x)$ is given real-valued function; $|\alpha_i| + |\beta_i| \neq 0$, $|\alpha_{ik}| + |\beta_{ik}| \neq 0$; $t(x) = t_1$ for $x \in [-1, d_1)$, $t(x) = t_2$ for $x \in (d_1, d_2)$, $t(x) = t_3$ for $x \in (d_2, 1]$, $x_{js} \in [-1, d_1) \cup (d_1, d_2) \cup (d_2, 1]$ are internal points; $k(x)$ is integrable function on $[-1, d_1) \cup (d_1, d_2) \cup (d_2, 1]$.
Such type nonstandard discontinuous boundary-value problems arise with the implementation of the method of separation of variables to the varied assortment of physical problems, namely, in heat and mass transfer problems (see, for example, [8, 18, 19, 24]), in diffraction problems (for example, [14]), in vibrating string problems, when the string loaded additionally with point masses (see, [20, 21]) and etc. Also, some discontinuous problems with manypoint and transmission conditions which arise in mechanics in the sense of thermal condition problem for a thin laminated plate were studied in (see, [18, 19, 24]). Spectral properties, coercive and solvability of boundary value problems in Sobolev spaces can be found in some works of Sadybekov, Agranovich, Imanbaev, Shakhmurov and Aliyev (for example, [1, 3, 9, 10, 13, 17, 22, 23, 26, 28]). Some boundary-value problems for differential equations with discontinuous coefficients were investigated by Rasulov in monographs (see, [21]). Various spectral properties of some transmission problems and its applications were investigated by authors Mukhtarov, Ya. Yakubov, Kandemir, Aydemir and some others (for example, [2, 4, 10–14, 19, 22, 27]).

From ([25], p. 186, Th. 2 formula 16) we consider in the following spaces:

Let s_0 and s_1 be non-negative integers $0 \leq s_0, s_1 < \infty$, $s_0 \neq s_1$, $1 < p < \infty$, $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \theta < 1$, $s = (1 - \theta)s_0 + \theta s_1$ and $\Omega \subset \mathbb{R}^n$. Then the following interpolation of Sobolev space is called Besov space.

$$B^s_{p,q}(\Omega) = (W^{s_0}_p(\Omega), W^{s_1}_p(\Omega))_{\theta,q}.$$

These spaces are Besov spaces with the definition by interpolation of Sobolev spaces ([25], p. 186, Th. 2 and [28]).

The Sobolev space $W^m_p(\Omega)$ is Banach space that consists of $u \in L_p(\Omega)$ for which the following norm

$$\|u\|_{W^m_p(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L_p(\Omega)} \right)^{\frac{1}{p}}.$$

Naturally $W^0_p(\Omega) = L_p(\Omega)$.

Let $0 \leq s_0, s_1$ are integers, $s_0 \neq s_1$, $1 < p < \infty$, $0 < \theta < 1$, $s = (1 - \theta)s_0 + \theta s_1$, then by virtue of from ([25], p. 186, Th. 2, formula 16)

$$B^s_p(\Omega) = W^s_p(\Omega) = (W^{s_0}_p(\Omega), W^{s_1}_p(\Omega))_\theta$$

and $1 < p_0, p_1 < \infty$,

$$L_p(\Omega) = (L_{p_0}(\Omega), L_{p_1}(\Omega))_{\theta,p}.$$

From ([25], p. 186, Remark 5). Where $1 < p_0, p_1 < \infty$

$$\|u\|_{B^s_p(\Omega)} \le c(\|u\|^{1-\theta}_{B^{s_0}_p(\Omega)} \|u\|^{\theta}_{B^{s_1}_p(\Omega)}, \ u \in B^{s_0}_p(\Omega) \cap B^{s_1}_p(\Omega),$$

and ([28], p. 20 Lemma 2.4]) for $0 \le s \le l$, $1 < p < \infty$ and $\lambda \in \mathbb{C}$

$$|\lambda|^{l-s} \|u\|_{W^s_p(\Omega)} \le C(\|u\|_{W^l_p(\Omega)} + |\lambda|^l \|u\|_{L_p(\Omega)}.$$

2. Solvability and Coerciveness of Homogeneous Equation with Nonhomogeneous Transmission Conditions

First, we will consider the following boundary value problem which consist of the homogeneous differential equation

$$L_0(\lambda)u := t(x)u''(x) - \lambda^2 u(x) = 0 \tag{2.1}$$

and nonhomogeneous boundary-transmission conditions

$$L_{10}u := \alpha_1 u^{(m_1)}(-1) + \beta_1 u^{(m_1)}(d_1 - 0) = h_1, \tag{2.2}$$

$$L_{20}u := \alpha_2 u^{(m_2)}(d_2 + 0) + \beta_2 u^{(m_2)}(1) = h_2, \tag{2.3}$$

$$\begin{array}{rcl} L_{j0}u & : & = \alpha_j u^{(m_j)}(d_k - 0) + \beta_j u^{(m_j)}(d_k + 0) = h_j, \\ k & = & 1 \text{ for } j = 3,4 \text{ and } k = 2 \text{ for } j = 5,6. \end{array} \tag{2.4}$$

We will consider the boundary value problem (2.1)-(2.4) with nonhomogeneous transmission conditions.

The notations and definitions which we consider throughout the paper are as following:
$\varphi_{jk} = (-1)^k \frac{1}{\sqrt{t_j}}$, $j = 1,2,3$, $k = 1,2$; $\underline{\varphi} := \min\{\arg t_1, \arg t_2, \arg t_3\}$, $\overline{\varphi} := \max\{\arg t_1, \arg t_2, \arg t_3\}$;

$$\theta = \begin{vmatrix} \alpha_1\varphi_{11}^{m_1} & \beta_1\varphi_{12}^{m_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2\varphi_{31}^{m_2} & \beta_2\varphi_{32}^{m_2} \\ 0 & \alpha_3\varphi_{12}^{m_3} & \beta_3\varphi_{21}^{m_3} & 0 & 0 & 0 \\ 0 & \alpha_4\varphi_{12}^{m_4} & \beta_4\varphi_{21}^{m_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5\varphi_{22}^{m_5} & \beta_5\varphi_{31}^{m_5} & 0 \\ 0 & 0 & 0 & \alpha_6\varphi_{22}^{m_6} & \beta_6\varphi_{31}^{m_6} & 0 \end{vmatrix},$$

$$D(\varepsilon,\lambda) := \left\{ \lambda \in \mathbb{C} \middle| \ \pi + \overline{\varphi} < \arg\lambda < 3\pi + \underline{\varphi} - \varepsilon \right\}.$$

