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Research Article

Marshall-Olkin Bilal distribution with associated minification process and acceptance sampling plans

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Abstract

In this paper, a new two-parameter lifetime distribution, called the Marshall-Olkin Bilal distribution is introduced and its various structural properties are discussed. The proposed model results from the Marshall-Olkin family of distributions with the baseline model as Bilal distribution. We examined different statistical aspects like moments, quantile function, order statistics and entropy. The hazard rate function of the proposed distribution can be increasing and upside-down bathtub shaped. The model parameter estimation is carried out by maximum likelihood, least squares, and weighted least squares methods, and a simulation study is performed. The flexibility of the proposed model is evaluated by two real data sets by comparing with different contending models. Its application in time series is studied by the associated autoregressive minification process, and the auto-correlation structure is derived. The acceptance sampling plans formulated for the proposed model and the characteristic results are illustrated.

Mathematics Subject Classification (2020). 60E05, 62M10, 62N05

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1. Introduction

In order to interpret real-world phenomena in a wide range of disciplines, many probability distributions have been proposed in the statistical literature. In particular, lifetime distributions play a fundamental role in several reliability fields, such as finance, manufacturing and biological sciences. Considerable effort has been expended in developing large classes of standard probability distributions along with relevant statistical methodologies. Using order statistics in lifetime distribution production, Abd-Elrahman [2] introduced the Bilal distribution providing better flexibility in modelling real data sets compared to the exponential and Lindley distributions.

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The Bilal distribution has less skewness and kurtosis than the exponential distribution and is a member of the class of new better than used in failure rates. As a potential baseline model in the construction of flexible lifetime distributions, recently, there has been a great interest in extensions, generalizations and related applications of the Bilal distribution. As a solution for the unimodal hazard rate function (HRF) of the Bilal distribution, Abd-Elrahman [1] introduced the two-parameter generalization called general Bilal distribution. A three-parameter generalization, called the Harris extended Bilal distribution was introduced by [27], and its various properties have been discussed. The application of U-statistics in the estimation of the scale parameter of Bilal distribution was discussed by [28]. To overcome the deficiencies of the existing distributions for modelling extremely skewed datasets, Altun [13] developed log-Bilal distribution and the associated regression. As a significant solution in associated regression and INAR(1) process for over-dispersed count datasets, Altun [12] introduced the Poisson-Bilal distribution. A bivariate version of the Bilal distribution, the Farlie-Gumbel-Morgenstern bivariate Bilal distribution and its inferential aspects using concomitants of order statistics were discussed by [29]. A new discrete Bilal distribution and its associated count time series model are proposed in the work by [4]. These studies have demonstrated the versatility of the Bilal distribution. There is, however, scope for reaching the goal of improved statistical modelling.

The extended distributions proposed by adding additional parameters generally provide improved flexibility. By adding an extra parameter to the baseline distribution, Marshall and Olkin [26] introduced the Marshall-Olkin family of distributions, providing more flexibility in analyzing real data sets. This new method has the stability property, and the resulting distribution has broad field behaviour in probability density function (PDF) and HRF with respect to the baseline distribution. Further, it offers an adaptable framework for modelling a wide range of phenomena and is applicable in various fields, including reliability analysis, finance, health science, and simulation studies. Using exponential distribution as a baseline model, Marshall and Olkin [26] developed the Marshall-Olkin exponential (MOE) distribution and its various properties have been discussed. Some notable contributions on the Marshall-Olkin family of distributions are given in [19], [21] and [24].

Thus, we develop a flexible lifetime model based on the Bilal distribution with more dynamic density and provide adequate real data modelling by introducing a shape parameter. The present study is motivated by the need for an improved model which also accommodates the properties of the one-parameter Bilal and its two-parameter generalization, namely general Bilal distribution. In addition to that, an empirical comparison study with respect to the MOE distribution is discussed. The prowess of the proposed model in real phenomena is demonstrated in the areas of time series analysis and reliability testing plans.

In time series analysis, autoregressive models occupy a prominent position. The autoregressive processes are widely used in time series models with non-Gaussian distributions due to naturally occurring non-Gaussian Markovian time series. The minification process has the capability to define an autoregressive process that can't be generated using linear random coefficient models, and constant HRF having marginals as non-Gaussian distribution. The use of acceptance sampling approaches in a quality control setting has given rise to a number of research interests in recent years. Accordingly, many authors have developed several kinds of sampling plans assuming that the lifetime of the units follows the distribution under consideration. Due to certain restrictions, examining the whole production unit is impossible. So, the acceptance sampling plan (ASP) acts as a decision rule for the acceptance of a lot from a sample of products. It arose from a consideration of both consumer and producer risks, representing a middle ground between complete inspection and no inspection.

In this article, we establish a two-parameter generalization of Bilal distribution, using the vision of [26], the Marshall-Olkin Bilal (MOB) distribution, followed by its statistical properties and associated application in various fields. The main contribution of this paper is that it establishes the practical usefulness of the proposed distribution. We tried to compare the proposed model with the Bilal distribution, the two-parameter general Bilal distribution and the MOE distribution, which is demonstrated through various statistical properties and real data analysis. The application in time series modelling is evaluated by the autoregressive minification process with the MOB distribution as marginals. We propose the reliability test plan for accepting or rejecting a lot, where the lifetime of the product follows the MOB distribution, and its several characteristic results are investigated.

The remaining part of the paper is organized in the following order. Section 2 describes the nature of the PDF of the MOB distribution. In Section 3, we introduce the associated statistical properties like HRF, moments, moment generating function (MGF), quantile function, mean residual life function, Rényi entropy, order statistics and stress-strength parameter. The parameter estimation of the model under maximum likelihood (ML), least squares (LS), and weighted least squares (WLS) methods and the Fisher information matrix is discussed in Section 4. The asymptotic behaviour of the MOB distribution, with the help of certain simulated data sets, is discussed in Section 5. Section 6 assesses the performance of the proposed distribution with two real data sets. The autoregressive minification process with the MOB distribution as marginals is discussed in Section 7. The ASP associated with the MOB distribution is developed, and the characteristics are presented in Section 8. Finally, the study is concluded in Section 9.

2. The Marshall-Olkin Bilal distribution

In this section, we describe the MOB distribution and investigate some of its structural properties. Following [26], for a baseline distribution having survival function (SF) $\overline{G}_1(x)$ and PDF $g_1(x)$, adding a tilt parameter θ gives rise to Marshall-Olkin extended family of distribution with the SF

$$\overline{G}_{MO}(x) = \frac{\theta\ \overline{G}_1(x)}{1-\overline{\theta}\ \overline{G}_1(x)}, \tag{2.1}$$

where $\theta > 0$ $(\overline{\theta} = 1-\theta)$. The corresponding PDF is given by

$$g_{MO}(x) = \frac{\theta\ g_1(x)}{\left(1-\overline{\theta}\ \overline{G}_1(x)\right)^2}.$$

On the other hand, Abd-Elrahman [2] introduced the Bilal distribution, with the PDF

$$f(x;\lambda) = \frac{6}{\lambda}\left(e^{-\frac{2x}{\lambda}} - e^{-\frac{3x}{\lambda}}\right), \quad x > 0 \tag{2.2}$$

and SF

$$\overline{F}(x;\lambda) = \left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right),\ x > 0, \tag{2.3}$$

where $\lambda > 0$ is the scale parameter. Taking $\overline{G}_1(x)$ as $\overline{F}(x;\lambda)$ in (2.1) results in the MOB distribution, denoted by MOB (λ,θ). The exact definition is formalized below.

Definition 2.1
A continuous random variable X is said to follow MOB (λ,θ) *if it's PDF and cumulative distribution function (CDF) is given by*

$$g(x;\lambda,\theta) = \frac{6\ \theta\ \left(e^{-\frac{2x}{\lambda}} - e^{-\frac{3x}{\lambda}}\right)}{\lambda\ \left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]\right)^2},\ x > 0 \tag{2.4}$$

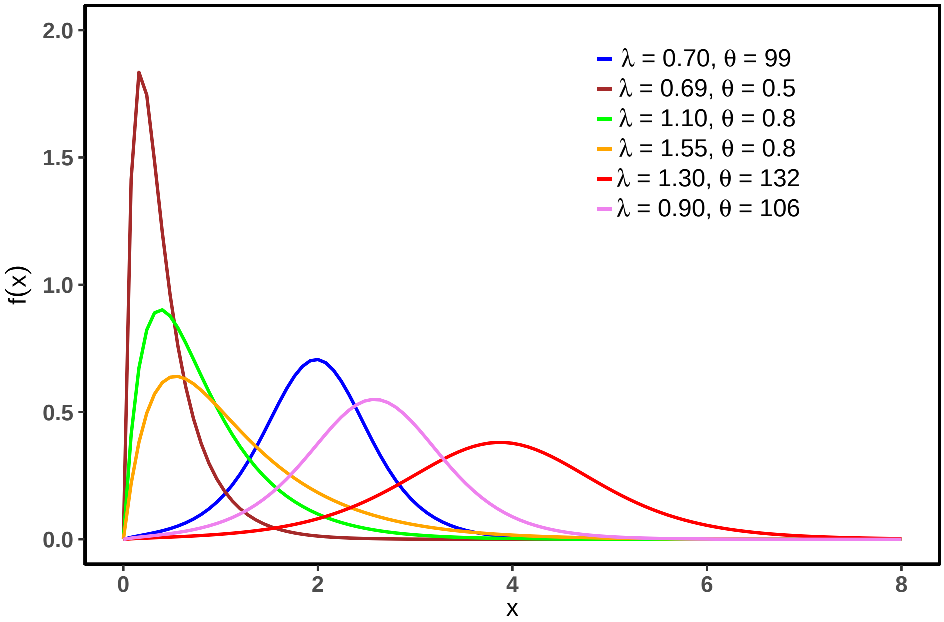

Figure 1. Density plot of the MOB distribution.

and

$$G(x;\lambda,\theta)=\frac{1-\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)}{\left(1-\overline{\theta}\left[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right]\right)},\ x>0, \tag{2.5}$$

respectively, where $\lambda > 0$ is the scale parameter and $\theta > 0$ is the shape parameter.

Remark 2.1

If $\theta = 1$, the MOB distribution reduces to the Bilal distribution. (Hence $\theta = 1$ is omitted voluntarily afterwards).

Figure 1 displays the density plot of the MOB distribution for different parameter values. We can see that the density of the MOB distribution is unimodal and mainly right-skewed or almost symmetrical, with a certain flexibility in the median and kurtosis.

Theorem 2.1

Marshall-Olkin Bilal distribution is geometric extreme stable.

Proof.

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed (iid) random variables with SF $\overline{G}_2(x)$ and N follows geometric (p) such that $X_i's$ are independent of N. Let $U_N = min(X_1, X_2, ..., X_N)$ and $V_N = max(X_1, X_2, ..., X_N)$.

Then the SF of U_N is given by

$$\overline{M}(x) \;=\; P(U_N > x) = \sum_{n=1}^{\infty} \overline{G}_2(x)^{n-1}\ (1-p)^{n-1}\ p\ \overline{G}_2(x) = \frac{p\ \overline{G}_2(x)}{1-(1-p)\ \overline{G}_2(x)}.$$

If X_i follows MOB distribution, then

$$\overline{M}(x)=\frac{p\theta\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)}{1-(1-p\theta)\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)}.$$

Thus, the distribution of U_N and X_i belongs to the same family. Therefore, we can say that the MOB distribution is geometric minimum stable.

Now, the SF of V_N is given by

$$\overline{N}(x) = 1 - P(V_N < x) = 1 - \frac{p\ G_2(x)}{1-(1-p)\ G_2(x)} = \frac{p^{'}\ \overline{G}_2(x)}{1-(1-p^{'})\ \overline{G}_2(x)},$$

where $p^{'} = \frac{1}{p}$. If X_i follows MOB distribution, then

$$\overline{N}(x) = \frac{p^{'}\theta\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)}{1-(1-p^{'}\theta)\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)}.$$

Thus, the distribution of V_N and X_i belongs to the same family. Hence, we can say that the MOB distribution is geometric maximum stable. Thus, the MOB distribution is geometric extreme stable.

