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New Traveling Wave Solutions for the Sixth-order Boussinesq Equation

He Yang

Department of Mathematics, Augusta University, Augusta, GA 30912, USA

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Abstract

In this paper, we investigate the new traveling wave solutions for the sixth-order Boussinesq equation using the tanh-coth method. Such a method is a type of expansion method that represents the solutions of partial differential equations as polynomials of tanh and coth functions. We discover several new traveling wave solutions for the sixth-order Boussinesq equation with different parameters, which are of fundamental importance for various applications.

1. Introduction

In this paper, we consider the following sixth-order Boussinesq equation (1.1)

$$u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} + (u^2)_{xx} = 0, \tag{1.1}$$

where $\beta = 1$ or -1. The Boussinesq approximation for water waves was originally derived by Joseph Boussinesq in 1871 [1]. The fourth-order Boussinesq equations were then introduced in the following year [2]. Since then, a great number of mathematical models have been referred as Boussinesq equations, which are usually called Boussinesq-type equations. Among the wide range of Boussinesq-type equations, the sixth-order Boussinesq equations have attracted great attentions from the researchers all over the world. In particular, the Boussinesq-type equations with linear strong damping and nonlinear source [3], fourth-order dispersion term and nonlinear source [4], cubic nonlinearity [5], and the linear Boussinesq-type equation [6] have been considered. In addition to the aforementioned work, Christov, Maugin and Velarde [7] reexamined the Boussinesq-type equations for the shallow fluid layers and derived equation (1.1). The exact controllability and stability of the equation has been studied in [8]. However, the traveling wave solutions for (1.1) has not been considered. In this paper, we will fill in the gap by discussing the traveling wave solutions in the closed form.

The methodology that we use for the derivation of the traveling wave solutions is called the tanh-coth method, which belongs to the broader category of expansion methods. The expansion methods are analytical methods that look for a summation of finite terms in specific forms, including the tanh function and extended tanh expansion method, Jacobi elliptic functions method, extended direct algebraic method, sine-cosine method, and modified (G'/G)-expansion method. In particular, Amirov and Anutgan [9] applied the tanh function and polynomial function methods to derive the analytical solitary wave solutions for the sixth-order modified Boussinesq equation. A similar method named tanh-coth method has also been used to find the exact solutions for various partial differential equations. In [10], the author used tanh-coth method to derive the solitons and kink solutions for nonlinear parabolic equations, including the Fisher equation, Newell-Whithead equation, Allen-Cahn equation, FitzHugh-Nagumo equation and the Burgers-Fisher equation. The tanh-coth method for some nonlinear pseudo-parabolic equations, including the Benjamin-Bona-Mahony-Peregrine-Burgers equation, the Oskolkov-Benjamin-Bona-Mahony-Burgers equation, the Oskolkov equation and the generalized hyperelastic-rod wave equation, were discussed