Below, the direct sum of Sobolev spaces $W^k_q(-1,d_1)\dot{+}W^k_q(d_1,d_2)\dot{+}W^k_q(d_2,1)$ (for an integer $k \ge 0$ and real $q > 1$) is defined as Banach space of complex-valued functions $u = u(x)$ defined on $[-1,d_1) \cup (d_1,d_2) \cup (d_2,1]$ which belong to $W^k_q(-1,d_1)$, $W^k_q(d_1,d_2)$ and $W^k_q(d_2,1)$ on intervals $(-1,d_1),(d_1,d_2)$ and $(d_2,1)$,respectively, with the norm

$$\|u\|_{W^k_q(-1,1)} = \|u\|_{W^k_q(-1,d_1)} + \|u\|_{W^k_q(d_1,d_2)} + \|u\|_{W^k_q(d_2,1)}$$

Here, as usual, $W^k_q(a,b)$ is the Sobolev space, i.e. the Banach space consisting of all measurable functions $u(x)$ that have generalized derivatives up to k-th order inclusive on the interval (a,b) with the infinite norm

$$\|u\|_{W^k_q(a,b)} = \sum_{i=0}^{k} \left(\int_a^b \left|u^{(i)}(x)\right|^q dx \right)^{\frac{1}{q}}, \ k \ge 0 \text{ and } q > 1.$$

Theorem 2.1. *If $\theta \ne 0$ then, for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that for all $\lambda \in D(\varepsilon,\lambda)$ for which $|\lambda| > r_\varepsilon$, the problem (2.1)- (2.4) has a unique solution $u(x,\lambda)$ that belongs to $W^k_q(-1,d_1)\dot{+}W^k_q(d_1,d_2)\dot{+}W^k_q(d_2,1)$ for arbitrary $n \ge \max\{2, \max\{m_1,m_2,m_3,m_4,m_5,m_6\} + 1\}$ and for these λ the coercive estimate*

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u\|_{W^k_q(-1,1)} \le C(\varepsilon) \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|h_j\right|$$

is valid.

Proof. We shall define six basic solutions $u_{jk} = u_{jk}(x,\lambda)$ of the equation (2.1) as

$$u_{jk} := \begin{cases} e^{\varphi_{jk}\lambda(x-\xi_{jk})}, \text{for } x \in I_j \\ 0 \quad , \text{for } x \notin I_j \end{cases} \qquad j = 1,2,3, k = 1,2,$$

where $\xi_{11} = -1$, $\xi_{12} = \xi_{21} = d_1$, $\xi_{22} = \xi_{31} = d_2$, $\xi_{32} = 1$; $I_1 = [-1, d_1)$, $I_2 = (d_1, d_2)$, $I_3 = (d_2, 1]$. Then, the general solution of the equation (2.1) can be written in the form

$$u(x,\lambda) = \sum_{j=1}^{3}\left(C_{j1}u_{j1}(x,\lambda) + C_{j2}u_{j2}(x,\lambda)\right) \tag{2.5}$$

Substituting (2.5) into boundary-transmission conditions (2.2)-(2.4) yields a system of linear homogeneous equations with respect to variables $C_{11}, C_{12}, C_{21}, C_{22}, C_{31}, C_{32}$ as

$$\begin{aligned}
&C_{11}\left(\alpha_1 + \beta_1 e^{\varphi_{11}\lambda(d_1+1)}\right)(\varphi_{11}\lambda)^{m_1} + C_{12}\left(\alpha_1 e^{\varphi_{12}\lambda(-1-d_1)} + \beta_1\right)(\varphi_{12}\lambda)^{m_1} = h_1 \\
&+C_{31}\left(\alpha_2 + \beta_2 e^{\varphi_{31}\lambda(1-d_2)}\right)(\varphi_{31}\lambda)^{m_2} + C_{32}\left(\alpha_2 e^{\varphi_{32}\lambda(d_2-1)} + \beta_2\right)(\varphi_{32}\lambda)^{m_2} = h_2 \\
&C_{11}\alpha_3 e^{\varphi_{11}\lambda(d_1+1)}(\varphi_{11}\lambda)^{m_3} + C_{12}\alpha_3(\varphi_{12}\lambda)^{m_3} + C_{21}\beta_3(\varphi_{21}\lambda)^{m_3} \\
&+C_{22}\beta_3 e^{\varphi_{22}\lambda(d_1-d_2)}(\varphi_{22}\lambda)^{m_3} = h_3 \\
&C_{11}\alpha_4 e^{\varphi_{11}\lambda(d_1+1)}(\varphi_{11}\lambda)^{m_4} + C_{12}\alpha_4(\varphi_{12}\lambda)^{m_4} + C_{21}\beta_4(\varphi_{21}\lambda)^{m_4} \\
&+C_{22}\beta_4 e^{\varphi_{22}\lambda(d_1-d_2)}(\varphi_{22}\lambda)^{m_4} = h_4 \\
&C_{21}\alpha_5 e^{\varphi_{21}\lambda(d_2-d_1)}(\varphi_{21}\lambda)^{m_5} + C_{22}\alpha_5(\varphi_{22}\lambda)^{m_5} + C_{31}\beta_5(\varphi_{31}\lambda)^{m_5} \\
&+C_{32}\beta_5 e^{\varphi_{32}\lambda(d_2-1)}(\varphi_{32}\lambda)^{m_5} = h_5 \\
&C_{21}\alpha_6 e^{\varphi_{21}\lambda(d_2-d_1)}(\varphi_{21}\lambda)^{m_6} + C_{22}\alpha_6(\varphi_{22}\lambda)^{m_6} + C_{31}\beta_6(\varphi_{31}\lambda)^{m_6} \\
&+C_{32}\beta_6 e^{\varphi_{32}\lambda(d_2-1)}(\varphi_{32}\lambda)^{m_6} = h_6
\end{aligned} \tag{2.6}$$

From $\lambda \in D(\varepsilon,\lambda)$, let

$$\frac{\pi}{2} + \varepsilon < \arg \lambda\varphi_{jk} < \frac{3\pi}{2} - \varepsilon,\ j = 1,2,3,\ k = 1,$$
$$-\frac{\pi}{2} + \varepsilon < \arg \lambda\varphi_{jk} < \frac{\pi}{2} - \varepsilon,\ j = 1,2,3,\ k = 2.$$