The following theorem shows that the MOB distribution can be derived by the compounding argument.

Theorem 2.2
Let $X|\psi$ have a (conditional) survival function

$$\overline{G}(x;\lambda|\psi) = exp\left\{-\left[\frac{1}{\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)} - 1\right]\psi\right\},\ x>0,\ \lambda>0,$$

and ψ have (unconditional) probability density function

$$p(\psi;\theta) = \theta\ e^{-\theta\psi},\ \psi>0,\ \theta>0.$$

Then, the (unconditional) distribution of X is MOB (λ,θ).

Proof.
The theorem is trivial, since

$$\begin{aligned}\overline{G}(x;\lambda,\theta) &= \int_0^\infty \overline{G}(x;\lambda|\psi)\ p(\psi;\theta)d\psi \\ &= \int_0^\infty \theta\ exp\left\{-\frac{\left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]\right)}{\theta\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)}\psi\right\}d\psi \\ &= \frac{\theta\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)}{\left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]\right)}.\end{aligned}$$

Hence proved.

Proposition 2.1
Consider a sequence of iid random variables $\{X_i, i\geq 1\}$ with common survival function $\overline{F}(x)$ and N be a geometric random variable with parameter θ, which is independent of $\{X_i\}$ for all $i\geq 1$. Let $U_N = \min_{1\leq i\leq N} X_i$. Then, $\{U_N\}$ is distributed as MOB (λ,θ) if and only if $\{X_i\}$ follows the Bilal distribution.

3. Statistical properties

3.1. Hazard rate function

From (2.5), the SF of the MOB distribution is given by

$$\overline{G}(x;\lambda,\theta) = \frac{\theta\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)}{\left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]\right)},\ x>0. \tag{3.1}$$

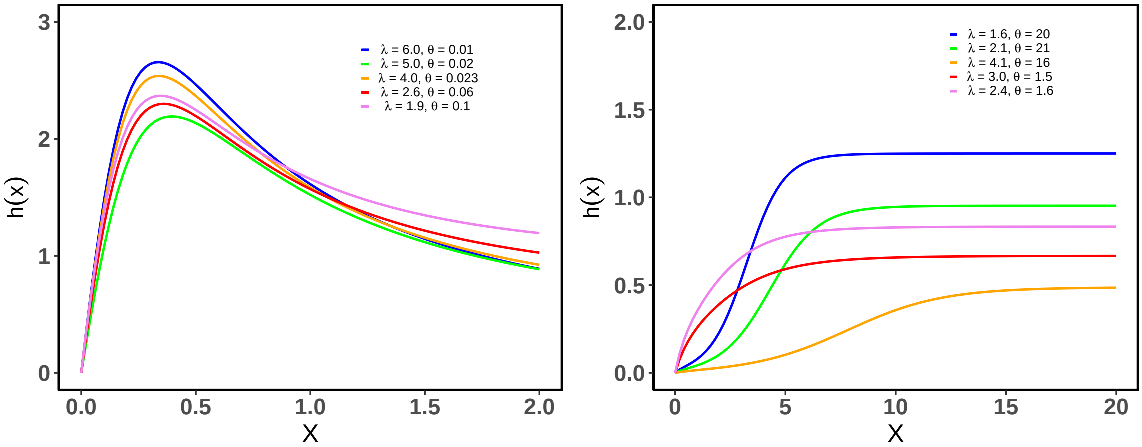

Figure 2. Plots of the HRF of the MOB distribution.

The odds function of the MOB distribution is given by

$$\lambda_G(x;\lambda,\theta) = \frac{G(x;\lambda,\theta)}{\overline{G}(x;\lambda,\theta)} = \frac{1-\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)}{\theta\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)}. \tag{3.2}$$

The HRF of the MOB distribution is given by

$$h(x;\lambda,\theta) = \frac{6\ (1-e^{\frac{-x}{\lambda}})}{\lambda\ \left(3-2e^{-\frac{x}{\lambda}}\right)\left[1-\overline{\theta}\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)\right]},\ x>0. \tag{3.3}$$

Figure 2 displays HRF of the MOB distribution for different parameter values. It shows that the HRF can be increasing and upside-down bathtub, which is not shared by the Bilal distribution.

The cumulative HRF of the MOB distribution is given by

$$\Lambda(x;\lambda,\theta) = \int_0^x h(t;\lambda,\theta)dt \quad = \quad \sum_{i=0}^{\infty}\sum_{j=0}^{i-1} c_{i,j}\ \beta^*(x,2i+j+1,2), \tag{3.4}$$

where $\beta^*(x,2i+j+1,2)$ is the incomplete beta function in terms of $e^{\frac{-t}{\lambda}}$ and

$$c_{i,j} = \frac{6\ (-1)^j\ 3^{i-j}\ 2^j\ \overline{\theta}^i}{\lambda}.$$

The derivation is given in Appendix A.
The corresponding reverse HRF is given by

$$r(x;\lambda,\theta) = \frac{g(x;\lambda,\theta)}{G(x;\lambda,\theta)} = \frac{6\ \theta\ \left(e^{-\frac{2x}{\lambda}}-e^{-\frac{3x}{\lambda}}\right)}{\lambda\ \left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right]\right)\left[1-\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)\right]}. \tag{3.5}$$

3.2. Moment generating function and moments

Theorem 3.1
The MOB (λ,θ) *has finite moments of all positive orders.*

Proof.
Since

$$G(0) = \ 0 \text{ and } \lim_{x\to\infty} inf\ x\ h(x;\lambda,\theta) = \infty,$$

following [15], we can conclude that MOB (λ,θ) has finite moments of all positive orders.

Although the Marshall-Olkin extended family has finite moments of all positive order, it cannot be expressed in closed form. An infinite series expansion is given in the following proposition.

Proposition 3.1

(1) *The* MOB *density can be expressed as a linear combination of Lehman Type-II-Bilal distribution ([18]) as follows.*

$$g(x;\lambda,\theta)=\sum_{i=0}^{\infty}\frac{\lambda w_i}{6(i+1)}\ (i+1)\ f(x;\lambda)\ [\overline{F}(x;\lambda)]^i,$$

where $f(x;\lambda)$ & $\overline{F}(x;\lambda)$ *are given in (2.2) and (2.3).*

(2) *Let* $X\sim$ MOB (λ,θ). *Then, the MGF of X is given by*

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}\ \lambda\ \beta(2+2i+j-\lambda t,2),\ t<\frac{2}{\lambda}.$$

(3) *Let* $X\sim$ MOB (λ,θ) *and r be an integer. Then the* r^{th} *raw moment of X is given by*

$$\mu_r^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}\ r!\ \lambda^r\ \left[\frac{1}{(2+2i+j)^r}-\frac{1}{(3+2i+j)^r}\right].$$

(4) *The* r^{th} *incomplete moment,* $m_r(y)=\int_0^y g(x;\lambda,\theta)dx$, *for the MOB distribution is given by*

$$m_r(y)=\sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}\ \left[\gamma\left(r+1,\frac{(2+2i+j)}{\lambda},y\right)-\gamma\left(r+1,\frac{(3+2i+j)}{\lambda},y\right)\right],$$

where

$$z_{ij}=w_i\binom{i}{j}\ 3^{i-j}\ (-2)^j,\ w_i=\begin{cases}\binom{i+1}{i}\overline{\theta}^i\ \frac{6}{\lambda}\ \theta & if\ 0<\theta<1\\ \frac{6}{\lambda\theta}(-1)^i\left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l\ \binom{l}{i}\ \binom{l+1}{l}\right] & if\ \theta>1\end{cases},\quad (3.6)$$

$$\beta(a,b)\ =\ \int_0^1 v^{a-1}(1-v)^{b-1}dv,\ \text{and}\ \gamma(r+1,a,y)=\int_0^y\ x^r\ e^{-ax}\ dx.$$

The derivation is given in Appendix B.

3.3. Quantile function

The following proposition will give the quantile function of the MOB distribution.

Proposition 3.2
The quantile function of the MOB distribution is given by $Q(u;\lambda,\theta)=F^{-1}(x;\lambda,\theta)$ *and can be expressed as*

$$Q(u;\lambda,\theta)=-\lambda\ \log[\gamma(u)],\quad (3.7)$$

where

$$\gamma(u) = \begin{cases} 0.5+\sin(\alpha_u+\frac{\pi}{6}) & if \quad 0<\alpha<0.5 \\ 0.5 & if \quad \alpha=0.5 \\ 0.5-\cos(\alpha_u+\frac{\pi}{3}) & if \quad 0.5<\alpha<1 \end{cases},$$

$$\alpha_u = \frac{1}{3}\, tan^{-1}\left(\frac{2\,\sqrt{\alpha\,(1-\alpha)}}{2\,\alpha-1}\right) \text{ and } \alpha = 1-\frac{(1-u)}{[\theta+\overline{\theta}\,(1-u)]}.$$

Proof.
Using (2.5), we have

$$G(x,\lambda,\theta)=u \implies 1-\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)=1-\frac{(1-u)}{[\theta+\overline{\theta}\,(1-u)]}. \tag{3.8}$$

The LHS of (3.8) is the CDF of the Bilal distribution. Hence, the proof follows from the quantile function of Bilal distribution [2].

Since the MOB distribution has a closed-form quantile function, it has the variate generation property, which is very useful in simulation studies.

3.4. Mean residual life function

Theorem 3.2
Let $X \sim$ MOB (λ,θ)*. Then the mean residual life function of* X*,* $m(t)=\frac{\int_t^\infty \overline{G}(x;\lambda,\theta)\ dx}{\overline{G}(t;\lambda,\theta)}$*, is given by*

$$m(t)=\left(\left[3-2e^{-\frac{t}{\lambda}}\right]^{-1}-\overline{\theta}\ e^{-\frac{2t}{\lambda}}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{1+i}\ z_{i,j}^m\ \frac{\lambda\ e^{\frac{-t(2i+j)}{\lambda}}}{(2+2i+j)},$$

where

$$z_{i,j}^m = w_i^m\binom{1+i}{j}\ 3^{1+i-j}\ (-1)^j\ 2^j$$

$$\text{and } w_i^m = \begin{cases} \overline{\theta}^i & if\ 0<\theta<1 \\ \frac{1}{\theta^2}(-1)^i\left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l\ \binom{l}{i}\ \right] & if\ \theta>1 \end{cases}.$$

The derivation is given in Appendix C.

3.5. Order statistics

Suppose $X_1, X_2, \ldots, X_n$ be a random sample from the MOB distribution with PDF $g(x;\lambda,\theta)$ and CDF $G(x;\lambda,\theta)$. Let $X_{j:n}$ be the j^{th} order statistic. Then, the PDF of $X_{j:n}$ is given by

$$g_{j:n}(x;\lambda,\theta)=\binom{n}{j}\ j\ g(x;\lambda,\theta)[\overline{G}(x;\lambda,\theta)]^{n-j}[1-\overline{G}(x;\lambda,\theta)]^{j-1}.$$

Using the expansion of the MOB distribution given in the Proposition [3.1], we have

$$\begin{aligned} g_{j:n}(x;\lambda,\theta) &= \sum_{h=0}^{j-1} c_{j,h} \frac{6\ \theta^{1+n-j+h}\left(e^{-\frac{2x}{\lambda}} - e^{-\frac{3x}{\lambda}}\right)\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]^{n-j+h}}{\lambda\ \left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]\right)^{2+n-j+h}} \\ &= \sum_{h=0}^{j-1}\sum_{i=0}^{\infty} c_{j,h}\ w_i^* e^{-\frac{x}{\lambda}\ (2[1+n-j+h+i])} \left(1-e^{-\frac{x}{\lambda}}\right)\ \left(3-2e^{-\frac{x}{\lambda}}\right)^{i+n-j+h} \\ &= \sum_{h=0}^{j-1}\sum_{i=0}^{\infty}\sum_{m=0}^{i+n-j+h} z^*_{i,m,j,h} e^{-\frac{x}{\lambda}\ (2[1+n-j+h+i])} \left(1-e^{-\frac{x}{\lambda}}\right)^{m+1}, \end{aligned}$$

where

$$z^*_{i,m,j,h} = c_{j,h}\ w_i^* \binom{i+n-j+h}{m} 3^{i+n-j+h-m}\ (-1)^m\ 2^m,\ c_{j,h} = \binom{n}{j}\ j\ (-1)^h \binom{j-1}{h}$$

$$\text{and } w_i^* = \begin{cases} \binom{i+1+n-j+h}{i}\overline{\theta}^i\ \frac{6}{\lambda}\ \theta^{1+n-j+h} & if\ 0<\theta<1 \\ \frac{6}{\lambda\theta}(-1)^i \left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l\ \binom{l}{i}\ \binom{l+1+n-j+h}{l}\right] & if\ \theta>1 \end{cases}.$$

The distribution of sample minima and maxima, and L-moments follow from the density of order statistics.