Email address and ORCID number: **hyang1@augusta.edu, 0000-0001-9608-4920**
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in [11]. Recently, the method has also been successfully applied to stochastic differential equations [12, 13] and fractional differential equations [14, 15]. Some extended methods including the extended tanh method [16, 17] and the modified tanh-coth method [18], were developed for the Zakharov-like equation, fourth-order Boussinesq equation, the Klein-Gordon equations, the Khokhlov-Zabolotskaya-Kuznetsov, the Newell-Whitehead-Segel and the Rabinovich wave equations.
Other than the tanh related methods, the Jacobi elliptic function method has also been applied to find the traveling wave and soliton solutions for partial differential equations and fractional differential equations. In [19], the authors used the F-expansion technique to solve the sine-Gordon equation in terms of the Jacobi elliptic functions. Also in [20], Fang and Dai discussed three different approaches for obtaining the bright and dark soliton solutions for a time-fractional higher-order nonlinear Schrodinger equation. More specifically, the Jacobi elliptic function method, Riccati equation method and the double function method have been used to study the time-fractional Schrodinger equation with Kerr law, power law and log law of nonlinearity. Similar to the aforementioned methods, the extended direct algebraic method also assumes that the solution to a given differential equation can be expressed as a finite sum of certain functions. But it requires that each of the function satisfies a specific first order differential equation with parameters. The extended direct algebraic method has been used to find the traveling wave solutions for the coupled systems of KdV equations, the variant Boussinesq equations and the coupled Burgers equations [21]. An alternative method named sine-cosine method was also employed to construct the traveling wave solutions for nonlinear Schrodinger equations [22]. Instead of looking for an analytical solution in the form of a summation of some particular functions, the sine-cosine method simply looks for an ansatz in the form of a power of a truncated sine or cosine function with some unknown parameters. Such a method has been successfully utilized to obtain some traveling wave solutions for several nonlinear Schrodinger equations. Another popular method called modified (G'/G)-expansion method has also been developed for finding exact wave solutions of various PDEs. The main idea of the method is to assume that the exact solution can be expressed as a polynomial in (G'/G) and that G satisfies a specific second-order ODE with parameters to be determined by balancing the derivatives and nonlinear terms in the given PDE. Interest readers can check the work by Bansal and Gupta in [23] where they used such a method to solve the Klein-Gordon-Schrodinger equation.
In this paper, we investigate the traveling wave solutions for the sixth-order Boussinesq equation (1.1) by utilizing the tanh-coth method due to its powerfulness and simplicity. The rest of the paper is organized as follows: in section 2, we describe the framework of the tanh-coth method for general PDEs. In section 3-5, we establish the procedure of finding the traveling wave solutions for the sixth-order Boussinesq equation and discuss different cases for the values of parameters β in (1.1). In particular, we discuss the Boussinesq equation with $\beta = 1$ and $\beta = -1$ in section 4 and section 5, respectively. Some concluding remarks are given in section 6.

2. Description of the tanh-coth method

Consider a PDE in the following form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, u_{xxx}, \ldots) = 0, \tag{2.1}$$

where P is a polynomial in terms of the unknown function $u(x,t)$ and its various derivatives. We look for a traveling wave solution $u(\xi)$ with $\xi = x - vt$, where v is the wave speed. Then equation (2.1) can be written as

$$P(u, u', u'', u''', \ldots) = 0, \tag{2.2}$$

which is an ODE with respect to $u(\xi)$, the traveling wave solution.
Next, we let $Y = \tanh(\mu\xi)$ and assume that $u(\xi)$ can be expressed as a finite expansion given in the following equation

$$u(\xi) = a_0 + \sum_{i=1}^{M} a_i Y^i(\xi) + \sum_{i=1}^{M} b_i Y^{-i}(\xi). \tag{2.3}$$

Here a_i for $0 \le i \le M$ and b_j for $1 \le j \le M$ are unknown constants to be determined, and we assume that $a_M \neq 0$. We then substitute (2.3) into (2.2) and balance the coefficients of the various powers of Y. One key component in such a process is to apply the following equality for Y:

$$Y' = \mu - \mu Y^2, \tag{2.4}$$

so that the various derivatives of Y can be converted to powers of Y. Also note that we need to consider the change of variables before we apply (2.4). That is, when we calculate $u'(\xi)$, the following change of derivative is needed:

$$u'(\xi) = \mu(1-Y^2)\frac{du}{dY} = \mu(1-Y^2)\left(\sum_{i=1}^{M} i a_i Y^{i-1} - \sum_{i=1}^{M} i b_i Y^{-i-1}\right).$$

Note that the highest power of Y in $u'(\xi)$ is $(M+1)$ which is one more than the highest power of Y in $u(\xi)$. In addition, we can further calculate the second derivative of $u(\xi)$ to get

$$u''(\xi) = \mu(1-Y^2)\frac{du'(\xi)}{dY} = \mu^2(1-Y^2)\left(-2Y\frac{du}{dY} + (1-Y^2)\frac{d^2u}{dY^2}\right).$$