Consequently, for these λ and for $\varepsilon > 0$, we have

$$(-1)^{k+1} Re\lambda\varphi_{jk} \le -|\lambda|\left|\varphi_{jk}\right| \sin\varepsilon,\ j = 1,2,3,\ k = 1,2.$$

Let $\omega_{11} := d_1 + 1, \omega_{12} := -1 - d_1, \omega_{21} := d_2 - d_1, \omega_{22} := d_1 - d_2, \omega_{31} := 1 - d_2, \omega_{32} := d_2 - 1$. Hence, the determinant of the system (2.6) has the form

$$\begin{aligned}
\Delta(\lambda) &= \lambda^{\sum_{j=1}^{6} m_j}\left(\begin{vmatrix}
\alpha_1\varphi_{11}^{m_1} & \beta_1\varphi_{12}^{m_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2\varphi_{31}^{m_2} & \beta_2\varphi_{32}^{m_2} \\
0 & \alpha_3\varphi_{12}^{m_3} & \beta_3\varphi_{21}^{m_3} & 0 & 0 & 0 \\
0 & \alpha_4\varphi_{12}^{m_4} & \beta_4\varphi_{21}^{m_4} & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_5\varphi_{22}^{m_5} & \beta_5\varphi_{31}^{m_5} & 0 \\
0 & 0 & 0 & \alpha_6\varphi_{22}^{m_6} & \beta_6\varphi_{31}^{m_6} & 0
\end{vmatrix}\right. \\
&\quad \left. + e^{\sum_{j=1}^{3}\sum_{k=1}^{2}\lambda\varphi_{jk}\omega_{jk}}\begin{vmatrix}
\beta_1\varphi_{11}^{m_1} & \alpha_1\varphi_{12}^{m_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2\varphi_{31}^{m_2} & \beta_2\varphi_{32}^{m_2} \\
\alpha_3\varphi_{11}^{m_3} & 0 & 0 & \beta_3\varphi_{22}^{m_3} & 0 & 0 \\
\alpha_4\varphi_{11}^{m_4} & 0 & 0 & \beta_4\varphi_{22}^{m_4} & 0 & 0 \\
0 & 0 & \alpha_5\varphi_{21}^{m_5} & 0 & 0 & \beta_5\varphi_{32}^{m_5} \\
0 & 0 & \alpha_6\varphi_{21}^{m_6} & 0 & 0 & \beta_6\varphi_{32}^{m_6}
\end{vmatrix}\right) \\
&= \lambda^m\left(\theta + \sigma(\lambda)\right),
\end{aligned}$$

where $m = \sum_{j=1}^{6} m_j$. It is easy to see that $\sigma(\lambda) \to 0$ if $\lambda \in D(\varepsilon, \lambda)$ and $|\lambda| \to \infty$, since $\varphi_{jk}\omega_{jk} < 0$ for each $j = 1, 2, 3,\ k = 1, 2$. Because of the fact that $\theta \neq 0$, there exists $r_\varepsilon > 0$ such that for all complex numbers λ satisfying $\lambda \in D(\varepsilon, \lambda)$ and $|\lambda| > r_\varepsilon$, we have $\Delta(\lambda) \neq 0$. So, the system (2.6) has a unique solution

$$C_{jk}(\lambda) = \frac{1}{\Delta(\lambda)} \sum_{i=1}^{6} \Delta_{ijk}(\lambda) h_i,\ j = 1, 2, 3,\ k = 1, 2,$$

where the determinants $\Delta_{ijk}(\lambda)$ have the representation given by

$$\Delta_{ijk}(\lambda) = \left(\theta_{ijk} + \sigma_{ijk}(\lambda)\right) \lambda^{\sum_{\upsilon=1}^{6} m_\upsilon - m_i},\ j = 1, 2, 3,\ k = 1, 2,$$

where θ_{ijk} are complex numbers and $\sigma_{ijk} \to 0$ as $|\lambda| \to \infty$ in the angle $D(\varepsilon, \lambda)$. Then, we have

$$C_{jk}(\lambda) = \sum_{i=1}^{6} \lambda^{-m_i} \frac{\theta_{ijk} + \sigma_{ijk}(\lambda)}{\theta + \sigma(\lambda)} h_i,\ j = 1, 2, 3,\ k = 1, 2.$$

Thus, the solution of the problem (2.1)-(2.4) has the form

$$\begin{aligned} u(x, \lambda) &= \sum_{j=1}^{3} \sum_{k=1}^{2} C_{jk}(\lambda) u_{jk}(x, \lambda) \\ &= \sum_{j=1}^{3} \sum_{k=1}^{2} \sum_{i=1}^{6} \lambda^{-m_i} \frac{\theta_{ijk} + \sigma_{ijk}(\lambda)}{\theta + \sigma(\lambda)} h_i u_{jk}(x, \lambda). \end{aligned}$$

In that case, for each integer $n \geq 0$ and $\frac{\pi + \overline{\varphi}}{2} + \varepsilon \leq \arg \lambda \leq \frac{3\pi + \underline{\varphi}}{2} - \varepsilon,\ |\lambda| \to \infty$, we have the estimate

$$\left\| u^{(n)} \right\|_{L_q(-1,1)} \leq C \sum_{i=1}^{6} \left(|\lambda|^{n-m_i} |h_i| \sum_{j=1}^{3} \sum_{k=1}^{2} \left\| u_{jk}(., \lambda) \right\|_{L_q(I_j)} \right), \tag{2.7}$$

or

$$\begin{aligned} \left\| u^{(n)} \right\|_{L_q(-1,1)} \leq\ & C \sum_{i=1}^{6} |\lambda|^{n-m_i} |h_i| \Big(\left\| e^{\varphi_{11}\lambda(x+1)} \right\|_{L_q(-1,d_1)} + \left\| e^{\varphi_{12}\lambda(x-d_1)} \right\|_{L_q(-1,d_1)} \\ & + \left\| e^{\varphi_{21}\lambda(x-d_1)} \right\|_{L_q(d_1,d_2)} + \left\| e^{\varphi_{22}\lambda(x-d_2)} \right\|_{L_q(d_1,d_2)} \\ & + \left\| e^{\varphi_{31}\lambda(x-d_2)} \right\|_{L_q(d_2,1)} + \left\| e^{\varphi_{32}\lambda(x-1)} \right\|_{L_q(d_2,1)} \Big). \end{aligned}$$

