Avrupa Bilim ve Teknoloji Dergisi
Özel Sayı 39, S. 1-5, Temmuz 2022

Araştırma Makalesi

www.ejosat.com ISSN:2148-2683

European Journal of Science and Technology
Special Issue 39, pp. 1-5, July 2022

Research Article

On The Geometric Approach of Differential Equations

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(3rd International Conference on Applied Engineering and Natural Sciences ICAENS 2022, July 20-23, 2022)

(**DOI:** 10.31590/ejosat.1142206)

ATIF/REFERENCE: Bulut, V. (2022). On The Geometric Approach of Differential Equations. *European Journal of Science and Technology*, (39), 1-5.

Abstract

Expressing physical facts around us possesses to use mathematical models in both theory and application. Since these models usually involves differential equations, we present a novel geometric approach to these models using developable ruled surfaces and line congruences.

In this paper, a tangent developable ruled surface according to first order differential equation and its solution function are presented. Moreover, we expressed a line congruence based on the solution of exact differential equation and its solution function. Also, some examples of tangent developable ruled surfaces and line congruences are given.

Keywords: Developable ruled surface, tangent developable ruled surface, first-order differential equation, exact differential equation, line congruence.

Diferansiyel Denklemlerin Geometrik Yaklaşımı Üzerine

Öz

Çevremizdeki fiziksel gerçekleri ifade etmek, matematiksel modelleri hem teoride hem de uygulamada kullanmayı gerektirir. Bu modeller genellikle diferansiyel denklemleri içerdiğinden, açılabilir regle yüzeyler ve doğru kongrüanslarını kullanarak bu modellere yeni bir geometrik yaklaşım sunuyoruz.

Bu makalede, birinci mertebeden diferansiyel denkleme göre teğet açılabilir bir regle yüzey ve çözüm fonksiyonu sunulmaktadır. Ayrıca, tam diferansiyel denklemin çözümüne ve çözüm fonksiyonuna dayalı bir doğru kongrüansı ifade edilmektedir. Ayrıca, teğet açılabilir regle yüzeyler ve doğru kongrüanslar ile ilgili bazı örnekler verilmektedir.

Anahtar Kelimeler: Açılabilir regle yüzey, teğet açılabilir regle yüzey, birinci mertebeden diferansiyel denklem, tam diferansiyel denklem, doğru kongrüansı.

1. Introduction

Developable surfaces are very special surfaces in manufacturing industry such as ship hulls (P´erez and Su´arez, 2007), car bodies and airplane skins (Frey and Bindschadler, 1993), and clothing (Chen and Tang, 2010), since these surfaces can be unfolded into a plane without stretching or tearing.

A two-parameter set of lines is called a line congruence. Line congruences has been gained big importance in theory but in present congruences have started to attract attention in practice (Bottema and Roth, 1979; Odehnal and Pottmann, 2001; Pottman and Wallner, 2001).

Mathematical models are widely used for understanding of physical phenomena in engineering, the natural sciences, economics, and even business. These models often contains unknown function and its derivatives. This kind of an equation is called a differential equation. If a mathematical model involves the rate of change of one variable with respect to another, this model involves a differential equation (Nagle, Saff and Snider, 2012).

2. Preliminaries

2.1. Developable Ruled Surface and Line Congruence

A one parameter family of straight lines is called ruled surface and defined by;

$$\boldsymbol{R}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{d}(v) \qquad (1)$$

where, $\boldsymbol{\alpha}(v)$ is the directrix curve and $\boldsymbol{l}(u) = \boldsymbol{\alpha}(v_0) + u\boldsymbol{d}(v_0)$ is a line and called as the generator or the ruling of the ruled surface $\boldsymbol{R}(u,v)$ for a constant value $v = v_0$.