3.6. Entropy

Entropy is one of the measures of uncertainty about a random variable. One of the most common measures of entropy, Rényi entropy, is given in the following proposition.

Proposition 3.3
Let $X \sim$ MOB (λ,θ). Then the Rényi entropy of X is given by $I_R(x) = (1-v)^{-1} logE[g(X;\lambda,\theta)^{v-1}]$, with $v>0$ and $v\neq 1$, and can be expressed as

$$I_R(x) = (1-v)^{-1}\ log\left[\sum_{i=0}^{\infty}\ \sum_{j=0}^{i}\ z^v_{i,j}\ \lambda\ \beta(2v+2i+j, v+1)\right],$$

where $z^v_{i,j}$ is obtained by replacing w_i by w_i^v in (3.6) and it is given by

$$w_i^v = \begin{cases} \binom{-2v}{i}(-1)^i\ \overline{\theta}^i\ \frac{6^v}{\lambda^v}\ \theta^v & if\ 0<\theta<1 \\ \frac{6^v}{\lambda^v\theta^v}(-1)^i\ \left[\sum_{l=i}^{\infty}(\frac{1}{\theta}-1)^l\ \binom{l}{i}\ \binom{-2v}{l}\right] & if\ \theta>1 \end{cases}.$$

The derivation is given in Appendix D.

3.7. Stress-strength parameter

Let X and Y be two independent random variables following MOB (λ,θ). Then the stress-strength parameter, $R = 1-\int_0^{\infty} g(x;\lambda,\theta)\ \overline{G}(x;\lambda,\theta)\ dx$, is given by

$$R = P(Y<X) = 1-\int_0^{\infty} g(x;\lambda,\theta)\ \overline{G}(x;\lambda,\theta)dx = 1-\sum_{i=0}^{\infty}\ \sum_{j=0}^{1+i}\ z^{ss}_{i,j}\ \lambda\beta(4+2i+j,2),$$

where

$$z_{i,j}^{ss} = w_i^{ss} \binom{1+i}{j} 3^{1+i-j} (-1)^j 2^j$$

$$\text{and } w_i^{ss} = \begin{cases} \binom{i+2}{i}\overline{\theta}^i \frac{6}{\lambda} \theta^2 & if\ 0<\theta<1 \\ \frac{6}{\lambda\theta}(-1)^i \left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l \binom{l}{i} \binom{l+2}{l}\right] & if\ \theta>1 \end{cases}.$$

The derivation is given in Appendix E.

4. Parameter estimation

Here, we estimate the unknown parameters of the MOB distribution using ML, LS and WLS methods.

4.1. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from MOB (λ, θ). From (2.4), the log-likelihood function is given by

$$\log L_{(\lambda,\theta)} = n\log\left(\frac{6\,\theta}{\lambda}\right) - \sum_{i=1}^{n}\frac{2x_i}{\lambda} + \sum_{i=1}^{n}\log\left(1-e^{-\frac{x_i}{\lambda}}\right) - 2\sum_{i=1}^{n}\log\left(1-\overline{\theta}[3e^{-\frac{2x_i}{\lambda}} - 2e^{-\frac{3x_i}{\lambda}}]\right). \tag{4.1}$$

The ML estimates $(\widehat{\lambda}, \widehat{\theta})$ of the parameters (λ, θ) are now obtained from the solution of following equations:

$$\frac{\partial \log L_{(\lambda,\theta)}}{\partial\lambda} = 0 \text{ and } \frac{\partial \log L_{(\lambda,\theta)}}{\partial\theta} = 0\ .$$

Now

$$\frac{\partial \log L_{(\lambda,\theta)}}{\partial\lambda} = -\frac{n}{\lambda} + \sum_{i=1}^{n}\frac{2x_i}{\lambda^2} - \sum_{i=1}^{n}\frac{x_i e^{-\frac{x_i}{\lambda}}}{(1-e^{-\frac{x_i}{\lambda}})\lambda^2} + \overline{\theta}\sum_{i=1}^{n}\frac{6\ x_i\left(e^{-\frac{2x_i}{\lambda}} - e^{-\frac{3x_i}{\lambda}}\right)}{\lambda^2\left(1-\overline{\theta}[3e^{-\frac{2x_i}{\lambda}} - 2e^{-\frac{3x_i}{\lambda}}]\right)}$$

and

$$\frac{\partial \log L_{(\lambda,\theta)}}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\frac{[3e^{-\frac{2x_i}{\lambda}} - 2e^{-\frac{3x_i}{\lambda}}]}{\left(1-\overline{\theta}[3e^{-\frac{2x_i}{\lambda}} - 2e^{-\frac{3x_i}{\lambda}}]\right)}.$$

Since we can't find the solution in an explicit form when equating to zero, we would go for direct maximization of (4.1) using numerical methods with the help of statistical software.

The inferential analysis on the parameters can be performed using the asymptotic properties of ML estimates. For the parameter estimate $\widehat{\phi} = (\widehat{\lambda}, \widehat{\theta})$, assuming underlying regularity conditions

$$\widehat{\phi} \sim N_2(0_2, K^{-1}),$$

where K^{-1} is the information matrix of the parameters and for large values of n, $K = n^{-1}J_n$ where

$$J_n = \begin{bmatrix} \frac{\partial^2 \log L_{(\lambda,\theta)}}{\partial\lambda^2} & \frac{\partial^2 \log L_{(\lambda,\theta)}}{\partial\lambda\partial\theta} \\ \frac{\partial^2 \log L_{(\lambda,\theta)}}{\partial\theta\partial\lambda} & \frac{\partial^2 \log L_{(\lambda,\theta)}}{\partial\theta^2} \end{bmatrix}_{(\lambda,\theta)=(\widehat{\lambda},\widehat{\theta})}.$$

4.2. Least squares and weighted least squares estimation

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the order statistics of a random sample of size n from the MOB (λ, θ). By minimizing the following equation, we obtain the LS estimates of the parameters (λ, θ):

$$LS_{(\lambda,\theta)} = \sum_{i=1}^{n} \left[\frac{1 - \left(3e^{-\frac{2x_{(i)}}{\lambda}} - 2e^{-\frac{3x_{(i)}}{\lambda}}\right)}{\left(1 - \bar{\theta}\left[3e^{-\frac{2x_{(i)}}{\lambda}} - 2e^{-\frac{3x_{(i)}}{\lambda}}\right]\right)} - \frac{i}{n+1} \right]^2. \tag{4.2}$$

The WLS estimates of the parameters (λ, θ) are obtained by minimizing the following equation:

$$WLS_{(\lambda,\theta)} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\frac{1 - \left(3e^{-\frac{2x_{(i)}}{\lambda}} - 2e^{-\frac{3x_{(i)}}{\lambda}}\right)}{\left(1 - \bar{\theta}\left[3e^{-\frac{2x_{(i)}}{\lambda}} - 2e^{-\frac{3x_{(i)}}{\lambda}}\right]\right)} - \frac{i}{n+1} \right]^2. \tag{4.3}$$

5. Simulation

The model performance is analysed by means of a simulation study. The simulation is carried out by 1,000 replications for sample of sizes $n=$ 50, 100, 150, 200 and 250 with the following parameter values: $(\lambda, \theta) = (1.5, 0.4)$, $(1.8, 0.64)$, $(0.76, 1.41)$ and $(2.2, 2.7)$. The parameter estimation is carried out by ML, LS and WLS methods, and the following quantities were computed.

(1) Bias of the parameters,

$\text{Bias}(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha), \ \alpha \in \{\lambda, \theta\}$.

(2) Root mean square error (RMSE) of the parameters,

$\text{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2}, \ \alpha \in \{\lambda, \theta\}$.

Table 1 lists the bias and RMSE for the ML, LS and WLS estimates of the parameters λ and θ.

In general, we can conclude that ML, LS, and WLS estimations performed very well. The RMSE and bias decrease as n increases.

6. Data analysis

We assess the performance of the proposed model by two real datasets, the carbon fibres data set provided by [25], and the cancer patient data set given in [31]. We compare the performance of the MOB distribution with some other competing models such as Bilal (B) distribution, General Bilal (GB) distribution introduced by [1], Exponentiated Exponential (EE) distribution defined by [20], Gamma (G) distribution, Weibull (W) distribution and Exponentiated Lindley (EL) distribution discussed by [30].

We estimate the unknown parameters of the MOB distribution by the ML method and assess the model performance by information criteria and goodness-of-fit statistics. The smaller values of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) and the larger value of log likelihood (logL) indicate the model adequacy. The goodness of fit of the model is evaluated by employing the Kolmogorov-Smirnov (KS) statistic and associated p value, Anderson-Darling (AD), Cramer-von Mises (CM) and average scaled absolute error (ASAE) ([17]) statistics. The smaller the goodness-of-fit measures, the better the fit.

Table 1. Simulation results of the MOB distribution.