Further,

$$\begin{aligned} \|u_{11}(., \lambda)\|^q_{L_q(-1,d_1)} &= \int_{-1}^{d_1} e^{\mathrm{Re}\varphi_{11}\lambda(x+1)} dx \leq \int_{-1}^{d_1} e^{-q|\lambda||\varphi_{11}| \sin\frac{\varepsilon}{2}(x+1)} dx \\ &= \left(-q\,|\lambda|\,|\varphi_{11}| \sin\frac{\varepsilon}{2} \right)^{-1} \left(e^{-q|\lambda||\varphi_{11}| \sin\frac{\varepsilon}{2}(d_1+1)} - 1 \right) \\ &\leq C(\varepsilon)\,|\lambda|^{-1}, \end{aligned}$$

$$\begin{aligned} \|u_{12}(., \lambda)\|^q_{L_q(-1,d_1)} &= \int_{-1}^{d_1} e^{\mathrm{Re}\varphi_{12}\lambda(x-d_1)} dx \leq \int_{-1}^{d_1} e^{-q|\lambda||\varphi_{12}| \sin\frac{\varepsilon}{2}(x-d_1)} dx \\ &= \left(-q\,|\lambda|\,|\varphi_{12}| \sin\frac{\varepsilon}{2} \right)^{-1} \left(1 - e^{q|\lambda||\varphi_{12}| \sin\frac{\varepsilon}{2}(1+d_1)} \right) \\ &\leq C(\varepsilon)\,|\lambda|^{-1}, \end{aligned}$$

$$\begin{aligned}
\|u_{21}(.,\lambda)\|^q_{L_q(-1,d_1)} &= \int_{d_1}^{d_2} e^{\mathrm{Re}\varphi_{21}\lambda(x-d_1)}dx \leq \int_{d_1}^{d_2} e^{-q|\lambda||\varphi_{21}|\sin\frac{\varepsilon}{2}(x-d_1)}dx \\
&= \left(-q\,|\lambda|\,|\varphi_{21}|\sin\frac{\varepsilon}{2}\right)^{-1}\left(e^{-q|\lambda||\varphi_{21}|\sin\frac{\varepsilon}{2}(d_2-d_1)}-1\right) \\
&\leq C(\varepsilon)\,|\lambda|^{-1},
\end{aligned}$$

as $|\lambda|\to\infty$ in the angle $\lambda\in D(\varepsilon,\lambda)$. In a similar way, we have

$$\left\|u_{jk}(.,\lambda)\right\|^q_{L_q(I_j)} \leq C(\varepsilon)\,|\lambda|^{-1},\ \ j=2,3, k=1,2$$

as $|\lambda|\to\infty$ in the angle $\lambda\in D(\varepsilon,\lambda)$. Substituting these inequalities in 2.7), we have

$$\left\|u^{(n)}\right\|_{L_q(-1,1)} \leq C(\varepsilon)\sum_{i=1}^{6}|\lambda|^{n-m_i-\frac{1}{q}}\,|h_i|,$$

which, in turn, gives us the needed estimation (2.7). The proof is complete. □

3. Fredholm Property of the Main Problem

Let X and Y be Banach spaces and Y^* be the adjoint of Y. The linear operator $A : X\to Y$ is called a Fredholm operator if the following conditions are satisfied:
1) The range $R(A)=\{Au|\,u\in D(A)\}$ is closed in Y,
2) $\ker A=\{u|\,u\in D(A)\,\text{and}\,Au=0\}$ and
$co\ker A=\{u^*|\,u^*\in Y^*\,\text{and}\,u^*(Au)=0\,\text{for all}\ u\in D(A)\}$ are finite dimensional subspaces in X and Y^*, respectively,
3) $\dim\ker A=\dim co\ker A$.

Suppose that $n\geq\max\left\{2,\max\left\{m_j : j=1,2,...,6\right\}+1\right\}$ and define a linear operator $\mathcal{L}$ from $W_q^n(-1,d_1)\dot{+}W_q^n(d_1,d_2)$ $\dot{+}W_q^n(d_2,1)$ into $W_q^n(-1,d_1)\dot{+}W_q^n(d_1,d_2)\dot{+}W_q^n(d_2,1)+\mathbb{C}^6$ by action law

$$\mathcal{L}u=(L(\lambda)u,L_1u,L_2u,L_3u,L_4u,L_5u,L_6u).$$

Theorem 3.1. *Let the following conditions be satisfied:*
1) $t(x)=t_1$ on $[-1,d_1)$, $t(x)=t_2$ on (d_1,d_2), $t(x)=t_3$ at $x\in(d_2,1]$, $t_1\neq 0$, $t_2\neq 0$, $t_3\neq 0$; $m_k\geq 0$; $\theta\neq 0$.
2) $k(x)$ is measurable function on $[-1,d_1)\cup(d_1,d_2)\cup(d_2,1]$.
Then, the linear operator

$$\mathcal{L}: u\longrightarrow\mathcal{L}u:=(t(x)u''(x)+k(x)u,L_1u,L_2u,L_3u,L_4u,L_5u,L_6u\,)$$

from $W_q^n(-1,d_1)\dot{+}W_q^n(d_1,d_2)\dot{+}W_q^n(d_2,1)$ onto $W_q^{n-2}(-1,d_1)\dot{+}W_q^{n-2}(d_1,d_2)\dot{+}W_q^{n-2}(d_2,1)\dot{+}\mathbb{C}^6$ is bounded and Fredholm.

Proof. The operator $\mathcal{L}$ can be rewritten in the form $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2$, where

$$\begin{aligned}
\mathcal{L}_1u &= (t(x)u''(x),u(-1),u'(-1),u(-d_1)-u(+d_1),u'(-d_1)-u'(+d_1), \\
&\quad\ u(-d_2)-u(+d_2),u'(-d_2)-u'(+d_2)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_2u &= (k(x),L_1u-u(-1),L_2u-u'(-1),L_3u-u(-d_1)+u(+d_1)\,, \\
&\quad\ L_4u-u'(-d_1)+u'(+d_1),L_5u-u(-d_2)+u(+d_2),L_6u-u'(-d_2)+u'(+d_2)).
\end{aligned}$$

Let $f\in L_q(-1,1)$. Then, from condition (1) and $\frac{1}{p}+\frac{1}{q}=1$, it follows that the function $t^{-1}(x)f(x)\in L_1(-1,1)\cap L_q(-1,1)$.

