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A TILING APPROACH TO FIBONACCI p-NUMBERS

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Abstract. In this paper, we introduce tiling representations of Fibonacci p-numbers, which are generalizations of the well-known Fibonacci and Narayana numbers, and generalized in the distance sense. We obtain that Fibonacci p-numbers count the number of distinct ways to tile a $1 \times n$ board using various $1 \times r$, r-ominoes from $r = 1$ up to $r = p + 1$. Moreover, sum formulas and product identities of these numbers with special subscripts are given by tiling interpretations that allow the derivation of their properties.

1. Introduction

The well-known Fibonacci numbers are given by recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$. Fibonacci numbers have many interesting generalizations, interpretations and properties in almost every modern science. Most generalizations are presented by using the recurrence relation of Fibonacci numbers with the definition of a distance between numbers or by changing the coefficients of the added terms [1]. For example, Falcon and Plaza, considering the k parameter in the recurrence relation of the Fibonacci numbers, determined k-Fibonacci numbers, $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ with the initial values $F_{k,0} = 0$ and $F_{k,1} = 1$ [2]. More details and properties of these numbers can be seen in [3, 4]. Narayana numbers are defined by $N_n = N_{n-1} + N_{n-3}$ for $n \geq 3$ and $N_0 = 0$, $N_1 = N_2 = 1$ as a generalization in the distance sense [5]. For other example, Stakhov, taking into account the p parameter for distance between of the added terms in the recurrence relation of the Fibonacci numbers, defined the Fibonacci p-numbers as

$$F_p(n) = F_p(n-1) + F_p(n-p-1), \quad n > p+1 \tag{1.1}$$

with initial values $F_p(1) = F_p(2) = \ldots = F_p(p+1) = 1$ for $p \geq 0$, and obtained $(p \times 1) \times (p \times 1)$ companion matrix for these numbers [6, 7]. Some authors have studied the fundamental identities of Fibonacci p-numbers similar to the well-known properties of Fibonacci numbers, providing various general formulas for these numbers [8, 9]. Using various properties of Pascal's triangle, Fibonacci p-numbers can be derived, and their matrix representations are given [10]. For more

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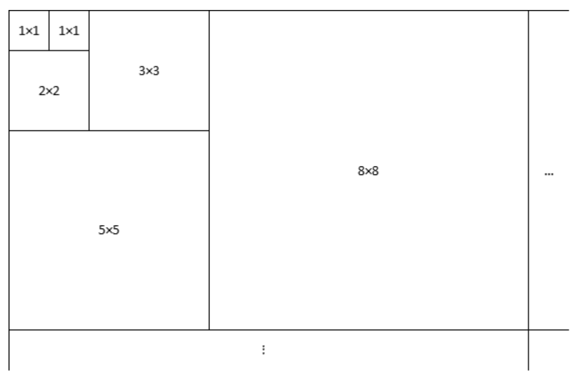
generalizations, see [11, 12, 13]. In [14], Fibonacci numbers are generalized by $F(k,n) = F(k,n-1)+F(k,n-k)$ for $n \geq k+1$ and $F(k,n) = n+1$ for $0 \leq n \leq k$, the integer $k \geq 1$ by changing initial values in the distance sense. In [15, 16], the authors introduced distance Fibonacci numbers and a new kind of these numbers.

The classical method of tiling the plane using appropriate geometric figures has been extensively used to express Fibonacci numbers, their generalizations and properties. Fibonacci numbers are associated with various rectangles and used to obtain well-known relations between Fibonacci numbers. Brother Alfred showed that the sum of the squares of the first n Fibonacci numbers is proven using the geometric argument: $F_1^2 + F_2^2 + \ldots + F_n^2 = F_nF_{n+1}$ [17]. For example, the area of the rectangle below is the summing of the squares of first 6 Fibonacci numbers, $F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2$, or the multiplying of height and width, $F_6(F_5 + F_6) = F_6F_7$.