$\lambda = 1.5, \theta = 0.4$										
	n	MLE			LSE			WLSE		
		Estimate	Bias	RMSE	Estimate	Bias	RMSE	Estimate	Bias	RMSE
λ	50	1.5671	0.0671	0.5481	1.9059	0.4059	0.8622	1.8398	0.3398	0.8043
	100	1.5237	0.0237	0.3993	1.7683	0.2683	0.6995	1.6924	0.1924	0.5970
	150	1.4945	0.0055	0.3187	1.6984	0.1984	0.5928	1.6119	0.1119	0.4639
	200	1.5149	0.0149	0.2824	1.6666	0.1666	0.5372	1.5882	0.0882	0.3969
	250	1.5211	0.0211	0.2771	1.5000	0.0000	0.0000	1.5688	0.0688	0.3615
θ	50	0.4822	0.0822	0.2564	0.4081	0.0081	0.2773	0.4206	0.0206	0.2698
	100	0.4643	0.0643	0.2160	0.4113	0.0113	0.2447	0.4202	0.0202	0.2300
	150	0.4554	0.0554	0.1833	0.4122	0.0122	0.2238	0.4241	0.0241	0.2048
	200	0.4347	0.0347	0.1583	0.4140	0.0140	0.2091	0.4220	0.0220	0.1868
	250	0.4257	0.0257	0.1493	0.4000	0.0000	0.0000	0.4211	0.0211	0.1753
$\lambda = 1.8, \theta = 0.64$										
	n	MLE			LSE			WLSE		
		Estimate	Bias	RMSE	Estimate	Bias	RMSE	Estimate	Bias	RMSE
λ	50	1.8765	0.0765	0.4248	2.0401	0.2401	0.5394	2.0000	0.2000	0.5039
	100	1.8445	0.0445	0.3274	1.9680	0.1680	0.4504	1.9236	0.1236	0.3927
	150	1.8209	0.0209	0.2860	1.9338	0.1338	0.3974	1.8880	0.0880	0.3419
	200	1.8310	0.0310	0.2585	1.9166	0.1166	0.3719	1.8751	0.0751	0.3092
	250	1.8166	0.0166	0.2402	1.8000	0.0000	0.0000	1.8622	0.0622	0.2925
θ	50	0.6686	0.0286	0.2378	0.5961	0.0439	0.2629	0.6123	0.0277	0.2541
	100	0.6594	0.0194	0.2104	0.6085	0.0315	0.2396	0.6238	0.0162	0.2260
	150	0.6646	0.0246	0.1937	0.6152	0.0248	0.2222	0.6326	0.0074	0.2075
	200	0.6558	0.0158	0.1776	0.6245	0.0155	0.2117	0.6382	0.0018	0.1942
	250	0.6593	0.0193	0.1698	0.6400	0.0000	0.0000	0.6398	0.0002	0.1867
$\lambda = 0.76, \theta = 1.41$										
	n	MLE			LSE			WLSE		
		Estimate	Bias	RMSE	Estimate	Bias	RMSE	Estimate	Bias	RMSE
λ	50	0.7682	0.0082	0.1052	0.7917	0.0317	0.1090	0.7863	0.0263	0.1059
	100	0.7638	0.0038	0.0807	0.7803	0.0203	0.0907	0.7748	0.0148	0.0843
	150	0.7613	0.0013	0.0745	0.7775	0.0175	0.0827	0.7719	0.0119	0.0774
	200	0.7652	0.0052	0.0680	0.7768	0.0168	0.0784	0.7719	0.0119	0.0725
	250	0.7623	0.0023	0.0645	0.7600	0.0000	0.0000	0.7693	0.0093	0.0685
θ	50	1.4576	0.0476	0.3487	1.3766	0.0334	0.3579	1.3970	0.0130	0.3543
	100	1.4369	0.0269	0.3238	1.3831	0.0269	0.3398	1.4008	0.0092	0.3321
	150	1.4393	0.0293	0.3108	1.3847	0.0253	0.3213	1.4030	0.0070	0.3151
	200	1.4296	0.0196	0.2876	1.3962	0.0138	0.3124	1.4109	0.0009	0.3024
	250	1.4322	0.0222	0.2778	1.4100	0.0000	0.0000	1.4103	0.0003	0.2907
$\lambda = 2.2, \theta = 2.7$										
	n	MLE			LSE			WLSE		
		Estimate	Bias	RMSE	Estimate	Bias	RMSE	Estimate	Bias	RMSE
λ	50	2.2106	0.0106	0.2241	2.2462	0.0462	0.2196	2.2377	0.0377	0.2159
	100	2.2005	0.0005	0.1656	2.2232	0.0232	0.1737	2.2161	0.0161	0.1655
	150	2.1969	0.0031	0.1516	2.2199	0.0199	0.1564	2.2112	0.0112	0.1489
	200	2.2063	0.0063	0.1374	2.2213	0.0213	0.1437	2.2149	0.0149	0.1372
	250	2.2012	0.0012	0.1295	2.2000	0.0000	0.0000	2.2113	0.0113	0.1296
θ	50	2.7582	0.0582	0.4575	2.6634	0.0366	0.4607	2.6901	0.0099	0.4584
	100	2.7294	0.0294	0.4366	2.6692	0.0308	0.4474	2.6893	0.0107	0.4443
	150	2.7304	0.0304	0.4286	2.6689	0.0311	0.4299	2.6930	0.0070	0.4240
	200	2.7211	0.0211	0.4041	2.6844	0.0156	0.4187	2.7022	0.0022	0.4136
	250	2.7222	0.0222	0.3906	2.7000	0.0000	0.0000	2.6963	0.0037	0.4029

6.1. The cancer patient data set

We consider a sample of 128 bladder cancer patients and their emission times (in months) given in [31]. Figure 3 displays the total time on test (TTT) plot and box plot for the

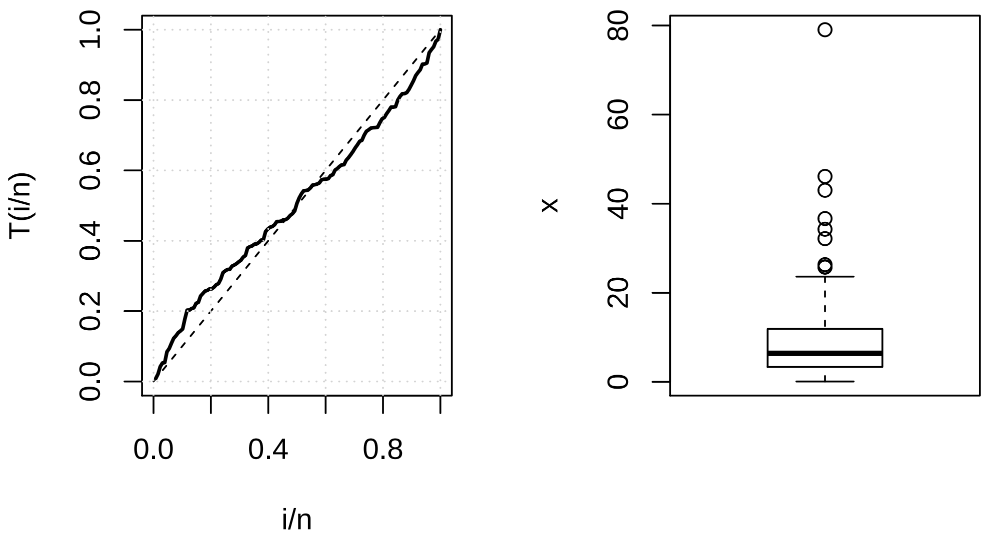

Figure 3. The TTT plot (left) and box plot (right) for the cancer patient data set.

data, and we can see that the observations are right-skewed and have an upside-down bathtub HRF, thus indicating the applicability of the proposed model.

Table 2. ML estimates with SE (in parentheses) and goodness-of-fit-measures for the cancer patient data set.

Distribution	Estimates		logL	AIC	BIC	KS	p value	AD	CM	ASAE
MOB (λ, θ)	30.669 (12.6342)	0.094 (0.0742)	-411.9	827.8153	833.5194	0.0426	0.97	0.3960	0.0297	0.0081
B (λ)	11.088 (0.7081)		-424.7	851.3509	854.2029	0.1303	0.03	4.5073	0.6902	0.0261
MOE (λ, θ)	0.110 (0.0199)	1.056 (0.3215)	-414.3	832.6523	838.3564	0.0812	0.37	1.1130	0.1700	0.0147
GB (θ, λ)	1.015 (0.0555)	11.182 (0.7988)	-424.6	853.2829	858.9869	0.1278	0.03	4.3549	0.6575	0.0271
EE (α, β)	1.218 (0.1488)	8.254 (0.9245)	-413.1	830.1552	835.8592	0.0725	0.51	0.7137	0.1278	0.0000
EL (α, β)	0.733 (0.0911)	0.165 (0.0166)	-416.3	836.5719	842.2759	0.0927	0.22	1.3228	0.2465	0.0188
W (θ, λ)	1.048 (0.0676)	9.560 (0.8529)	-414.1	832.1738	837.8778	0.0700	0.56	0.9578	0.1537	0.0152
G (α, β)	1.173 (0.1308)	0.125 (0.0173)	-413.4	830.7356	836.4396	0.0732	0.50	0.7706	0.1353	0.0153

Table 2 lists the ML estimates with SE (in parentheses) for different models. We can see that the MOB distribution has the maximum logL and the least AIC and BIC values. Moreover, the KS statistic is minimum with a large p value, and the AD, CM and ASAE statistics have the smallest values. We can conclude that the MOB distribution performs well among the considered competitive models.

The fitted PDF and CDF plots, Q-Q and P-P plots are given in Figure 4. We can see that the MOB distribution provides a better fit in the estimated density and CDF plots.

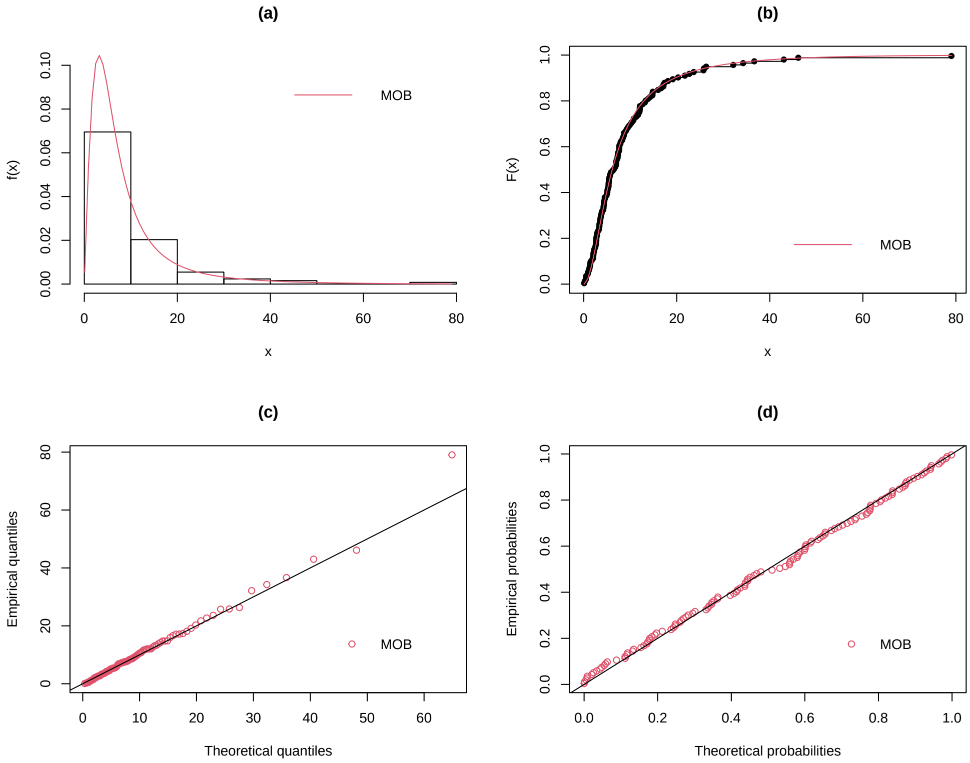

Figure 4. The fitted PDF (a) and CDF plots (b), Q-Q plot (c) and P-P plot (d) for the cancer patient data set.

The points in the Q-Q and P-P plots are almost in a straight line. Hence, we can infer that the MOB distribution yields the best fit for the cancer patient data.

6.2. The carbon fibres data set

Here, we consider the uncensored 60 real observations on the breaking stress of carbon fibres given in [25]. Figure 5 displays the TTT plot and box plot for the data, and we can see that the observations are symmetrical and have an increasing HRF, thus indicating the applicability of the proposed model.

Table 3 lists the ML estimates with standard errors (SE) for different models. We can see that the MOB distribution has the maximum logL and the least AIC and BIC values. Moreover, the KS statistic is minimum with a large p value, and the AD, CM and ASAE statistics have the smallest values. Here too, the MOB distribution performs well among the considered competitive models.

The fitted PDF and CDF plots, Q-Q and P-P plots are given in Figure 6. We can see that the MOB distribution provides a better fit in the estimated density and CDF plots. The points in the Q-Q and P-P plots are almost in a straight line. Hence, we can infer that the MOB distribution yields the best fit for the carbon fibres data.

7. Minification process

Now, we look forward to the application of the MOB distribution in the field of time series analysis. For this, we are going to create a minification process with marginals as the MOB distribution.

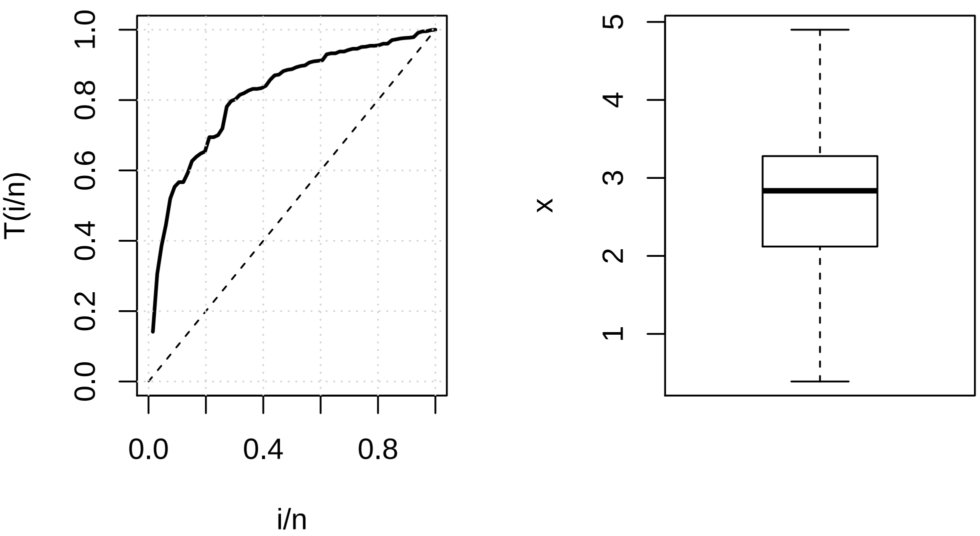

Figure 5. The TTT plot (left) and box plot (right) for the carbon fibres data set.