Indeed, because of Schwartz inequality, we have

$$\begin{aligned}
\int_{d_2}^{1}\left|t^{-1}(x)f(x)\right|dx &\leq \left(\int_{d_2}^{1}x^{-p}(x)f(x)dx\right)^{\frac{1}{p}}\left(\int_{d_2}^{1}x^q(x)\,|f(x)|^q\,dx\right)^{\frac{1}{q}} \\
&\leq C\,x^{\frac{1}{p}-1}\Big|_{d_2}^{1}\left(\int_{d_2}^{1}x^q(x)\,|f(x)|^q\,dx\right)^{\frac{1}{q}} \\
&\leq C\,\|f\|_{L_q(d_2,1)}\,. \qquad (3.1)
\end{aligned}$$

Consequently, a solution of the problem

$$t(x)u''(x) = f(x),\ x \in (-1, d_1) \cup (d_1, d_2) \cup (d_2, 1),$$

$$\begin{aligned} u(-1) &= g_1,\ u'(-1) = g_2, \\ u(-d_1) - u(+d_1) &= g_3,\ u'(-d_1) - u'(+d_1) = g_4, \\ u(-d_2) - u(+d_2) &= g_5,\ u'(-d_2) - u'(+d_2) = g_6, \end{aligned}$$

has the form

$$\begin{aligned} u(x) &= \int_{-1}^{x} (x-y)t^{-1}(y)f(y)dy + g_1 + (x+1)g_2,\quad x \in (-1, d_1), \\ u(x) &= \int_{d_1}^{x} (x-y)t^{-1}(y)f(y)dy + \int_{-1}^{d_1} (x-y)t^{-1}(y)f(y)dy + g_1 + g_2 - g_3 \\ &\quad + \left(\int_{-1}^{d_1} t^{-1}(y)f(y)dy + g_2 - g_4\right)(x - d_1),\ x \in (d_1, d_2)\ , \\ u(x) &= \int_{d_2}^{x} (x-y)t^{-1}(y)f(y)dy + \int_{d_1}^{d_2} (x-y)t^{-1}(y)f(y)dy + \int_{-1}^{d_1} (x-y)t^{-1}(y)f(y)dy + g_1 + g_2 - g_3 \\ &\quad + \left(\int_{-1}^{d_1} t^{-1}(y)f(y)dy + g_2 - g_4\right)(d_2 - d_1) - g_5 \\ &\quad + \left(\int_{d_1}^{d_2} t^{-1}(y)f(y)dy + 2\int_{-1}^{d_1} t^{-1}(y)f(y)dy + g_2 - g_4 - g_6\right)(x - d_2),\ x \in (d_2, 1)\ . \end{aligned} \tag{3.2}$$

If $f \in W_q^{l-2}(-1, d_1) \dot{+} W_q^{l-2}(d_1, d_2) \dot{+} W_q^{l-2}(d_2, 1)$, then (3.2) implies $u''(x) = t^{-1}(x)f(x)$, $u^{(p+2)}(x) = t^{-1}(x)f^{(p)}(x)$, $0 \le p \le l-2$. Thus, from condition 1, the inequality (3.1) and Theorem 2.1, we obtain that the operator $\mathcal{L}_1$ from $W_q^n(-1, d_1) \dot{+} W_q^n(d_1, d_2) \dot{+} W_q^n(d_2, 1)$ onto $W_q^{n-2}(-1, d_1) \dot{+} W_q^{n-2}(d_1, d_2) \dot{+} W_q^{n-2}(d_2, 1) \dot{+} \mathbb{C}^6$ is isomorphism. Also, it is easy to see that the linear operator $\mathcal{L}_2$ acts compactly from $W_q^n(-1, d_1) \dot{+} W_q^n(d_1, d_2) \dot{+} W_q^n(d_2, 1)$ onto $W_q^{n-2}(-1, d_1) \dot{+} W_q^{n-2}(d_1, d_2) \dot{+} W_q^{n-2}(d_2, 1) \dot{+} \mathbb{C}^6$. Consequently, we can apply the theorem of Fredholm operator perturbation [15] to the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, from which it follows that the operator $\mathcal{L}$ is Fredholm. Moreover, it is obvious that the operator $\mathcal{L}$ is bounded. Thus the proof is completed. □

4. Isomorphism and Coerciveness of the Principal Part of the Problem

Consider the problem (1.1)-(1.4) without internal points, namely,

$$L_0(\lambda)u := t(x)u''(x) - \lambda^2 u(x) = f(x),\ x \in [-1, d_1) \cup (d_1, d_2) \cup (d_2, 1], \tag{4.1}$$

$$L_{01}u := \alpha_1 u^{(m_1)}(-1) + \beta_1 u^{(m_1)}(d_1 - 0) = h_1, \tag{4.2}$$

$$L_{02}u := \alpha_2 u^{(m_2)}(d_2 + 0) + \beta_2 u^{(m_2)}(1) = h_2, \tag{4.3}$$

$$L_{0j}u := \alpha_j u^{(m_j)}(d_k - 0) + \beta_j u^{(m_j)}(d_k + 0) = h_j,\quad k = 1 \text{ for } j = 3, 4 \text{ and } k = 2 \text{ for } j = 5, 6. \tag{4.4}$$

The operator corresponding to this problem is

$$\widetilde{\mathcal{L}}_0 u = (L_0(\lambda)u, L_{01}(\lambda)u, L_{02}, L_{03}, L_{04}, L_{05}, L_{06})\,.$$

Theorem 4.1. *Let the following conditions be satisfied:*

1) $t_1 \neq 0, t_2 \neq 0, t_3 \neq 0\,; m_k \ge 0; \theta \neq 0,$

2) $n \ge \max\{2, \max\{m_1, m_2, m_3, m_4, m_5, m_6\} + 1\}$.

Then, for each $\varepsilon > 0$, *there exists* $\rho_\varepsilon > 0$ *such that for all complex numbers* λ *satisfying* $\lambda \in D(\varepsilon, \lambda)$, $|\lambda| > \rho_\varepsilon$ *the operator* $\widetilde{\mathcal{L}}_0(\lambda)$ *from* $W_q^n(-1, d_1) \dot{+} W_q^n(d_1, d_2) \dot{+} W_q^n(d_2, 1)$ *onto* $W_q^{n-2}(-1, d_1) \dot{+} W_q^{n-2}(d_1, d_2) \dot{+} W_q^{n-2}(d_2, 1) \dot{+} \mathbb{C}^6$ *is an isomorphism and for these* λ *the following coercive estimate holds for the solution of the problem (4.1)-(4.4),*

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u\|_{W_q^k} \le C(\varepsilon)\left(\|f\|_{W_q^{n-2}} + |\lambda|^{n-2}\|f\|_{L_{q,0}} + \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|h_j\right|\right). \tag{4.5}$$