Since the leading terms in $Y\frac{du}{dY}$ and $(1-Y^2)\frac{d^2u}{dY^2}$ are both Y^M, the highest power of Y in $u''(\xi)$ is $(M+2)$, which is two more than the highest power of Y in $u(\xi)$. Similarly, one can show that if the highest order of derivatives for all the linear terms in (2.2) is K, then the leading term for all the linear terms in the equation is a constant times Y^{M+k}. Usually, one can calculate the value of M by balancing the linear terms of the highest order and the leading nonlinear terms. For example, if P in (2.1) is defined to be $P(u,u_t,u_{xx}) = u_t - u_{xx} + u - u^3$, then the linear term of the highest order in (2.2) is u'' which leads to Y^{M+2} terms, and the leading nonlinear term is u^3 which leads to Y^{3M} terms. By matching the highest power of these two terms, we can get $M = 1$.
Once the value of M is determined, we can rewrite (2.2) as a finite expansion in terms of Y using (2.4) and (2.3). We then further collect all the coefficients of Y^i for all i and derive a system of equations by setting theses coefficients to be equal to zero. By solving the algebraic system, we can obtain the values of a_i (for $0 \le i \le M$), b_j (for $1 \le j \le M$), μ and v, which leads to an analytical solution in the form of (2.3). Note that if we assume that $b_j = 0$ for $1 \le j \le M$ in (2.3), then the method recovers the standard tanh method. The tanh-coth method works very well for PDEs in the form of (2.1). Even for PDEs that are not in the given form as in (2.1), we may still apply the tanh-coth method if the PDEs can be transformed to (2.1). Interested readers can refer to [24] for a thorough discussion about finding the exact solutions of the sine-Gordon and the sinh-Gordon equations using the tanh method.

3. The tanh-coth Method for the sixth-order Boussinesq equation

We now discuss how to solve the sixth-order Boussinesq equation (1.1) using the tanh-coth method. Let $u(x,t) = u(\xi)$ be the traveling wave solution to (1.1) where $\xi = x - vt$ with v being the constant speed of the traveling wave. Then, equation (1.1) becomes

$$v^2u'' - u'' + \beta u^{(4)} - u^{(6)} + (u^2)'' = 0. \tag{3.1}$$

Here u'', $u^{(4)}$ and $u^{(6)}$ represent $\frac{d^2u}{d\xi^2}$, $\frac{d^4u}{d\xi^4}$ and $\frac{d^6u}{d\xi^6}$, respectively. We then integrate (3.1) with respect to ξ twice, and set the integration constants to zero, to obtain the following equation

$$(v^2-1)u + \beta u'' - u^{(4)} + u^2 = 0. \tag{3.2}$$

We now use the tanh-coth method by letting

$$u(\xi) = \sum_{i=0}^{M} a_i Y^i(\xi) + \sum_{i=1}^{M} b_i Y^{-i}(\xi), \tag{3.3}$$

where $Y = \tanh(\mu\xi)$ satisfies

$$Y' = \mu - \mu Y^2, \tag{3.4}$$

and a_i for $i = 0,1,\ldots,M$ and b_j for $j = 1,2,\ldots,M$ are constants to be determined. Based on the ansarz of $u(\xi)$ given in (3.3) and the derivative of Y in (3.4), as well as the description of the tanh-coth method in section 2, we can balance the highest power of Y in the leading nonlinear term u^2 with the power of Y in the linear term of the highest order, i.e.,$u^{(4)}$, in (3.2). Thus, we can get

$$2M = M + 4,$$

which leads to $M = 4$. Therefore, equation (3.3) becomes

$$u(\xi) = \sum_{i=0}^{4} a_i Y^i(\xi) + \sum_{i=1}^{4} b_i Y^{-i}(\xi). \tag{3.5}$$