If the tangent plane is constant at all points along a given ruling of the ruled surface, then this surface is called a developable ruled surface (Kreyszig, 1991). A developable ruled surface can be a plane, a conical surface, a cylindrical surface or tangential developable surface (see Figure 1). A ruled surface is called as developable ruled surface if the following relation is satisfied.

$$(\boldsymbol{\alpha}', \boldsymbol{d}, \boldsymbol{d}') = 0 \qquad (2)$$

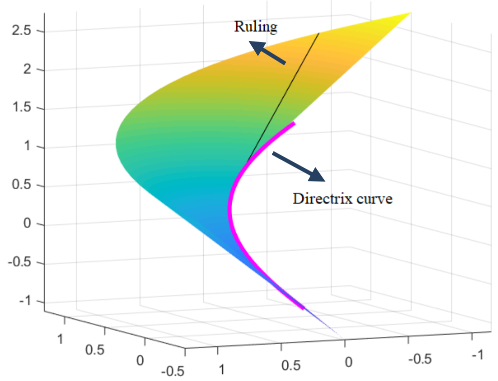

Fig. 1: A tangential developable surface

If the ruled surface in (1) takes one more parameter, therefore a ruled surface family which is called line congruence is obtained. We can express a line congruence with following equation.

$$\boldsymbol{R}(u,v,w) = \boldsymbol{\alpha}(v,w) + u\boldsymbol{d}(v,w) \qquad (3)$$

in which $\boldsymbol{\alpha}(v,w)$ is the base surface and $\boldsymbol{d}(v,w)$ is the generator vector field.

2.2. First-Order Differential Equations

An equation contains an unknown function and its derivatives is called differential equation. If the unknown function depends on only one independent variable, the differential equation is an ordinary differential equation (ODE). If the unknown function depends on two or more independent variables, it is called a partial differential equation (PDE). A solution of a differential equation is a function that satisfies the differential equation identically for all values of the independent variable in the interval I.

Many physical problems can be solved by using first-order differential equations when these problems formulated mathematically. Standard form of the first-order differential equation is as follows:

$$y' = \frac{dy}{dx} = f(x,y) \qquad (4)$$

in which, the dependent variable is y and the independent variable is x. As known from calculus, a derivative $\frac{dy}{dx}$ of a differentiable function $y = y(x)$ yields slopes of tangent lines at points on its graph (Zill, 2001).

3. Geometric Application of First-Order Differential Equations

Let's suppose that we have the solution of given first-order differential equation. Since the first-derivative of a differentiable function gives slopes of tangent lines at points on its graph, a developable surface can be expressed according to tangential developable surface. For this purpose, a developable ruled surface can be expressed according to the first-order differential equation in (4) and the solution function $y = y(x)$ of this equation by

$$\boldsymbol{D}(t,u) = \boldsymbol{r}(t) + u\boldsymbol{r}'(t) \qquad (5)$$

where $\begin{cases} \boldsymbol{r}(t) = (t, \quad y(t), \quad g(t, y(t))) \\ \boldsymbol{r}'(t) = \left(1, \quad y'(t), \quad g_t + g_y \frac{dy}{dt}\right) \end{cases}$

Example 1: A first-order differential equation is $\frac{dy}{dt} = \frac{t}{\sqrt{25-t^2}}$. Solution of this equation is $y(t) = -\sqrt{25-t^2} + c$, in which c is the integral constant. Let's write the equation of the developable ruled surface as

$$\boldsymbol{D}(t,u) = \Big[t+u,\ \left(-\sqrt{25-t^2}+c\right)+u\frac{t}{\sqrt{25-t^2}},\ t-5\left(-\sqrt{25-t^2}+c\right)+7+u\left(1-5\left(\frac{t}{\sqrt{25-t^2}}\right)\right)\Big]$$

Developable ruled surfaces belong to ebove equation by using the values of the integral constant can be seen in Figure 2.