Proof. It is obvious that, the linear operator $\widetilde{\mathcal{L}}_0(\lambda)$ is continuous from $W_q^n(-1,d_1)\dot{+}W_q^n(d_1,d_2)\dot{+}W_q^n(d_2,1)$ to $W_q^{n-2}(-1,d_1)\dot{+}W_q^{n-2}(d_1,d_2)\dot{+}W_q^{n-2}(d_2,1)\dot{+}\mathbb{C}^6$. Let $(f(x),h_1,h_2,h_3,h_4,h_5,h_6)\in W_q^{n-2}(-1,d_1)\dot{+}W_q^{n-2}(d_1,d_2)\dot{+}W_q^{n-2}(d_2,1)\dot{+}\mathbb{C}^6$ be any elements. We shall look for the solution $u(x,\lambda)$ of problem (15)-(18) in the type of the sum $u(x,\lambda)=u_1(x,\lambda)+u_2(x,\lambda)+u_3(x,\lambda)$ where $u_1(x,\lambda)=u_{11}(x,\lambda)+u_{12}(x,\lambda);\ u_2(x,\lambda)=u_{21}(x,\lambda)+u_{22}(x,\lambda);\ u_3(x,\lambda)=u_{31}(x,\lambda)+u_{32}(x,\lambda)$.

By $f_j(x)$ $(j=1,2,3)$, we shall denote the restriction of $f(x)$ on the interval I_j $(j=1,2,3)$. Let $\widetilde{f_j}(x)\in W_q^{n-2}(\mathbb{R})$ be an extension of $f_j(x)\in W_q^{n-2}(I_j)$ such that the extension operator $T_jf_j:=\widetilde{f_j}$ from $W_q^{n-2}(I_j)$ to $W_q^{n-2}(\mathbb{R})$ is bounded for $j=1,2,3$ ([23], Lemma 1.7.6), where as usual $\mathbb{R}=(-\infty,\infty)$. First, consider the equations

$$t(x)u''(x)-\lambda^2u(x)=\widetilde{f_j}(x),x\in\mathbb{R}$$

for $j=1,2,3$. Applying the ([23], Theorem 3.2.1), it is seen that, this equation has a unique solution $\widetilde{u}_{j1}=\widetilde{u}_{j1}(x,\lambda)\in W_q^n(\mathbb{R})$ and for $u_{j1}(x,\lambda)$ (i.e. the restriction of $\widetilde{u}_{j1}(x,\lambda)$ on interval I_j) the estimate

$$\sum_{k=0}^{n}|\lambda|^{n-k}\left\|u_{j1}(x,\lambda)\right\|_{W_q^k(I_j)}\le C(\varepsilon)\left(\|f\|_{W_q^{n-2}(I_j)}+|\lambda|^{n-2}\|f\|_{L_q(I_j)}\right)\tag{4.6}$$

for $j=1,2,3$, is valid for all complex numbers satisfying $\lambda\in D(\varepsilon,\lambda)$. Consequently, the function

$$u_{10}(x,\lambda):=\begin{cases}u_{11}(x,\lambda), & x\in(-1,d_1)\\ u_{21}(x,\lambda), & x\in(d_1,d_2)\\ u_{31}(x,\lambda), & x\in(d_2,1)\end{cases}$$

satisfies the equation(4.1). In terms of this solution, we construct the boundary-value problem

$$L_0(\lambda)u:=t(x)u''(x)-\lambda^2u(x)=f(x),\ x\in[-1,d_1)\cup(d_1,d_2)\cup(d_2,1],$$

$$L_{0j}u=h_j-L_ju,\quad j=1,2,3,4,5,6.$$

By Theorem 1, this problem has a unique solution $u_{20}(x,\lambda)=:u_{12}(x,\lambda)+u_{22}(x,\lambda)+u_{32}(x,\lambda)$ that belongs to $W_q^n(-1,d_1)\dot{+}W_q^n(d_1,d_2)\dot{+}W_q^n(d_2$ for all complex numbers λ sufficiently large in modulus satisfying $\lambda\in D(\varepsilon,\lambda)$, and for these λ the estimate

$$\sum_{k=0}^{n}|\lambda|^{n-k}\|u_{20}(x,\lambda)\|_{q,k}\le C(\varepsilon)\sum_{j=1}^{6}|\lambda|^{n-m_j-\frac{1}{q}}\left(\left|h_j\right|+\left|L_{0j}u_{10}\right|\right)\tag{4.7}$$

holds. By applying Theorem 1 and taking into account ([23], Theorem 1.7.7/2), we have that for all $\lambda\in D(\varepsilon,\lambda)$ and $n\ge\max\{2,\max\{m_1,m_2,m_3,m_4,m_5,m_6\}+1\}$ the following estimates hold.

$$\begin{aligned}|\lambda|^{n-m_j-\frac{1}{q}}\left|L_{0j}u_{10}\right| &\le C|\lambda|^{n-m_j-\frac{1}{q}}\|u_{10}\|_{C^{m_j}[-1,d_1]+C^{m_j}[d_1,d_2]+C^{m_j}[d_2,1]}\\ &\le C\left(|\lambda|\|u_{10}\|_{q,0}+\|u_{10}\|_{q,n}\right)\\ &\le C\left(\|f\|_{q,n-2}+|\lambda|^{n-2}\|f\|_{q,0}\right).\end{aligned}\tag{4.8}$$

From (4.7) and (4.8), we have the following inequality

$$\sum_{k=0}^{n}|\lambda|^{n-k}\|u_{20}(x,\lambda)\|_{q,k}\le C(\varepsilon)\left(\|f\|_{q,n-2}+|\lambda|^{n-2}\|f\|_{q,0}+\sum_{j=1}^{6}|\lambda|^{n-m_j-\frac{1}{q}}\left|h_j\right|\right).\tag{4.9}$$

It is easy to see that the function $u(x,\lambda)$ defined as $u(x,\lambda)=u_{10}(x,\lambda)+u_{20}(x,\lambda)=\sum_{i=1}^{3}\sum_{j=1}^{2}u_{ij}(x,\lambda)$ is the solution of the considered problem (4.1)-(4.4). Taking into account the estimates (4.6) and (4.9), it is seen that the required estimation (4.5) is valid for this solution . Furthermore, from estimate (4.5), the uniqueness of the solution is obvious. Meanwhile by Theorem 2, the operator $\mathcal{L}$ is Fredholm from $W_q^n(-1,d_1)\dot{+}W_q^n(d_1,d_2)\dot{+}W_q^n(d_2,1)$ to $W_q^{n-2}(-1,d_1)\dot{+}W_q^{n-2}(d_1,d_2)\dot{+}W_q^{n-2}(d_2,1)\dot{+}\mathbb{C}^6$. The fact that the operator is an isomorphism is obvious, since the operator is a Fredholm. Thus, we completed the proof. □

5. Isomorphism and Coerciveness of the Main Problem

We shall study the main problem(1.1)-(1.4) now.