Detailed calculations show that

$$\begin{aligned} u'' \quad = \quad & 20\mu^2a_4Y^6 + 12\mu^2a_3Y^5 + (6\mu^2a_2 - 32\mu^2a_4)Y^4 + (2\mu^2a_1 - 18\mu^2a_3)Y^3 + (12\mu^2a_4 - 8\mu^2a_2)Y^2 \\ & +(6\mu^2a_3 - 2\mu^2a_1)Y + (2\mu^2a_2 + 2\mu^2b_2) + (6\mu^2b_3 - 2\mu^2b_1)Y^{-1} + (12\mu^2b_4 - 8\mu^2b_2)Y^{-2} \\ & +(2\mu^2b_1 - 18\mu^2b_3)Y^{-3} + (6\mu^2b_2 - 32\mu^2b_4)Y^{-4} + 12\mu^2b_3Y^{-5} + 20\mu^2b_4Y^{-6}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} u^{(4)} \quad = \quad & 840\mu^4a_4Y^8 + 360\mu^4a_3Y^7 + (120\mu^4a_2 - 2080\mu^4a_4)Y^6 + (24\mu^4a_1 - 816\mu^4a_3)Y^5 \\ & +(1696\mu^4a_4 - 240\mu^4a_2)Y^4 + (576\mu^4a_3 - 40\mu^4a_1)Y^3 + (136\mu^4a_2 - 480\mu^4a_4)Y^2 \\ & +(16\mu^4a_1 - 120\mu^4a_3)Y + (24\mu^4a_4 - 16\mu^4a_2 - 16\mu^4b_2 + 24\mu^4b_4) \\ & +(16\mu^4b_1 - 120\mu^4b_3)Y^{-1} + (136\mu^4b_2 - 480\mu^4b_4)Y^{-2} + (576\mu^4b_3 - 40\mu^4b_1)Y^{-3} \\ & +(1696\mu^4b_4 - 240\mu^4b_2)Y^{-4} + (24\mu^4b_1 - 816\mu^4b_3)Y^{-5} + (120\mu^4b_2 - 2080\mu^4b_4)Y^{-6} \\ & +360\mu^4b_3Y^{-7} + 840\mu^4b_4Y^{-8}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} u^2 &= a_4^2Y^8+2a_3a_4Y^7+(a_3^2+2a_2a_4)Y^6+(2a_1a_4+2a_2a_3)Y^5+(a_2^2+2a_0a_4+2a_1a_3)Y^4 \\ &+(2a_0a_3+2a_1a_2+2a_4b_1)Y^3+(a_1^2+2a_0a_2+2a_3b_1+2a_4b_2)Y^2 \\ &+(2a_0a_1+2a_2b_1+2a_3b_2+2a_4b_3)Y+(a_0^2+2a_1b_1+2a_2b_2+2a_3b_3+2a_4b_4) \\ &+(2a_0b_1+2a_1b_2+2a_2b_3+2a_3b_4)Y^{-1}+(b_1^2+2a_0b_2+2a_1b_3+2a_2b_4)Y^{-2} \\ &+(2a_0b_3+2a_1b_4+2b_1b_2)Y^{-3}+(b_2^2+2a_0b_4+2b_1b_3)Y^{-4}+(2b_1b_4+2b_2b_3)Y^{-5} \\ &+(b_3^2+2b_2b_4)Y^{-6}+2b_3b_4Y^{-7}+b_4^2Y^{-8}. \end{aligned} \tag{3.8}$$

We then substitute (3.5), (3.6), (3.7) and (3.8) into (3.2), collect all the coefficients of Y^i for $i=-8,-7,\dots,8$, and set them equal to zero so that we can obtain a system of equations. Next, we discuss the results for $\beta=1$ and -1.