As an extension of this, in [18, 19], the authors presented tiling interpretations for generalized Fibonacci numbers. In [20], nth Fibonacci number is interpreted as the number of ways to tile a $1 \times n$ board with cells labeled $1, 2, \ldots, n$ using 1×1 squares and 1×2 dominoes. Benjamin and Quinn have used these interpretations to provide tiling proofs of many Fibonacci type relations [21, 22]. For example, the number ways to tile $1 \times n$ boards for $n = 1, 2, 3, 4, 5$ using 1×1 squares and 1×2 dominoes are F_1, F_2, F_3, F_4 and F_5 as provided in the figure 1, respectively.

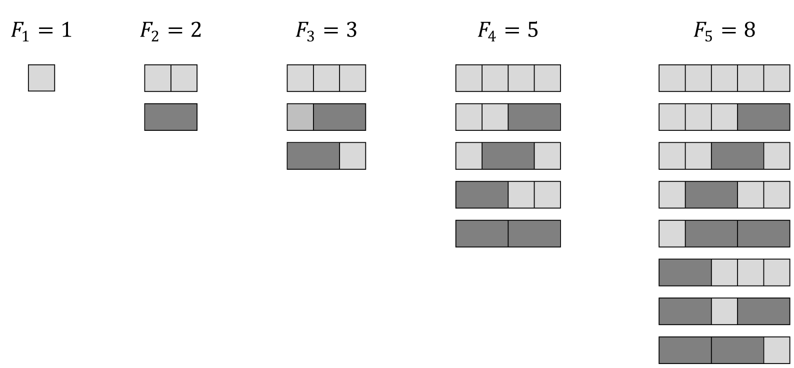


FIGURE 1. Tile interpretations of the first few Fibonacci numbers

The aim of this study is to introduce the Fibonacci p-numbers with the tiling approach. The $(n+1)$th Fibonacci p-number is expressed by tiling interpretations which allows one to derive properties of them via tiling proof. It is also to obtain product identities and sum formulas of these numbers with special subscripts.

2. TILING REPRESENTATIONS OF THE FIBONACCI p-NUMBERS

In this section, we introduce tiling approach to Fibonacci p-numbers using set of n integers. We explore the $(n+1)$th Fibonacci p-number, $F_p(n+1)$ with the tiling interpretations which are related to special set decomposition.

Suppose that $X = \{1, 2, \ldots, n\}$ is the set of n integers. First, we give an interpretation of Fibonacci p-numbers using the number of all p-decompositions of the set X which has 1 and $p+1$ element subsets, such that all $p+1$ subsets contain consecutive integers. Let $\mathcal{T} = \{T_i \mid i \in I\}$ be the family of subsets satisfying the following conditions of the set X

i: $|T_i| = \{1, p+1\}$ for $i \in I$
ii: $T_i \bigcap T_j = \emptyset$ for $i \neq j$, $i, j \in I$
iii: $\left|\bigcup_{i\in I} T_i\right| = n$

where $n \geq 1$ and $p \geq 0$. Each the set $\mathcal{T}$ is called as a p-decomposition of the set X.

Theorem 2.1. *For integers $n \geq 1$ and $p \geq 0$, the number of all p-decompositions of the set X is equal to $F_p(n+1)$.*

Proof. Let $n \geq 1$, $p \geq 0$ be integers and $X = \{1, 2, \ldots, n\}$. Denote by $c_p(n)$ the number of all p-decompositions of the set X. If $n \leq p+1$, then $\mathcal{T}$ can be obtained in only one p-decomposition which is $\{\{1, 2, \ldots, p+1\}\}$. Then we get $c_p(n) = 1 = F_p(n+1)$. Now let us assume that $n > p+1$, and suppose that $c_p(n) = F_p(n+1)$ satisfies for arbitrary n. We will show that $c_p(n+1) = F_p(n+2)$. Let $c_{p,1}(n+1)$ denote the number of all p-decompositions of the set X such that $\{1\} \in \mathcal{T}$ and $c_{p,p+1}(n+1)$ denote the number of all p-decompositions of the set X such that $\{1, 2, \ldots, p+1\} \in \mathcal{T}$. Hence $c_p(n+1) = c_{p,1}(n+1) + c_{p,p+1}(n+1)$. If $\{1\} \in \mathcal{T}$ then the number of all p-decompositions of the set $\{2, 3, \ldots, n+1\}$ is $c_p(n+1-1) = c_{p,1}(n+1)$. If $\{1, 2, \ldots, p+1\} \in \mathcal{T}$ then the number of all p-decompositions of the set $\{p+2, p+3, \ldots, n+1\}$ is $c_p(n+1-p-1) = c_{p,p+1}(n+1)$. By the induction's hypothesis and recurrence relation 1.1 we have