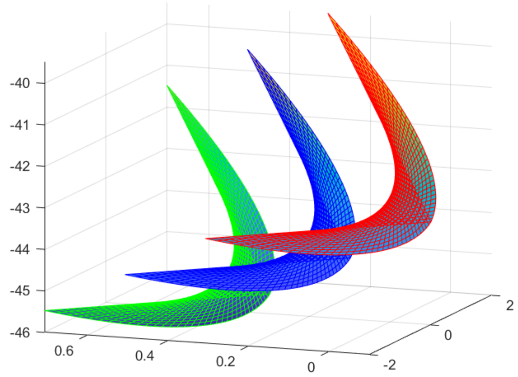

Fig. 2: Developable ruled surfaces according to the values of the integral constant

Example 2: A first-order differential equation is $\frac{dy}{dt} = \sin 5t$. Solution of this equation is $y(t) = -\frac{1}{5}\cos 5t + c$. We can express the equation of the developable ruled surface as

$$\boldsymbol{D}(t,u) = \Big[t+u,\ \left(-\frac{1}{5}\cos 5t+c\right)+u\sin 5t,\ t+3\left(\frac{1}{5}\cos 5t+c\right)+5+u\left(1-3(\sin 5t)\right)\Big]$$

Developable ruled surfaces belong to ebove equation by using the values of the integral constant can be seen in Figure 3.

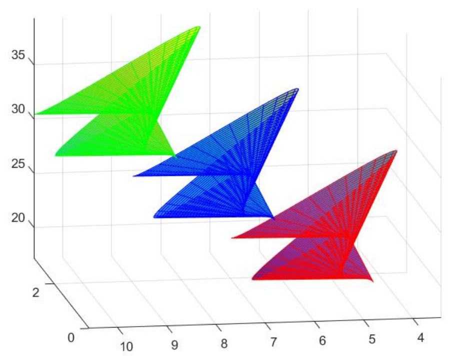

Fig. 3: Developable ruled surfaces according to the values of the integral constant

Example 3: A first-order differential equation is $\frac{dy}{dt} = -\frac{2t}{y^2}$. Solution of this equation is $y(t) = -\frac{1}{t^2+c}$. The equation of the developable ruled surface is given as follows:

$$\boldsymbol{D}(t,u) = \Big[t+u,\ \left(-\frac{1}{t^2+c}\right)+u\left(-\frac{2t}{y^2}\right),\ t-4\left(-\frac{1}{t^2+c}\right)+9+u\left(1-4\left(-\frac{2t}{y^2}\right)\right)\Big]$$

Developable ruled surfaces can be seen in Figure 4.

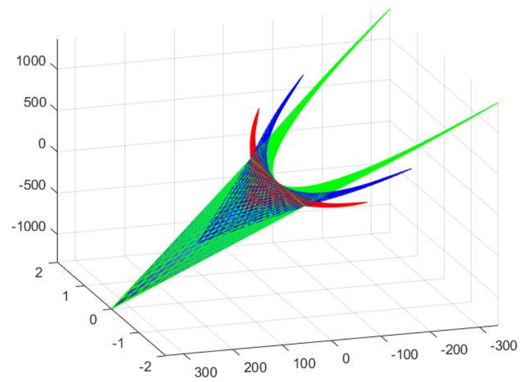

Fig. 4: Developable ruled surfaces according to the values of the integral constant

4. Geometric Application of Exact Equations

Differential of a function of two variables as $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ with continuous first partial derivatives in a region $\boldsymbol{R}$ of the $\boldsymbol{xy}$-plane as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \qquad (6)$$

If the function is given in the form of $f(x,y) = c$, then a first-order differential equation can be generated by using the equation (6).

Definition: A differential expression $M(x,y)dx + N(x,y)dy$ is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined in R. A first order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (7)$$

is called exact equations if the expression on the left-hand side is an exact differential (Zill, 2001).