Theorem 5.1. *Let*

1. $t_1 \neq 0, t_2 \neq 0, t_3 \neq 0, m_k \geq 0$ *and* $\theta \neq 0$
2. $n \geq \max\{2, \max\{m_1, m_2, m_3, m_4, m_5, m_6\} + 1\}$.

Then, for any $\varepsilon > 0$ *there exists* $r_\varepsilon > 0$ *such that for all complex numbers* $\lambda \in D(\varepsilon, \lambda)$ *for which* $|\lambda| > r_\varepsilon$ *the operator*

$$\mathcal{L}u = (L(\lambda)u, L_1(\lambda)u, L_2, L_3, L_4, L_5, L_6)$$

is an isomorphism from $W_q^n(-1, d_1)\dot{+}W_q^n(d_1, d_2)\dot{+}W_q^n(d_2, 1)$ *onto* $W_q^{n-2}(-1, d_1)\dot{+}W_q^{n-2}(d_1, d_2)\dot{+}W_q^{n-2}(d_2, 1)\dot{+}\mathbb{C}^6$ *and for these* λ, *the following coercive estimate holds for the solution of main problem (1.1)-(1.4)*

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u(x,\lambda)\|_{q,k} \leq C(\varepsilon)\left(\|f\|_{q,n-2} + |\lambda|^{n-2}\|f\|_{q,0} + \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|h_j\right|\right), \tag{5.1}$$

where $C(\varepsilon)$ *is a constant which depend on only* ε.

Proof. Let $(f(x), h_1, h_2, h_3, h_4, h_5, h_6)$ be any element of $W_q^{n-2}(-1, d_1)\dot{+}W_q^{n-2}(d_1, d_2)\dot{+}W_q^{n-2}(d_2, 1)\dot{+}\mathbb{C}^6$. Assume that there exists a solution $u(x, \lambda)$ of problem (1.1)-(1.4) corresponding to this element. Then, this solution satisfies the following equalities

$$L_0(\lambda)u = L(\lambda)u - k(x)u$$
$$L_{0j}u = L_ju - A_ju, \;\; j = 1, 2, 3, 4, 5, 6,$$

where $A_ju = \sum_{s=1}^{n_j} a_{js}u^{(m_j)}(x_{js})$. By applying Theorem 3 to the problem (4.1)- (4.4), we have that for this solution the following a priory estimate holds

$$\begin{aligned}
\sum_{k=0}^{n} |\lambda|^{n-k} \|u(x,\lambda)\|_{q,k} \;\; &\leq \;\; C(\varepsilon)\Big(\|L(\lambda)u - k(x)u\|_{q,n-2} + |\lambda|^{n-2}\|L(\lambda)u - k(x)u\|_{q,0} \\
&\quad + \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|L_ju - A_ju\right| \\
&\leq \;\; C(\varepsilon)\Big(\|f\|_{q,n-2} + |\lambda|^{n-2}\|f\|_{q,0} + \|k(x)u\|_{q,n-2} + |\lambda|^{n-2}\;\|k(x)u\|_{q,0} \\
&\quad + \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|h_j\right| + \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|A_ju\right|.
\end{aligned} \tag{5.2}$$

Let μ be any real number satisfying

$$\mu := \min\left\{\min_{x_{ij}\in(-1,d_1)}\left\{-1 + x_{ij}, d_1 - x_{ij}\right\}, \min_{x_{ij}\in(d_1,d_2)}\left\{d_1 + x_{ij}, d_2 - x_{ij}\right\}, \min_{x_{ij}\in(d_2,1)}\left\{d_2 + x_{ij}, 1 - x_{ij}\right\}\right\}.$$

By applying the same approach as in [19], it is easy to construct a function $\zeta_\mu(x) \in C_0^\infty[-1, 1]$ such that

$$\zeta_\mu(x) = 1 \text{ for } x \in [-1 + \mu, d_1 - \mu] \cup [d_1 + \mu, d_2 - \mu] \cup [d_2 + \mu, 1 - \mu],$$

$$\zeta_\mu(x) = 0 \text{ for } x \in [-1, -1 + \frac{\mu}{2}] \cup [d_1 - \frac{\mu}{2}, d_1 + \frac{\mu}{2}] \cup [d_2 - \frac{\mu}{2}, d_2 + \frac{\mu}{2}] \cup [1 - \frac{\mu}{2}, 1],$$

and $0 \leq \zeta_\mu(x) \leq 1$ for all $x \in [-1, 1]$. It is obvious that,

$$\left|A_ju\right| \leq C\left\|\left(\zeta_\mu u\right)^{(m_j)}\right\|_{C[-1,1]}. \tag{5.3}$$

By ([21], Theorem 3.10.4]), for $u \in W_q^n(-1, d_1)\dot{+}W_q^n(d_1, d_2)\dot{+}W_q^n(d_2, 1)$ the following estimate holds,

$$|\lambda|^{n-m_j-\frac{1}{q}} \left\|u^{(m_j)}\right\|_{C[-1,1]} \leq C\left(\|u\|_{q,n} + |\lambda|^n \|u\|_{q,0}\right). \tag{5.4}$$

By Theorem 3, from (5.3)-(5.4) it follows that for all $\lambda \in D(\varepsilon, \lambda)$ sufficiently large in modulus the following estimate holds,

$$
\begin{aligned}
|\lambda|^{n-m_j-\frac{1}{q}}\left|A_j u\right| &\leq C\,|\lambda|^{n-m_j-\frac{1}{q}}\left\|\left(\zeta_\mu u\right)^{(m_j)}\right\|_{C[-1,1]} \\
&\leq C\left(\left\|\zeta_\mu u\right\|_{q,n}+|\lambda|^n\left\|\zeta_\mu u\right\|_{q,0}\right) \\
&\leq C(\varepsilon)\left(\left\|L_0(\lambda)(\zeta_\mu u)\right\|_{q,n-2}+|\lambda|^{n-2}\left\|L_0(\lambda)(\zeta_\mu u)\right\|_{q,0}\right) \\
&\leq C(\varepsilon)\Big(\|L_0(\lambda)\|_{q,n-2}+|\lambda|^{n-2}\|L_0(\lambda)\|_{q,0} \\
&\quad+\|k(x)u\|_{q,n-2}+|\lambda|^{n-2}\|k(x)u\|_{q,0}+\sum_{k=0}^{n-1}|\lambda|^{n-1-k}\|u\|_{q,k}\Big) \\
&\leq C(\varepsilon)\Big(\|f\|_{q,n-2}+|\lambda|^{n-2}\|f\|_{q,0} \\
&\quad+\|k(x)u\|_{q,n-2}+|\lambda|^{n-2}\|k(x)u\|_{q,0}+\sum_{k=0}^{n-1}|\lambda|^{n-1-k}\|u\|_{q,k}\Big).
\end{aligned}
\tag{5.5}
$$