$$\begin{aligned} c_p(n+1) &= c_{p,1}(n+1) + c_{p,p+1}(n+1) \\ &= c_p(n) + c_p(n-p) \\ &= F_p(n+1) + F_p(n-p+1) \\ &= F_p(n+2) \end{aligned}$$

which ends the proof. □

Example 2.2. Let $c_2(n)$ denote the number of all p-decompositions of the set $X = \{1, 2, 3, 4, 5\}$ that contain consecutive integers, where $n \geq 0$. We define $X = \emptyset$ for $n = 0$. From Theorem 2.1 for $p = 2$, all p-decompositions are provided in the table 1.

TABLE 1

n	All p-decompositions of $X = \{1,2,3,4,5\}$	$c_3(n)$
0	$\emptyset$	1
1	$\{\{1\}\}$	1
2	$\{\{1\},\{2\}\}$	1
3	$\{\{1\},\{2\},\{3\}\},\{\{1,2,3\}\}$	2
4	$\{\{1\},\{2\},\{3\},\{4\}\},\{\{1\},\{2,3,4\}\},\{\{1,2,3\},\{4\}\}$	3
5	$\{\{1\},\{2\},\{3\},\{4\},\{5\}\},\{\{1\},\{2,3,4\},\{5\}\},\{\{1\},\{2\},\{3,4,5\}\},\{\{1,2,3\},\{4\},\{5\}\}$	4
		$\uparrow$ $F_2(n+1)$

This interpretation of sets allows a tiling approach for Fibonacci p-numbers. Assume a $1 \times n$ board, which is related to the $(n+1)$th Fibonacci p-number $F_p(n+1)$, is split as follows:

$1 \times p_1$	$1 \times p_2$	$\cdots$	$1 \times p_i$

Suppose the $1 \times n$ board has the tiling that satisfies the following conditions:

i: $p_i \in \{1, p+1\}$ for $i \in I$
ii: $\sum_{i \in I} p_i = n$

Let $F_p(n+1)$ represent the number of distinct ways to tile a $1 \times n$ board using 1×1 and $1 \times (p+1)$ tiles for $p \geq 0$. It is easy to obtain that $F_p(p+1) = 1$ for $n \leq p+1$. If $n > p+1$, we get there are $F_p(n)$ ways to tile in which the tiling ends in a 1×1 tile, and $F_p(n-p)$ ways if the tiling ends in a $1 \times (p+1)$ tile. Thus, $F_p(n+1) = F_p(n) + F_p(n-p)$, in equation 1.1 is obtained.

Theorem 2.3. *For integer $n \geq 1$, the number of distinct ways to tile a $1 \times n$ board with 1×1 and $1 \times (p+1)$ tiles is equal to $F_p(n+1)$.*

Proof. Let $n \geq 1$, $p \geq 0$ be integers and denote by $s(n)$ the number of distinct ways to tile a $1 \times n$ board with 1×1 and $1 \times (p+1)$ tiles. If $n \leq p+1$, then 1×1 tile can be tiled in exactly one way and $s(n) = 1 = F_p(n+1)$. Now let us assume that $n > p+1$, and suppose that $s(n) = F_p(n+1)$ holds for n. We will show that $s(n+1) = F_p(n+2)$. The first tile in all tilings is either 1×1 and $1 \times (p+1)$ tiles. If 1×1 tile is the first tile, then the number of distinct ways to tile a $1 \times (n+1-1)$ board is $s(n)$. If $1 \times (p+1)$ tile is the first tile, then the number of distinct ways to tile a $1 \times (n+1-p-1)$ board is $s(n-p)$. Thus $s(n+1) = s(n) + s(n-p)$. On the other hand, using the induction's hypothesis we have $s(n) = F_p(n+1)$ and $s(n-p) = F_p(n-p+1)$. By the recurrence relation 1.1, we obtain