On the other hand, due to the fact that $\boldsymbol{R}(u,v,w)$ in (3) indicates a line congruence in 3-D real space, the tangent plane of a regular surface forms a line congruence. Assume that the solution $F(x,y)$ of exact equation (7) is given. Since this solution function includes continuous first partial derivatives, a line congruence can be defined as

$$
\begin{aligned}
\boldsymbol{R}(u,v,w) &= \boldsymbol{S}(u,v) \\
&+w[\boldsymbol{S}(u,v) + u\boldsymbol{S}_{\boldsymbol{u}}(u,v) \\
&+v\boldsymbol{S}_{v}(u,v)].
\end{aligned} \tag{8}
$$

where $\boldsymbol{S}(u,v) = (u, \quad v, \quad F(u,v))$.

Example 4: Let a first-order exact differential equation be

$$(e^x + y)\,dx + (2 + x + ye^y)dy = 0.$$

Solution of this equation is found as

$$e^x + xy + 2y + ye^y - e^y = c.$$

Hence, the line congruence can be written by

$$\boldsymbol{R}(u,v,w) = \begin{bmatrix} u+2uw, & v+2vw, & e^u+uv+2v+ve^v-e^v-c \\ & & +w[e^u+uv+2v+ve^v-e^v-c \\ & & +u(e^u+v) \\ & & +v(2uv+u^2-1)] \end{bmatrix}$$

Line congruence can be seen in Figure 5.

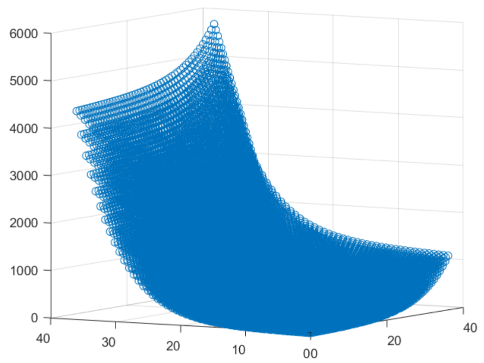

Fig. 5: The line congruence obtained from the exact equation $(e^x + y)dx + (2 + x + ye^y)dy = 0$.

Example 5: Let us given a first-order exact differential equation with

$$2xydx + (x^2 - 1)dy = 0.$$

We can determine the solution of this quation by

$$x^2y - y = c.$$

Therefore, we can express the line congruence as

$$\boldsymbol{R}(u,v,w) = \begin{bmatrix} u+2uw\,, & v+2vw\,, & u^2v-v-c \\ & & +w[u^2v-v-c \\ & & +u(2uv)+v(u^2-1)] \end{bmatrix}$$

Line congruence can be seen in Figure 6.

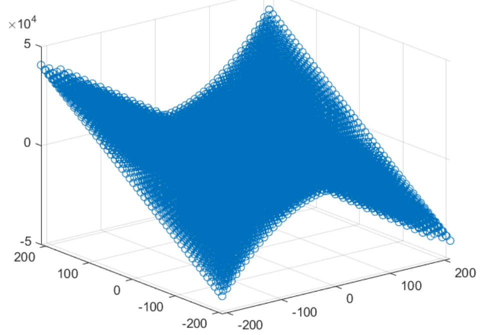

Fig. 6: The line congruence obtained from the exact equation $2xydx + (x^2 - 1)dy = 0$.

Example 6: A first-order exact differential equation given as in the form of below equation:

$$(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0.$$

If we solve this equation we will get solution as

$$x\sin y + y\cos x - \tfrac{1}{2}y^2 = c.$$

The line congruence can be presented with the following equation:

$$\boldsymbol{R}(u,v,w) = \begin{bmatrix} u+2uw, & v+2vw, & u\sin v + v\cos u - \frac{1}{2}v^2 - c \\ & & +w\left[u\sin v + v\cos u - \frac{1}{2}v^2 - c\right. \\ & & +u(\sin v - v\sin u) \\ & & +v(\cos u + u\cos v - v)] \end{bmatrix}$$

Line congruence can be seen from Figure 7.