By ([4], 2014, Theorem 1.3.3) there is a positive constant C such that for all u in the set $W_q^n(-1,d_1)\dotplus W_q^n(d_1,d_2)$ $\dotplus W_q^n(d_2,1)$ and for each $k=0,1,...,n-1$ the following inequality is valid

$$
\|u\|_{q,k}\leq\|u\|_{q,k}^{\frac{k}{k+1}}\,\|u\|_{q,0}^{\frac{1}{k+1}}\,. \tag{5.6}
$$

Applying the well-known Young inequality

$$
ab\leq\frac{1}{p}(ra)^p+\frac{1}{q}\left(\frac{b}{r}\right)^q,
$$

where $a>0,\ b>0,\ r>0,\ 1<p,q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$ to the right hand-side of (5.6) for $a=\|u\|_{q,k}^{\frac{k}{k+1}},\ b=\|u\|_{q,0}^{\frac{1}{k+1}},\ p=\frac{k+1}{k}$, it yields

$$
\|u\|_{q,k}\leq\left(\frac{k}{k+1}r^{\frac{k+1}{k}}\|u\|_{q,k+1}+\frac{1}{k+1}r^{-(k+1)}\|u\|_{q,0}\right),\ \text{for } k=0,1,...,n-1.
$$

We denote

$$
\begin{aligned}
M(r) &= \max\left(C\frac{k}{k+1}r^{\frac{k+1}{k}},\ k=0,1,...,n-1\right),\\
N(r) &= \max\left(C\frac{1}{k+1}r^{-(k+1)},\ k=0,1,...,n-1\right).
\end{aligned}
$$

Then, from inequality (5.5), we have

$$
\begin{aligned}
|\lambda|^{n-m_j-\frac{1}{q}}\left|A_j u\right| &\leq C(\varepsilon)\left(\|f\|_{q,n-2}+|\lambda|^{n-2}\|f\|_{q,0}\right) \\
&\quad+C(\varepsilon)\sum_{k=0}^{n-1}|\lambda|^{n-1-k}\left(M(r)\|u\|_{q,k+1}+N(r)\|u\|_{q,0}\right) \\
&\leq \left(C(\varepsilon)M(r)+T(r,\varepsilon)|\lambda|^{-1}\right)\sum_{k=0}^{n}|\lambda|^{n-k}\|u\|_{q,k}\,,
\end{aligned}
\tag{5.7}
$$

where $T(r,\varepsilon)$ is a constant which depends on only of r and ε. In view of ([28], Theorem 1.7.7/2), for any $\tau>0$ we obtain

$$
\|u\|_{q,k}\leq\tau\|u\|_{q,k+1}+C(\tau)\|u\|_{q,0}\,.
$$

From (5.7), we have

$$
\begin{aligned}
& \|k(x)u\|_{q,n-2} + |\lambda|^{n-2}\|k(x)u\|_{q,0} + \sum_{j=1}^{6} |\lambda|^{n-m_j-\frac{1}{q}} \left|A_j u\right| \\
\leq\ & C(\varepsilon)\left(\|f\|_{q,n-2} + |\lambda|^{n-2}\|f\|_{q,0}\right) + \tau\left(\|u\|_{q,k+1} + |\lambda|^{n-2}\|u\|_{q,0}\right) \\
& + C(\tau)|\lambda|^{n-2}\|u\|_{q,0} + \left(C(\varepsilon)M(r) + T(r,\varepsilon)|\lambda|^{-1}\right)\sum_{k=0}^{n}|\lambda|^{n-k}\|u\|_{q,k} + C\sum_{j=1}^{6}|\lambda|^{n-m_j-\frac{1}{q}}\|u\|_{q,k} \\
\leq\ & C(\varepsilon)\left(\|f\|_{q,n-2} + |\lambda|^{n-2}\|f\|_{q,0}\right) + \left(C(\varepsilon)M(r) + T(r,\varepsilon)|\lambda|^{-\frac{1}{q}}\right)\sum_{k=0}^{n}|\lambda|^{n-k}\|u\|_{q,k}\,. \qquad (5.8)
\end{aligned}
$$

Substituting (5.8) into (5.2), we obtain

$$
\begin{aligned}
\sum_{k=0}^{n}|\lambda|^{n-k}\|u(x,\lambda)\|_{q,k} \leq\ & C(\varepsilon)\left(\|f\|_{q,n-2} + |\lambda|^{n-2}\|f\|_{q,0} + \sum_{j=1}^{6}|\lambda|^{n-m_j-\frac{1}{q}}\left|h_j\right|\right) \\
& + \left(C(\varepsilon)M(r) + T(r,\varepsilon)|\lambda|^{-\frac{1}{q}}\right)\sum_{k=0}^{n}|\lambda|^{n-k}\|u\|_{q,k}\,.
\end{aligned}
$$

For a fixed $\varepsilon > 0$ we can choose $r > 0$ so small, and $|\lambda|$ so large that $C(\varepsilon)M(r) + T(r,\varepsilon)|\lambda|^{-\frac{1}{q}} < 1$. Thus, for $\lambda \in D(\varepsilon,\lambda)$ sufficiently large in modulus, we acquire a priori estimate (5.1). From this estimate, the uniqueness property of the solution of problem (1.1)- (1.4) is obtained, i.e. the operator $\mathcal{L}$ is one-to-one. Moreover, by Theorem 2 the operator $\mathcal{L}$ from $W_q^n(-1,d_1) \dot{+} W_q^n(d_1,d_2) \dot{+} W_q^n(d_2,1)$ onto $W_q^{n-2}(-1,d_1) \dot{+} W_q^{n-2}(d_1,d_2) \dot{+} W_q^{n-2}(d_2,1) \dot{+} \mathbb{C}^6$ is Fredholm. Consequently, the existence of a solution results in its uniqueness. This completes the proof. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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