Table 3. ML estimates with SE (in parentheses) and goodness-of-fit-measures for the carbon fibres data set.

Distribution	Estimates		logL	AIC	BIC	KS	p value	AD	CM	ASAE
MOB (λ, θ)	0.992 (0.1060)	92.939 (59.2891)	-85.1	174.1501	178.5294	0.061	0.97	0.3235	0.0513	0.0203
B (λ)	3.347 (0.2957)		-112.7	227.4284	229.6180	0.249	0.00	8.0711	1.5402	0.1833
MOE (λ, θ)	1.9899 (250.1131)	0.2153 (161.3181)	-85.2	174.3457	178.725	0.061	0.97	0.327	0.0523	0.0203
GB (θ, λ)	1.167 (0.0868)	3.680 (0.3541)	-110.7	225.3547	229.7340	0.280	0.00	9.2839	1.8605	0.1565
EE (α, β)	9.198 (2.1489)	0.993 (0.0987)	-95.4	194.7447	199.1240	0.155	0.08	2.0939	0.3731	
EL (α, β)	7.040 (1.6725)	1.246 (0.1090)	-93.8	191.5939	195.9733	0.147	0.12	1.8381	0.3286	0.0564
W (λ, θ)	3.441 (0.3309)	3.062 (0.1149)	-86.1	176.1352	180.5145	0.082	0.76	0.4859	0.0836	0.0229
G (α, β)	7.487 (1.2754)	2.713 (0.4780)	-91.2	186.3351	190.7144	0.133	0.20	1.3106	0.2461	0.0381

Theorem 7.1
Consider a first-order autoregressive minification process with the structure

$$X_n = \begin{cases} \epsilon_n & \textit{with probability } \theta \\ min(X_{n-1}, \epsilon_n) & \textit{with probability } 1-\theta \end{cases}, \tag{7.1}$$

where $0 < \theta < 1$ *and* $\{\epsilon_n, n \geq 1\}$ *are iid random variables and independent of* X_n*. Then* $\{X_n,\ n \geq 0\}$ *is a stationary Markovian first-order autoregressive model with marginals as the MOB distribution if and only if* $\{\epsilon_n\}$ *follows the Bilal distribution.*

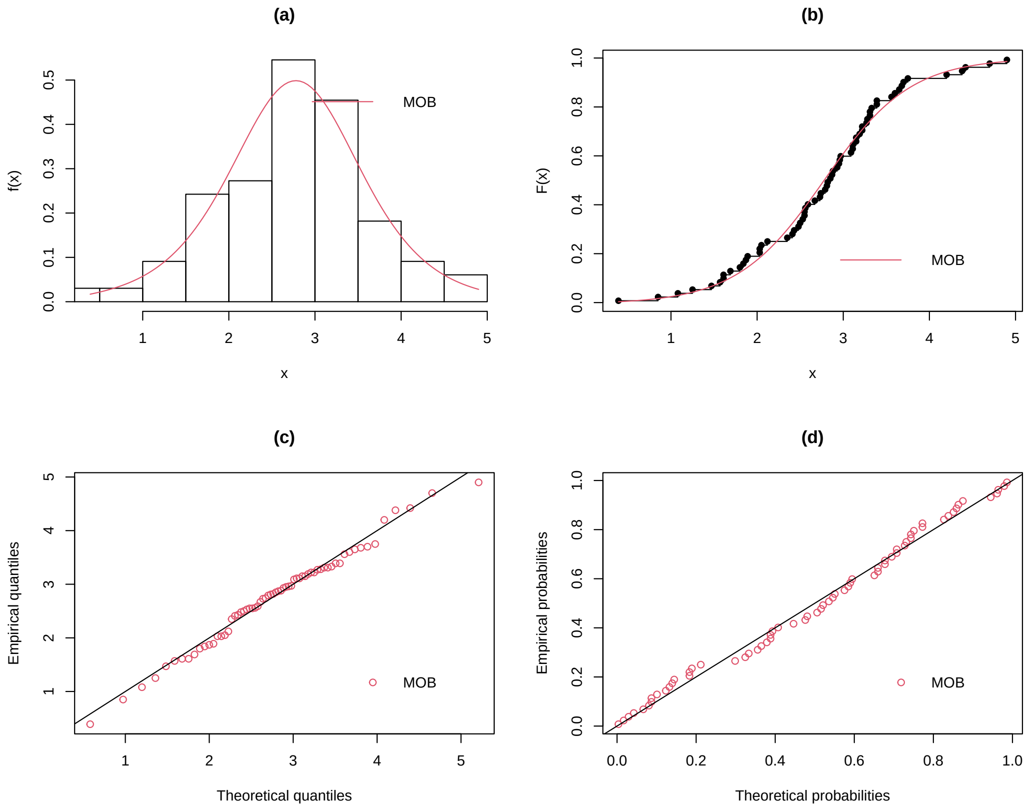

Figure 6. The fitted PDF (a) and CDF plots (b), Q-Q plot (c) and P-P plot (d) for the carbon fibres data set.

Proof.
From (7.1), the SF is given by

$$\overline{F}_{X_n}(x) = \theta\ \overline{F}_{\epsilon_n}(x) + (1-\theta)\ \overline{F}_{X_{n-1}}(x)\ \overline{F}_{\epsilon_n}(x). \tag{7.2}$$

Under the stationary equilibrium, it becomes

$$\overline{F}_X(x) \quad = \quad \theta\overline{F}_\epsilon(x) + (1-\theta)\overline{F}_X(x)\overline{F}_\epsilon(x) \implies \overline{F}_X(x) = \frac{\theta\overline{F}_\epsilon(x)}{\left[1-(1-\theta)\overline{F}_\epsilon(x)\right]}. \tag{7.3}$$

If we take $\overline{F}_\epsilon(x)$ in (7.3) as

$$\overline{F}_\epsilon(x) = \left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right),$$

then $\overline{F}_X(x)$ is the survival function of the MOB distribution.
Conversely, let

$$\overline{F}_{X_n}(x) = \frac{\theta\left(3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right)}{\left(1-\overline{\theta}\ \left[3e^{-\frac{2x}{\lambda}} - 2e^{-\frac{3x}{\lambda}}\right]\right)}. \tag{7.4}$$

Then, using (7.3), we can conclude that $\overline{F}_{\epsilon_n}(x)$ is distributed as Bilal, and it is easy to show that the process is stationary.

Remark 7.1
Here we assume that X_0 follows the MOB distribution. However, X_0 can have any distribution, and the result can be proved by mathematical induction.

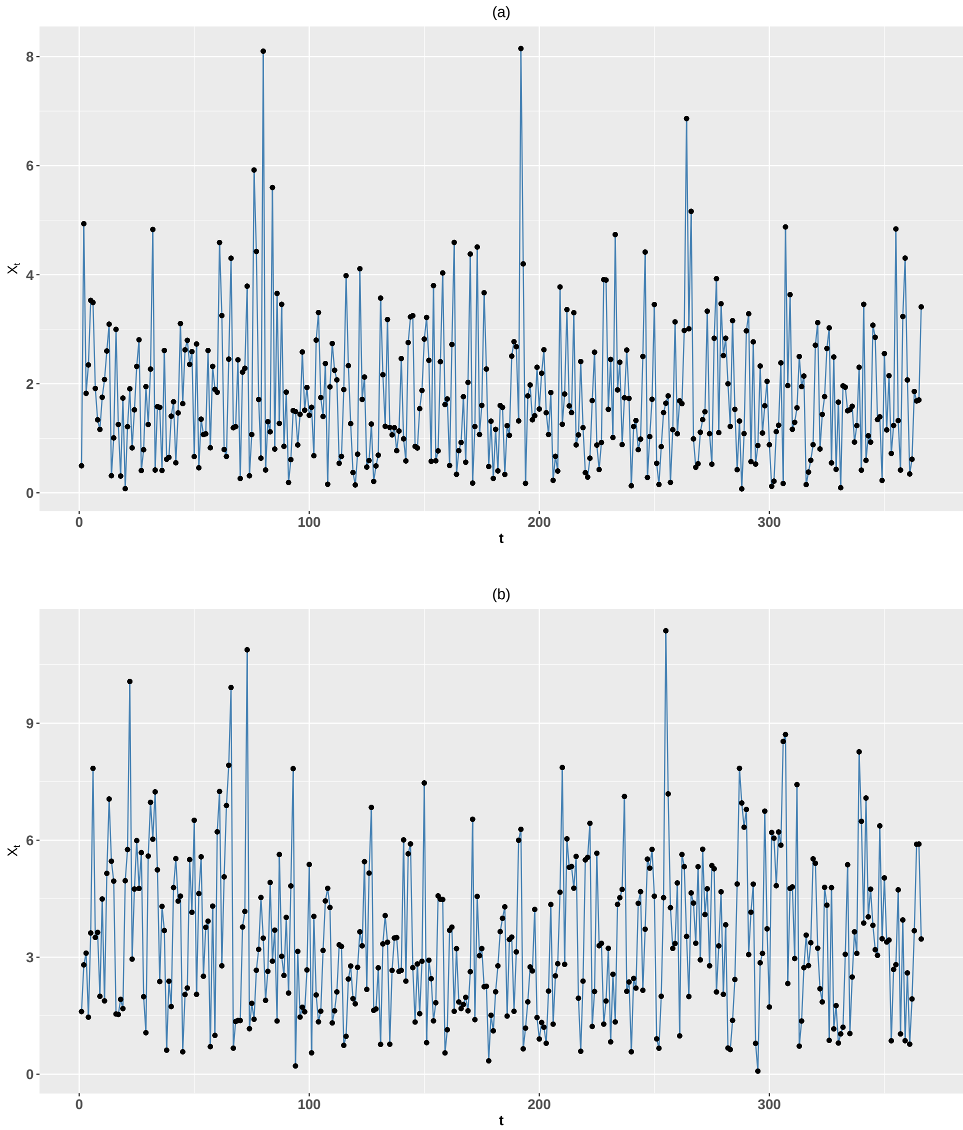

Figure 7. Sample path behaviour of the MOB minification process with parameter values $\lambda = 2.5,\ \theta = 0.73$ (a) and $\lambda = 5.6,\ \theta = 0.46$ (b).

Theorem 7.2
Consider a k^{th} order autoregressive minification process with structure

$$X_n = \begin{cases} \epsilon_n & with\ probability\ \theta_0 \\ min(X_{n-1}, \epsilon_n) & with\ probability\ \theta_1 \\ min(X_{n-2}, \epsilon_n) & with\ probability\ \theta_2 \\ : & : \\ . & . \\ min(X_{n-k}, \epsilon_n) & with\ probability\ \theta_k \end{cases}, \qquad (7.5)$$

where $0 < \theta_i < 1$, $\theta_1 + \theta_1 + ... + \theta_k = 1 - \theta_0$. Then $\{X_n\}$ is a Markovian k^{th} order autoregressive model with marginals as the MOB distribution if and only if $\{\epsilon_n\}$ is distributed as Bilal.

Table 4. The sample first-order auto-correlation with SD (in parentheses) for different values of parameters.