$$s(n+1) = s(n) + s(n-p) = F_p(n+1) + F_p(n-p+1) = F_p(n+2)$$

and the theorem is proved. □

Example 2.4. There are 5 different ways to tile a 1×7 board using 1×1 and 1×4 tiles from Theorem 2.3 for $p = 3$, all tilings are provided in the figure 2. So we have $F_3(8) = 5$.

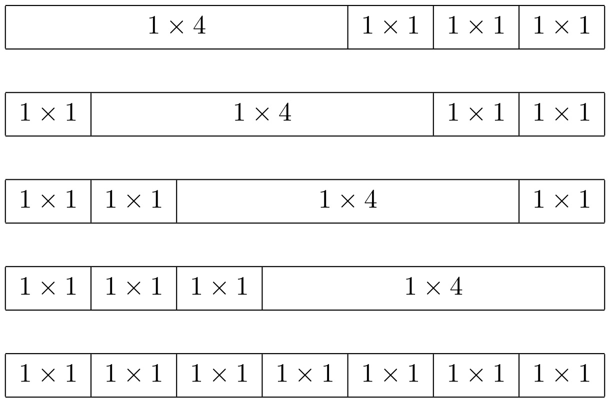


FIGURE 2. Tile interpretations of $F_3(8)$ number.

3. IDENTITIES OF THE FIBONACCI p-NUMBERS

In this section we present the sum formulas and product identities of the Fibonacci p-numbers by tiling interpretations that allow the derivation of their properties.

Theorem 3.1. *For integers $p \geq 0$ and $n \geq 1$, then*

$$\sum_{i=0}^{n} F_p(n+1-i) + 1 = F_p(n+p+2)$$

Proof. From Theorem 2.3, all tilings of a $1\times(n+p+1)$ board with squares, 1×1 tiles, and p-ominoes, $1\times(p+1)$ tiles, can be done in $F_p(n+p+2)$. There is exactly one tiling of all 1×1 tiles. Tilings containing a $1\times(p+1)$ tile can be partitioned according to the location of the first $1\times(p+1)$ tile, counting from the left. If $1\times(p+1)$ tile is the first tile, then all tilings of a $1\times n$ board can be done in $F_p(n+1)$. In other tilings, there are i times 1×1 before the $1\times(p+1)$ tile which covers from $(i+1)$th cell to the $(i+p+1)$th cell for $1 \leq i \leq n$ and we get $F_p(n+1-i)$ as tilings ways. Summing gives the desired result, $\sum_{i=0}^{n} F_p(n+1-i)+1 = F_p(n+p+2)$. So the proof is completed. □

Theorem 3.2. *Let $p \geq 0$ and $k \geq 1$ be integers. For $n+1 = (p+1)k + r$, $0 \leq r < p+1$,*

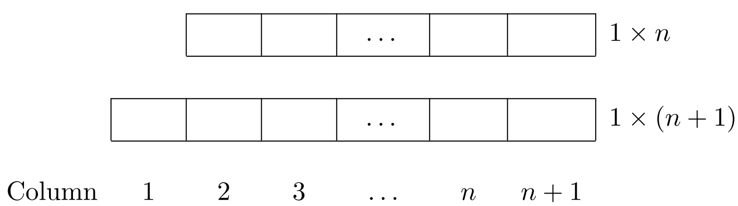
$$F_p(n+2) = \begin{cases} \sum_{i=0}^{\left\lceil \frac{n-p}{p+1} \right\rceil} F_p(n+1-i(p+1)) & , \ if \quad r \geq 1 \\ \sum_{i=0}^{\left\lceil \frac{n-p}{p+1} \right\rceil} F_p(n+1-i(p+1)) + 1 & , \ if \quad r = 0 \end{cases}$$