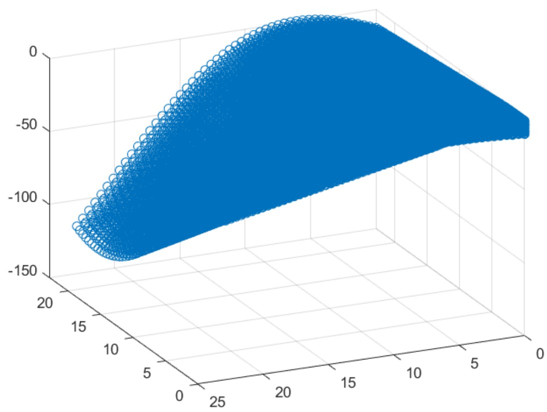

Fig. 7: The line congruence obtained from the exact equation $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$.

Example 7: Let us given a first-order exact differential equation by

$$\left(1 + \ln x + \tfrac{y}{x}\right)dx - (1 - \ln x)dy = 0.$$

When we solve ebove equation we have the following solution:

$$-y + y\ln x + x\ln x = c.$$

The line congruence can be written as

$$\boldsymbol{R}(u,v,w)=\begin{bmatrix} u+2uw, & v+2vw, & -v+v\ln u+u\ln u-c \\ & & +w[-v+v\ln u+u\ln u-c \\ & & +u\left(1+\ln u+\frac{v}{u}\right) \\ & & +v(-1+\ln u)] \end{bmatrix}$$

Line congruence can be seen from Figure 8.

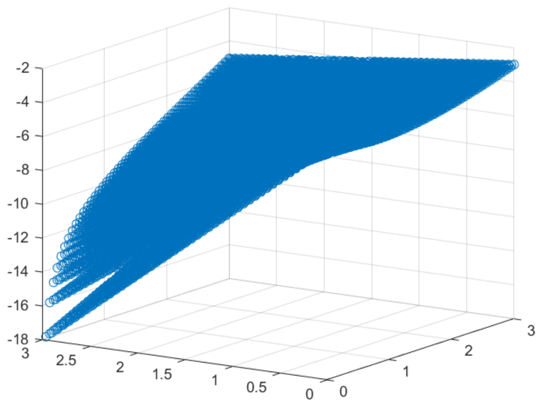

Fig. 8: The line congruence obtained from the exact equation $\left(1+\ln x+\frac{y}{x}\right)dx-(1-\ln x)dy=0$.

5. Conclusion

In this paper, we focused on the geometric approach of the first-order differential equations. For this purpose, we defined a tangent developable ruled surface which takes the solution of this differential equation as directrix curve and the equation itself as tangent vector of this directrix curve. Additionally, a line congruence is expressed based on exact differential equations. Also we presented some applications of developable ruled surfaces and line congruences.

References

Bottema, O. and Roth, B. (1979) Theoretical Kinematics, North-Holland Press, New York.

Chen, M. and Tang, K. (2010) A fully geometric approach for developable cloth deformation simulation. Vis. Computer 26 (2010), 853–863.

Frey, W. and Bindschadler, D. (1993) Computer aided design of a class of developable Bézier surfaces, vol. 8057, General Motors R & D Publication.

Nagle, K. R., Saff, E. B. and Snider, A. D. (2012) Fundamentals of Differential Equations. Boston: Addison-Wesley.

Kreyszig, E. (1991) Differential geometry, Dover Publications.

Odehnal, B. and Pottmann, H. (2001) Computing with discrete models of ruled surfaces and line congruences, Proceedings of the 2nd workshop on computational kinematics, Seoul.

P´erez, F. and Su´arez, J. A. (2007) Quasi-developable Bspline surfaces in ship hull design. Comp. Aided Geom. Design 39 (2007), 853–862.

Pottman, H. and Wallner, J. (2001) Computational Line Geometry, Springer-Verlag, Berlin, Heidelberg.

Zill, D. G. (2001) A first course in differential equations with modeling applications. Pacific Grove, CA: Brooks/Cole Thomson Learning.