λ \ θ	0.1	0.25	0.35	0.5	0.6
0.13	0.63818 (0.0444)	0.42909 (0.0513)	0.33234 (0.0536)	0.22040 (0.0555)	0.16079 (0.0559)
0.82	0.63834 (0.0463)	0.42898 (0.0506)	0.33226 (0.0524)	0.22058 (0.0537)	0.16117 (0.0538)
1.2	0.63769 (0.0453)	0.42893 (0.0523)	0.33233 (0.0550)	0.22028 (0.0568)	0.16050 (0.0569)
2.8	0.63833 (0.0439)	0.42915 (0.0528)	0.33231 (0.0553)	0.21998 (0.0565)	0.16012 (0.0563)
5.5	0.63900 (0.0427)	0.42886 (0.0502)	0.33224 (0.0534)	0.22046 (0.0558)	0.16108 (0.0561)
10.7	0.63865 (0.0443)	0.43061 (0.0521)	0.33436 (0.0545)	0.22253 (0.0557)	0.16299 (0.0556)
13.9	0.63731 (0.0425)	0.42677 (0.0495)	0.32997 (0.0524)	0.21816 (0.0543)	0.15888 (0.0545)
13.9	0.63691 (0.0434)	0.42810 (0.0503)	0.33165 (0.0525)	0.21982 (0.0541)	0.16038 (0.0543)
18.4	0.63724 (0.0451)	0.42798 (0.0520)	0.33178 (0.0543)	0.22019 (0.0556)	0.16079 (0.0555)
22.8	0.63846 (0.0444)	0.42911 (0.0534)	0.33195 (0.0564)	0.21939 (0.0580)	0.15966 (0.0576)

Proof.
The proof follows the same as in Theorem 7.1.

Figure 7 displays the time series plots of the simulated sample path of the MOB minification process with parameter values (a) $\lambda = 2.5,\ \theta = 0.73$ and (b) $\lambda = 5.6,\ \theta = 0.46$. Using the Monte Carlo method, we obtain the first-order auto-correlation of the MOB minification process. For different sets of parameter values of λ and θ, we simulate 365 observations from the process. For each sequence of observations, we obtain the first-order sample auto-correlation. The process is repeated 1000 times. Table 4 gives the averages of the first-order sample auto-correlation and associated standard deviation (SD) in brackets. We can see that as θ increases, the first-order autocorrelation decreases, irrespective of λ.

8. Acceptance sampling plans

The ASP plays a major role in statistical quality control. Several units that must be examined under the ASP are subject to intangible cumulative deterioration throughout the course of their existence. Therefore, it is necessary that the lifetime model accounts for this degradation and provides a flexible fit. The ASP based on the two-parameter quasi-Lindley distribution were discussed by [9]. Further, Tripathi et al. [32] introduced an improved attribute chain sampling plan for Darna distribution. The ASP for the truncated life test based on Tsallis q-exponential distribution and the power Lomax distribution were derived by [5] and [7], respectively. In addition to that, the ASP under beta binomial exponential II distribution was discussed by [22], and using hypergeometric theory for finite population under Q-Weibull distribution was developed by [6]. Some notable recent contributions to the area include [3], [8] and [11]. The proposed model is derived from the geometric compounding of random variables originating from an exponential family. The real life applicability and simple structure make the proposed model suitable for the sampling plans. So, we develop and discuss an ASP for a truncated life test on the MOB distribution and illustrate the results using lifetime data.

The major concerns of an ASP are the number of units to be reviewed (n) and the maximum possible number of failures (c) in the reviewed items for the acceptance of a lot. The process is terminated after a prefixed time, and the number of defects is recorded. If the number of defects out of n reviewed items does not exceed c at time t, then we accept the lot with the given probability of at least p^*. The lot is rejected when the number of defects exceeds c before time t. Thus it can be considered as a decision-making problem, whether to accept or reject a lot under consideration. So, the primary concern of our study is to determine the minimum number of units required for the decision rule for a given ASP.

Assume that the lifetime distribution follows the MOB distribution, with known θ and unknown λ, so that the average lifetime depends only on λ. Let λ_0 be the required minimum average lifetime. Then, from (2.5), we have

$$G(t;\lambda,\theta) \leq G(t;\lambda_0,\theta) \iff \lambda \geq \lambda_0.$$

The ASP is characterized by the following:

- the number of units n on the test,
- the acceptance number c,
- the maximum test duration t,
- the ratio $\frac{t}{\lambda_0}$, where λ_0 is the specified average lifetime.

Table 5. Minimum values of n for specified p^*, $\frac{t}{\lambda_0}$, $\lambda = 2.7$ and $\theta = 1.75$ for binomial approximation.

p^*	c	$\frac{t}{\lambda}$									
		0.68	0.84	0.99	1.1	1.3	1.42	1.62	1.81	2	2.5
0.75	0	4	3	2	2	2	1	1	1	1	1
	1	7	5	4	4	3	3	3	2	2	2
	2	11	8	6	6	5	4	4	4	3	3
	3	14	10	8	8	6	6	5	5	5	4
	4	17	13	10	9	8	7	7	6	6	5
	5	20	15	12	11	9	9	8	7	7	6
	6	23	18	14	13	11	10	9	8	8	7
	7	26	20	16	15	12	11	10	10	9	8
	8	29	22	18	16	14	13	12	11	10	10
	9	33	25	20	18	15	14	13	12	11	11
	10	36	27	22	20	17	16	14	13	13	12
0.95	0	7	5	4	4	3	3	2	2	2	1
	1	12	9	7	6	5	4	4	3	3	3
	2	16	12	9	8	7	6	5	5	4	4
	3	20	15	12	10	8	8	7	6	6	5
	4	24	17	14	12	10	9	8	7	7	6
	5	27	20	16	14	12	11	9	9	8	7
	6	31	23	18	16	13	12	11	10	9	8
	7	34	25	21	18	15	14	12	11	10	9
	8	38	28	23	20	17	15	14	12	12	10
	9	41	31	25	22	18	17	15	14	13	11
	10	45	33	27	24	20	18	16	15	14	13

For the well-being of consumers, the ASP must reject the lot with true average life λ less than λ_0. So, the probability of accepting a bad lot, the consumer's risk, should not exceed the value 1-p^*, where p^* is the lower bound for the probability that the ASP rejects a lot. The triplet $(n, c, \frac{t}{\lambda_0})$ characterize the ASP for a given p^*. Since the ASP can be considered as a Bernoulli trial, we can use binomial distribution for sufficiently large lots and obtain the acceptance probability. The main objective is to determine the minimum sample size n for known values of c and $\frac{t}{\lambda_0}$ such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i \, (1-p_0)^{n-i} \leq 1-p^*, \tag{8.1}$$

where $p_0 = G(t; \lambda_0, \theta)$ is the failure probability before time t. Table 5 displays the minimum values of n for $p^* = 0.75, 0.95$, $\frac{t}{\lambda_0}$= 0.68, 0.84, 0.99, 1.1, 1.3, 1.42, 1.62, 1.81, 2, 2.5, c=0,1,2,...,10, $\lambda = 2.7$ and $\theta = 1.75$.

For large values of n and small values of p_0, we can use Poisson approximation instead of Binomial with parameter $\alpha = np_0$ as

$$L_1(p_0) = \sum_{i=0}^{c} \frac{\alpha^i}{i!} \, e^{-\alpha} \leq 1-p^*. \tag{8.2}$$

The minimum values of n satisfying (8.2) are obtained the same as above and are given in Table 6.

Table 6. Minimum values of n for specified p^*, $\frac{t}{\lambda_0}$, $\lambda = 2.7$ and $\theta = 1.75$ for Poisson approximation.

p^*	c	$\frac{t}{\lambda}$									
		0.68	0.84	0.99	1.1	1.3	1.42	1.62	1.81	2	2.5
0.75	0	4	3	3	3	2	2	2	2	2	2
	1	8	6	5	5	4	4	4	4	3	3
	2	12	9	8	7	6	6	5	5	5	5
	3	15	12	10	9	8	7	7	6	6	6
	4	18	14	12	11	9	9	8	8	7	7
	5	21	17	14	12	11	10	9	9	9	8
	6	25	19	16	14	12	12	11	10	10	9
	7	28	21	18	16	14	13	12	11	11	11
	8	31	24	20	18	16	15	13	13	12	12
	9	34	26	22	20	17	16	15	14	14	13
	10	37	29	24	21	19	17	16	15	15	14
0.95	0	9	7	6	5	5	4	4	4	4	4
	1	14	11	9	8	7	7	6	6	6	5
	2	18	14	12	11	9	9	8	8	7	7
	3	22	17	14	13	11	11	10	9	9	9
	4	26	20	17	15	13	12	11	11	10	10
	5	30	23	19	17	15	14	13	12	12	11
	6	34	26	22	20	17	16	15	14	13	13
	7	38	29	24	22	19	18	16	15	15	14
	8	41	32	26	24	21	19	18	17	16	15
	9	45	34	29	26	22	21	19	18	18	17
	10	48	37	31	28	24	23	21	20	19	18

Table 7. The operating characteristic values for the ASP $(n, c, \frac{t}{\lambda})$.

p^*	n	c	$\frac{t}{\lambda_0}$	$\frac{\lambda}{\lambda_0}$					
				2	4	6	8	10	12
0.75	14	3	0.68	0.74542	0.98478	0.99801	0.99957	0.99988	0.99996
	10	3	0.84	0.74857	0.98435	0.99790	0.99954	0.99986	0.99995
	8	3	0.99	0.73698	0.98273	0.99761	0.99947	0.99984	0.99994
	8	3	1.1	0.64370	0.97216	0.99593	0.99907	0.99972	0.99990
	6	3	1.3	0.67988	0.97563	0.99639	0.99917	0.99974	0.99990
	6	3	1.42	0.59539	0.96419	0.99444	0.99868	0.99959	0.99985
	5	3	1.62	0.60509	0.96468	0.99442	0.99866	0.99958	0.99984
	5	3	1.81	0.48967	0.94401	0.99055	0.99766	0.99925	0.99971
	5	3	2	0.38138	0.91693	0.98502	0.99616	0.99874	0.99951
	4	3	2.5	0.35415	0.90359	0.98171	0.99515	0.99837	0.99935
0.95	20	3	0.68	0.53026	0.95983	0.99427	0.99873	0.99962	0.99986
	15	3	0.84	0.49275	0.95184	0.99280	0.99836	0.99951	0.99982
	12	3	0.99	0.46806	0.94571	0.99158	0.99804	0.99940	0.99978
	10	3	1.1	0.48437	0.94783	0.99183	0.99808	0.99941	0.99978
	8	3	1.3	0.46759	0.94297	0.99077	0.99779	0.99931	0.99974
	8	3	1.42	0.36902	0.91851	0.98597	0.99654	0.99890	0.99958
	7	3	1.62	0.32925	0.90464	0.98288	0.99567	0.99860	0.99946
	6	3	1.81	0.33348	0.90422	0.98256	0.99553	0.99854	0.99944
	6	3	2	0.23268	0.86165	0.97275	0.99275	0.99758	0.99905
	5	3	2.5	0.16749	0.81494	0.96017	0.98887	0.99616	0.99845

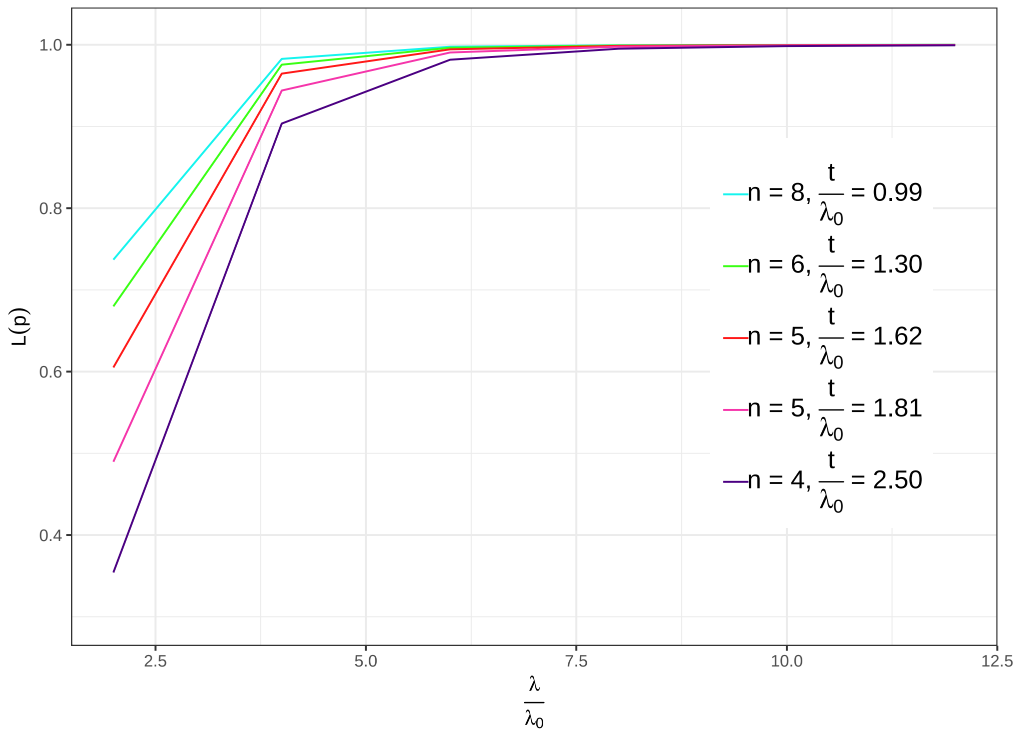

Figure 8. The OC curve of the ASP $(n, c, \frac{t}{\lambda})$ with $p^* = 0.75$.