Proof. From Theorem 2.3, all tilings of a $1\times(n+1)$ board with squares, 1×1 tiles, and p-ominoes, $1\times(p+1)$ tiles, can be done in $F_p(n+2)$. If $r \geq 1$, then all tilings have at least one 1×1 tile. Tilings containing a 1×1 tile can be partitioned according to the location of the first 1×1 tile, counting from the left. If 1×1 tile is the first tile, then all tilings of a $1\times n$ board can be done in $F_p(n+1)$. In other tilings, there are i times $1\times(p+1)$ before the 1×1 tile which covers $(i(p+1)+1)$th cell for $1 \leq i \leq \left\lceil \frac{n-p}{p+1} \right\rceil$ and we get $F_p(n+1-i(p+1))$ as tilings ways. Summing gives the desired result, $\sum_{i=0}^{\left\lceil \frac{n-p}{p+1} \right\rceil} F_p\left(n+1-i(p+1)\right) = F_p\left(n+2\right)$. Otherwise if $r=0$, there is exactly one more tile where all tiles are $1\times(p+1)$ and , $\sum_{i=0}^{\left\lceil \frac{n-p}{p+1} \right\rceil} F_p\left(n+1-i(p+1)\right)+1 = F_p\left(n+2\right)$ is obtained. So the proof is completed. □

Theorem 3.3. *For integers $p\geq 0$ and $k\geq 1$, then*

$$\begin{aligned} F_p\left(n+1\right)F_p\left(n+2\right) &= \sum_{i=0}^{\left\lceil \frac{n-p}{p+1} \right\rceil} F_p\left(n+1-i(p+1)\right)^2 \\ &+ \sum_{i=0}^{\left\lceil \frac{n-2p}{p+1} \right\rceil} F_p\left(n-i(p+1)\right)F_p\left(n+2-(i+1)(p+1)\right) \end{aligned}$$

Proof. We tile independently a $1\times n$ board and a $1\times(n+1)$ board which can be done in $F_p\left(n+1\right)F_p\left(n+2\right)$ ways as shown in following figure.



Number the columns of a $1\times n$ board and a $1\times(n+1)$ as shown. Let c be the smallest numbered column containing a 1×1 tile for any tiling. If $c=2+i(p+1)$ with $0\leq i\leq \left\lceil \frac{n-2p}{1+p} \right\rceil$ then the 1×1 tile appears on the top board and tilings can be done in $F_p\left(n-i(p+1)\right)F_p\left(n+2-(i+1)(p+1)\right)$ ways. If $c=1+i(p+1)$ with $0\leq i\leq \left\lceil \frac{n-p}{p+1} \right\rceil$ then the 1×1 tile appears on the bottom board and tilings can be done in $F_p\left(n+1-i(p+1)\right)^2$ ways. Summing gives the desired result, $F_p\left(n+1\right)F_p\left(n+2\right) = \sum_{i=0}^{\left\lceil \frac{n-p}{1+p} \right\rceil} F_p\left(n+1-i(p+1)\right)^2 + \sum_{i=0}^{\left\lceil \frac{n-2p}{1+p} \right\rceil} F_p\left(n-i(p+1)\right)F_p\left(n+2-(i+1)(p+1)\right)$. This completes the proof. □

4. Conclusion

Generalizations of Fibonacci numbers have been presented in many different ways, and the classical method of tiling the plane using appropriate geometric figures has been extensively used to express these numbers and their properties. Fibonacci p-numbers are presented with an arbitrary integer p parameter for the distance between added terms in the recurrence relation of Fibonacci numbers, and generalized in the distance sense of Fibonacci numbers. In this paper, Fibonacci p-numbers are expressed by tiling interpretations which are related to special set decomposition. It has been shown that Fibonacci p-numbers are the number of different ways to tile a $1\times n$ board with various $1\times r$, r-ominoes from $r=1$ to $r=p+1$ using tiling interpretations. Also, sum formulas and product identities and of these numbers with special subscripts are given by these interpretations. More general interpretations that allow us to calculate the Fibonacci p-numbers can be investigated.

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