The operating characteristic (OC) function of the ASP $(n, c, \frac{t}{\lambda})$ gives the probability of accepting the lot and is given by

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \tag{8.3}$$

where $p = G(t; \lambda, \theta)$ is considered as a function of λ. The OC function is built on the grounds of choices of n and c for given values of p^* and $\frac{t}{\lambda_0}$. By considering the fact that

$$\frac{t}{\lambda} = \frac{t}{\lambda_0} \Big/ \frac{\lambda}{\lambda_0},$$

the OC values for the ASP $(n, c, \frac{t}{\lambda})$ are obtained and given in Table 7. The OC curve for $p^* = 0.75$ is displayed in Figure 8.

For the sake of producers, the lot with λ greater than λ_0 should be accepted. The probability of rejecting a lot when λ is greater than λ_0, the producer's risk, can be found by determining $p = G(t; \lambda, \theta)$ and with the help of binomial distribution. For a specified producer's risk of 0.05, it would be interesting to know what value of $\frac{\lambda}{\lambda_0}$ will ensure that a producer's risk is less than or equal to 0.05, if the proposed sampling plan is adopted. The smallest value of $\frac{\lambda}{\lambda_0}$ must hold:

$$\sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \geq 0.95. \tag{8.4}$$

For the given ASP $(n, c, \frac{t}{\lambda})$ and prefixed p^*, Table 8 displays the minimum values of $\frac{\lambda}{\lambda_0}$

Table 8. Minimum values of $\frac{\lambda}{\lambda_0}$ for the ASP $(n, c, \frac{t}{\lambda})$, $\lambda = 2.7$ and $\theta = 1.75$ with producers risk of 0.05.

p^*	c	$\frac{t}{\lambda_0}$									
		0.68	0.84	0.99	1.1	1.3	1.42	1.62	1.81	2	2.5
	0	6.97	7.38	6.99	7.76	9.17	6.84	7.8	8.71	9.63	12.03
	1	3.23	3.24	3.3	3.67	3.55	3.88	4.42	3.57	3.95	4.93
	2	2.6	2.61	2.52	2.8	2.89	2.63	3	3.35	2.79	3.49
	3	2.21	2.18	2.18	2.43	2.28	2.49	2.4	2.68	2.96	2.87
	4	1.99	2.05	1.99	2.04	2.18	2.12	2.41	2.31	2.55	2.52
0.75	5	1.85	1.88	1.87	1.94	1.93	2.1	2.14	2.07	2.28	2.29
	6	1.76	1.83	1.78	1.87	1.91	1.91	1.95	1.9	2.1	2.13
	7	1.68	1.73	1.72	1.81	1.76	1.76	1.81	2.02	1.96	2.01
	8	1.63	1.65	1.67	1.68	1.76	1.79	1.88	1.9	1.85	2.31
	9	1.62	1.64	1.62	1.65	1.66	1.69	1.78	1.8	1.76	2.2
	10	1.58	1.58	1.59	1.63	1.67	1.71	1.69	1.72	1.9	2.11
	0	9.38	9.7	10.15	11.28	11.43	12.48	11.43	12.77	14.11	12.03
	1	4.44	4.64	4.71	4.76	5.01	4.73	5.4	4.94	5.46	6.82
	2	3.27	3.39	3.33	3.42	3.69	3.62	3.6	4.02	3.71	4.63
	3	2.77	2.85	2.91	2.85	2.87	3.13	3.22	3.17	3.5	3.7
	4	2.49	2.46	2.54	2.54	2.62	2.63	2.72	2.7	2.98	3.19
0.95	5	2.25	2.29	2.31	2.33	2.45	2.5	2.4	2.68	2.64	2.85
	6	2.14	2.17	2.15	2.19	2.2	2.25	2.38	2.43	2.41	2.62
	7	2.01	2.02	2.11	2.09	2.14	2.2	2.19	2.24	2.23	2.44
	8	1.95	1.96	2.01	2.01	2.09	2.05	2.19	2.1	2.32	2.31
	9	1.87	1.91	1.93	1.95	1.95	2.03	2.06	2.15	2.19	2.2
	10	1.83	1.83	1.87	1.89	1.93	1.92	1.95	2.04	2.09	2.37

required to satisfy (8.4).

8.1. Illustration

Let the lifetime follow the MOB distribution with parameters $\lambda = 2.7$ and $\theta = 1.75$. Suppose our interest is an ASP with an unknown average lifetime of 1000 hours and a termination time of 1100 hours. The consumer's risk is prefixed at $1 - p^{*}= 0.25$. From Table 5, the required number of n is 8 for an acceptance number $c= 3$ and $\frac{t}{\lambda_0} = 1.1$. Hence, the ASP under consideration is $(n = 8, c = 3, \frac{t}{\lambda_0} = 1.1)$. During the test time, we have a confidence level of 0.75 that the average lifetime is at least 1000 hours if, at most, three failures out of 8 are observed. For the Poisson approximation, the ASP under consideration becomes $(n = 9, c = 3, \frac{t}{\lambda_0} = 1.1)$. From the Table 7, we can see that, if $\frac{\lambda}{\lambda_0} = 2$, the producer's risk is 0.36. The producer's risk is negligible if it is 10 or 12. From Table 8, the minimum value of $\frac{\lambda}{\lambda_0}$ giving a producer's risk of 0.05 is 2.85. So if the consumer's risk is fixed at a specified level, then the quality can be reached by a predetermined ratio.

8.2. Application

Here, we consider the data regarding the time (in months) to the first failure of 20 small electric carts used for internal transportation and delivery in a large manufacturing facility and whose ASP was discussed by [10]. The data is as follows: 0.9, 1.5, 2.3, 3.2, 3.9, 5,6.2, 7.5, 8.3, 10.4, 11.1, 12.6, 15, 16.3, 19.3, 22.6, 24.8, 31.5, 38.1, 53.

Table 9 gives the goodness-of-fit statistics and information criterion of the MOB and Akash distribution, respectively. We can see that the MOB distribution has the maximum

Table 9. Goodness-of-fit-measures for the electric carts data.

Model	logL	AIC	BIC	KS	p value	AD	CM
MOB	-75.7932	153.5865	154.5822	0.1225	0.8901	0.9359	0.0854
Akash	-79.1776	160.3552	161.3510	0.2071	0.3130	2.4717	0.2528

logL and the least AIC and BIC values. Moreover, the KS statistic is minimum with a large p value, and the AD and CM statistics have the smallest values. Hence, the MOB distribution yields the best fit for the data than the Akash distribution.

We compare the performance of the ASP under the proposed model with respect to the Akash distribution given in [10] ($\delta = 0.2017$). The ASP is adopted under the assumption that the lifetime follows the MOB distribution ($\theta = 1.098544$). Let the prefixed average lifetime be 14.65 months and the testing time be 9.202 months. Thus, $n = 20$ and $\frac{t}{\lambda_0} = 0.678$. Using (8.1), the value of c is obtained as 6 for $p^{*} = 0.75$. Thus, we have the ASP $(n = 20,\ c = 6\ \frac{t}{\lambda_0} = 0.678)$. So we reject the lot. The ASP under Akash distribution is $(n = 20,\ c = 4\ \frac{t}{\lambda_0} = 0.678)$. Since the MOB distribution provided a better fit to the data than the Akash distribution, we accept the lot if and only if the number of failures is at most 5. Here, there are nine values which are less than t. So we reject the lot.

9. Conclusion

We proposed a two-parameter extension of the Bilal distribution by applying the Marshall-Olkin extended model, known as the Marshall-Olkin Bilal distribution, with the Bilal distribution as a sub-model. The addition of a shape parameter improved the flexibility of the proposed model. The MOB density is unimodal, and its different statistical structures are discussed. Various statistical properties can be determined using numerical approaches even though they lack closed-form expressions. The HRF of the MOB distribution can be increasing and upside-down bathtub shaped. The model parameter estimation was carried out using ML, LS and WLS methods, and the performance was assessed by a simulation study. The proposed MOB distribution provided the best fit for the real data sets when compared to the Bilal, general Bilal and Marshall-Olkin exponential and other distributions. The application in time series modelling was evaluated by the autoregressive minification process with the MOB distribution as marginals. The statistical properties and modelling of the autoregressive minification process are considered for further study. The ASP with the lifetime of units following the MOB distribution was established. The operating characteristic values and minimum sample size corresponding to the maximum possible defects and the minimum ratios of lifetime associated with the producer's risk were discussed. The real data modelling shows that ASP with respect to the proposed model is better than the existing model in the literature. Using the works done in [8], [14], [16] and [23], we look forward to extending the sampling plans of the proposed model with respect to neutrosophic statistics.

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Appendices

A. Cumulative hazard function

The cumulative hazard function is given by

$$\Lambda(x;\lambda,\theta) = \int_0^x \frac{6\ (1-e^{\frac{-t}{\lambda}})}{\lambda\ \left(3-2e^{-\frac{t}{\lambda}}\right)\left[1-\overline{\theta}\left(3e^{-\frac{2t}{\lambda}}-2e^{-\frac{3t}{\lambda}}\right)\right]}dt$$

Using the negative binomial power series

$$(1-z)^{-r} = \sum_{i=0}^{\infty}\binom{r+i-1}{i}z^i, |z|<1, \tag{A.1}$$

in the denominator, we have

$$\begin{aligned}\Lambda(x;\lambda,\theta) &= \sum_{i=0}^{\infty}\int_0^x \frac{6\ (1-e^{\frac{-t}{\lambda}})}{\lambda\ \left(3-2e^{-\frac{t}{\lambda}}\right)}\ \overline{\theta}^i\ \left(3e^{-\frac{2t}{\lambda}}-2e^{-\frac{3t}{\lambda}}\right)^i \\ &= \sum_{i=0}^{\infty}\int_0^x \frac{6\ 3^i\ \overline{\theta}^i}{\lambda}\ e^{-\frac{2ti}{\lambda}}(1-e^{\frac{-t}{\lambda}})\ \left(1-\frac{2}{3}e^{-\frac{t}{\lambda}}\right)^{i-1}\end{aligned}$$

Using the Binomial expansion

$$(1-z)^r = \sum_{i=0}^{r}(-1)^i\binom{r}{i}z^i, \tag{A.2}$$

we have,

$$\begin{aligned}\Lambda(x;\lambda,\theta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i-1}\int_0^x \binom{i-1}{j}\frac{6\ (-1)^j\ 3^{i-j}\ 2^j\ \overline{\theta}^i}{\lambda}\ e^{-\frac{t}{\lambda}(2i+j)}(1-e^{\frac{-t}{\lambda}}) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i-1} c_{i,j}\ \beta^*(x,2i+j+1,2),\end{aligned}$$

where $\beta^*(x,2i+j+1,2)$ is the incomplete beta function in terms of $e^{\frac{-t}{\lambda}}$ and

$$c_{i,j} = \frac{6\ (-1)^j\ 3^{i-j}\ 2^j\ \overline{\theta}^i}{\lambda}$$

B. Proposition 3.1

B.1.

Let us distinguish the case $0<\theta<1$ and the case $\theta>1$.
Case 1 0$<\theta<1$
Using the negative binomial power series given in A.1, the denominator of the MOB density given in (2.4) can be expanded as follows,

$$g(x;\lambda,\theta) = \sum_{i=0}^{\infty}\binom{i+1}{i}\overline{\theta}^i\ \frac{6}{\lambda}\ \theta\ \left(e^{-\frac{2x}{\lambda}}-e^{-\frac{3x}{\lambda}}\right)\left(3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right)^i. \tag{B.1}$$

Case 2 $\theta>1$
Let $\tau=\theta^{-1} \implies 0<\tau<1$.
Using A.1 on (2.4), we have

$$g(x;\lambda,\theta) = \frac{6}{\lambda}\tau\left(e^{-\frac{2x}{\lambda}}-e^{-\frac{3x}{\lambda}}\right)\sum_{l=0}^{\infty}\binom{l+1}{l}(1-\tau)^l\ (1-[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}])^l. \tag{B.2}$$

Using A.2, we have

$$\begin{aligned} g(x;\lambda,\theta) &= \sum_{i=0}^{\infty}\frac{6\ \tau(-1)^i}{\lambda}\left[\sum_{l=i}^{\infty}(1-\tau)^l\binom{l}{i}\binom{l+1}{l}\right]\left(e^{-\frac{2x}{\lambda}}-e^{-\frac{3x}{\lambda}}\right)\left[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right]^i \\ &= \sum_{i=0}^{\infty}w_i\left(e^{-\frac{2x}{\lambda}}-e^{-\frac{3x}{\lambda}}\right)\left[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right]^i. \end{aligned} \tag{B.3}$$

The Lehman type-II-G density for an arbitrary parent distribution $G_4(x)$, having PDF $g_4(x)$ is given by

$$g_\alpha(x)=\alpha\ g_4(x)\ [1-G_4(x)]^{\alpha-1}.$$

Thus,

$$g(x;\lambda,\theta)=\sum_{i=0}^{\infty}h_i(i+1)f(x;\lambda)[\overline{F}(x;\lambda)]^i,$$

where

$$h_i = w_i\frac{6}{\lambda(i+1)}\ \text{and}\ w_i=\begin{cases}\binom{i+1}{i}\overline{\theta}^i\ \frac{6}{\lambda}\ \theta & if\ 0<\theta<1\\ \frac{6}{\lambda\theta}(-1)^i\left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l\ \binom{l}{i}\ \binom{l+1}{l}\right] & if\ \theta>1\end{cases}.$$

B.2.

Now, by a suitable decomposition and the generalized binomial theorem, we have

$$\begin{aligned} \left(e^{-\frac{2x}{\lambda}}-e^{-\frac{3x}{\lambda}}\right)\left[3e^{-\frac{2x}{\lambda}}-2e^{-\frac{3x}{\lambda}}\right]^i &= 3^i\ e^{-2(1+i)\frac{x}{\lambda}}\left(1-e^{-\frac{x}{\lambda}}\right)\left(1-\frac{2}{3}e^{-\frac{x}{\lambda}}\right)^i \\ &= 3^i\ e^{-2(1+i)\frac{x}{\lambda}}\left(1-e^{-\frac{x}{\lambda}}\right)\sum_{j=0}^{i}\binom{i}{j}(-1)^j\frac{2^j}{3^j}e^{-j\frac{x}{\lambda}} \end{aligned}$$

Therefore, from B.3, we have

$$\begin{aligned} g(x;\lambda,\theta) &= \sum_{i=0}^{\infty}w_i 3^i\ e^{-2(1+i)\frac{x}{\lambda}}\left(1-e^{-\frac{3x}{\lambda}}\right)\sum_{j=0}^{i}(-1)^j\frac{2^j}{3^j}e^{-j\frac{x}{\lambda}} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\ z_{i,j}\ e^{-\frac{x}{\lambda}(2+2i+j)}\ \left(1-e^{-\frac{x}{\lambda}}\right), \end{aligned} \tag{B.4}$$

where

$$z_{i,j}=w_i\binom{i}{j}\ 3^{i-j}\ (-1)^j\ 2^j. \tag{B.5}$$

Thus,

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\ z_{i,j}\int_0^{\infty}\ e^{-\frac{x}{\lambda}(2+2i+j-\lambda t)}\ \left(1-e^{-\frac{x}{\lambda}}\right)dx.$$

Let $v=e^{\frac{-x}{\lambda}} \implies \frac{\partial v}{\partial x} = -\frac{e^{\frac{-x}{\lambda}}}{\lambda}$, then we have

$$\begin{aligned} M_X(t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\ z_{i,j}\ \lambda\int_0^1\ v^{(1+2i+j-\lambda t)}\ (1-v)\ dv \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\ z_{i,j}\ \lambda\ \beta(2+2i+j-\lambda t,2),\ t<\frac{2}{\lambda}. \end{aligned}$$

B.3.

Using B.4, the r^{th} raw moment of the MOB distribution is given by

$$\begin{aligned} \mu_r^{'} &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j} \int_0^{\infty} x^r\ e^{-\frac{x}{\lambda}(2+2i+j)}\ dx - \int_0^{\infty} x^r\ e^{-\frac{x}{\lambda}(3+2i+j)}\ dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}\ r!\ \lambda^r \left[\frac{1}{(2+2i+j)^r} - \frac{1}{(3+2i+j)^r}\right]. \end{aligned}$$

B.4.

The r^{th} incomplete moment of the MOB distribution is given by

$$\begin{aligned} m_r(y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j} \int_0^{y} x^r\ e^{-\frac{x}{\lambda}(2+2i+j)}\ dx - \int_0^{y} x^r\ e^{-\frac{x}{\lambda}(3+2i+j)}\ dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j} \left[\gamma\left(r+1, \frac{(2+2i+j)}{\lambda}, y\right) - \gamma\left(r+1, \frac{(3+2i+j)}{\lambda}, y\right)\right], \end{aligned}$$

where $\gamma(r+1, a, y) = \int_0^y\ x^r\ e^{-ax}\ dx.$

C. Mean residual life function

Following the procedure same as in appendix B, we have

$$\overline{G}(x;\lambda,\theta) = \sum_{i=0}^{\infty} w_i^m \left(3e^{\frac{-2x}{\lambda}} - 2e^{\frac{-3x}{\lambda}}\right)^{1+i} = \sum_{i=0}^{\infty}\sum_{j=0}^{1+i} z_{i,j}^m\ e^{-\frac{x}{\lambda}(2+2i+j)}\ (1-e^{-\frac{x}{\lambda}}). \quad \text{(C.1)}$$

Now

$$m(t) = \frac{\int_t^{\infty} \overline{G}(x;\lambda,\theta)\ dx}{\overline{G}(t;\lambda,\theta)} = \left(\left[3-2e^{-\frac{t}{\lambda}}\right]^{-1} - \overline{\theta}\ e^{-\frac{2t}{\lambda}}\right) \sum_{i=0}^{\infty}\sum_{j=0}^{1+i} z_{i,j}^m \frac{\lambda e^{\frac{-t(2i+j)}{\lambda}}}{(2+2i+j)},$$

where

$$\begin{aligned} z_{i,j}^m &= w_i^m \binom{1+i}{j} 3^{1+i-j}\ (-1)^j\ 2^j \\ \text{and } w_i^m &= \begin{cases} \overline{\theta}^i & if\ 0<\theta<1 \\ \frac{1}{\theta^2}(-1)^i \left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l\ \binom{l}{i}\ \right] & if\ \theta>1 \end{cases}. \end{aligned}$$

D. Proposition 3.3

Now, proceeding same as in appendix B and using the generalized binomial theorem, for any $r \in \mathbb{R}$,

$$(1-z)^{-r} = \sum_{i=0}^{\infty}\binom{-r}{i}(-1)^i\ z^i, |z|<1,$$

with $\binom{-r}{i} = \frac{(-r)(-r-1)...(-r-i+1)}{i!}$, we have

$$g^v(x,;\lambda,\theta) = \sum_{i=0}^{\infty} w_i^v e^{-\frac{2v+2i}{\lambda}} \left(1-e^{-\frac{x}{\lambda}}\right)^v \left(3-2e^{-\frac{x}{\lambda}}\right)^i = \sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}^v e^{-\frac{2v+2i+j}{\lambda}} \left(1-e^{-\frac{x}{\lambda}}\right)^v.$$

Let $d = e^{\frac{-x}{\lambda}} \implies \frac{\partial d}{\partial x} = -\frac{e^{\frac{-x}{\lambda}}}{\lambda}$, then we have

$$\begin{aligned} I_R(x) &= (1-v)^{-1}\ log\left[\int_0^1 \sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}^v\ \lambda\ d^{(2v+2i+j-1)}\ (1-d)^{(v)}\right] \\ &= (1-v)^{-1}\ log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{i} z_{i,j}^v\ \lambda\ \beta(2v+2i+j, v+1)\right], \end{aligned}$$

where $z_{i,j}^v$ is obtained by replacing w_i by w_i^v in B.5 and it is given by

$$w_i^v = \begin{cases} \binom{-2v}{i}(-1)^i\ \overline{\theta}^i\ \frac{6^v}{\lambda^v}\ \theta^v & if\ 0<\theta<1 \\ \\ \frac{6^v}{\lambda^v\theta^v}(-1)^i\ \left[\sum_{l=i}^{\infty}(\frac{1}{\theta}-1)^l\ \binom{l}{i}\ \binom{-2v}{l}\right] & if\ \theta>1 \end{cases}. \tag{D.1}$$

E. Stress-strength parameter

From (2.4) and (3.1) and proceeding similar as in appendix B

$$g(x;\lambda,\theta)\overline{G}(x;\lambda,\theta) = \sum_{i=0}^{\infty}\sum_{j=0}^{1+i} z_{i,j}^{ss}\ e^{-\frac{x}{\lambda}(4+2i+j)}\ \left(1-e^{-\frac{x}{\lambda}}\right).$$

Then,

$$\begin{aligned} R &= P(Y<X) = 1-\int_0^{\infty} g(x;\lambda,\theta)\ \overline{G}(x;\lambda,\theta)dx \\ &= 1-\sum_{i=0}^{\infty}\sum_{j=0}^{1+i} z_{i,j}^{ss}\int_0^{\infty} e^{-\frac{x}{\lambda}(4+2i+j)}\ \left(1-e^{-\frac{x}{\lambda}}\right)dx. \end{aligned}$$

Let $v = e^{\frac{-x}{\lambda}} \implies \frac{\partial v}{\partial x} = -\frac{e^{\frac{-x}{\lambda}}}{\lambda}$, then

$$\begin{aligned} R &= 1-\sum_{i=0}^{\infty}\sum_{j=0}^{1+i} z_{i,j}^{ss}\ \lambda\int_0^1 v^{(3+2i+j)}\ (1-v)\,dx. \\ &= 1-\sum_{i=0}^{\infty}\sum_{j=0}^{1+i} z_{i,j}^{ss}\ \lambda\beta(4+2i+j,2), \end{aligned}$$

where

$$\begin{aligned} z_{i,j}^{ss} &= w_i^{ss}\binom{1+i}{j}\ 3^{1+i-j}\ (-1)^j\ 2^j \text{and} \\ w_i^{ss} &= \begin{cases} \binom{i+2}{i}\overline{\theta}^i\ \frac{6}{\lambda}\ \theta^2 & if\ 0<\theta<1 \\ \\ \frac{6}{\lambda\theta}(-1)^i\left[\sum_{l=i}^{\infty}(1-\frac{1}{\theta})^l\ \binom{l}{i}\ \binom{l+2}{l}\right] & if\ \theta>1 \end{cases}. \end{aligned}$$