4. The Boussinesq equation with $\beta = 1$

For the case of $\beta=1$, we get the following system

$$\begin{aligned}
O(Y^8):&\quad a_4^2-840\mu^4a_4=0,\\
O(Y^7):&\quad -360a_3\mu^4+2a_3a_4=0,\\
O(Y^6):&\quad 2a_2a_4+20\mu^2a_4-120\mu^4a_2+2080\mu^4a_4+a_3^2=0,\\
O(Y^5):&\quad 2a_1a_4+2a_2a_3+12\mu^2a_3-24\mu^4a_1+816\mu^4a_3=0,\\
O(Y^4):&\quad 240\mu^4a_2-1696a_4\mu^4+6\mu^2a_2-32a_4\mu^2+a_2^2+a_4v^2-a_4+2a_0a_4+2a_1a_3=0,\\
O(Y^3):&\quad 2a_0a_3-a_3+2a_1a_2+2a_4b_1+2\mu^2a_1-18\mu^2a_3+40\mu^4a_1-576\mu^4a_3+a_3v^2=0,\\
O(Y^2):&\quad 2a_0a_2-a_2+2a_3b_1+2a_4b_2-8\mu^2a_2+12\mu^2a_4-136\mu^4a_2+480\mu^4a_4+a_2v^2+a_1^2=0,\\
O(Y):&\quad 2a_0a_1-a_1+2a_2b_1+2a_3b_2+2a_4b_3-2\mu^2a_1+6\mu^2a_3-16\mu^4a_1+120\mu^4a_3+a_1v^2=0,\\
O(Y^0):&\quad 2a_1b_1-a_0+2a_2b_2+2a_3b_3+2a_4b_4+2\mu^2a_2+16\mu^4a_2-24\mu^4a_4+2\mu^2b_2+16\mu^4b_2\\
&\qquad -24\mu^4b_4+a_0v^2+a_0^2=0,\\
O(Y^{-1}):&\quad 2a_0b_1-b_1+2a_1b_2+2a_2b_3+2a_3b_4-2\mu^2b_1+6\mu^2b_3-16\mu^4b_1+120\mu^4b_3+b_1v^2=0,\\
O(Y^{-2}):&\quad 2a_0b_2-b_2+2a_1b_3+2a_2b_4-8\mu^2b_2+12\mu^2b_4-136\mu^4b_2+480\mu^4b_4+b_2v^2+b_1^2=0,\\
O(Y^{-3}):&\quad 2a_0b_3-b_3+2a_1b_4+2b_1b_2+2\mu^2b_1-18\mu^2b_3+40\mu^4b_1-576\mu^4b_3+b_3v^2=0,\\
O(Y^{-4}):&\quad 240\mu^4b_2-1696b_4\mu^4+6\mu^2b_2-32b_4\mu^2+b_2^2+b_4v^2-b_4+2a_0b_4+2b_1b_3=0,\\
O(Y^{-5}):&\quad 2b_1b_4+2b_2b_3+12\mu^2b_3-24\mu^4b_1+816\mu^4b_3=0,\\
O(Y^{-6}):&\quad 2b_2b_4+20\mu^2b_4-120\mu^4b_2+2080\mu^4b_4+b_3^2=0,\\
O(Y^{-7}):&\quad -360b_3\mu^4+2b_3b_4=0,\\
O(Y^{-8}):&\quad -840\mu^4b_4+b_4^2=0.
\end{aligned}$$

4.1. When $a_4 = 0$

We can show that if $a_4=0$, then $a_3=a_2=a_1=0$ based on the coefficients of $O(Y^i)$ with $i=1,2,\dots,8$. Then the coefficient of $O(Y^0)$ leads to

$$-a_0+2\mu^2b_2+16\mu^4b_2-24\mu^4b_4+a_0v^2+a_0^2=0.$$

If we further assume $b_4=0$, then $b_3=b_2=b_1=0$, and the equation above leads to the trivial solutions to (3.2), namely, $u=0$ or $u=1-v^2$. Therefore, for the case of $a_4=0$, we assume $b_4\neq 0$ so that the coefficient of Y^{-8} gives

$$b_4=840\mu^4.$$

Thus we can solve for b_3, b_2, b_1 using the coefficients of Y^{-i} for $i=7,6$ and 5 to get

$$b_3=0,\quad b_2=-\frac{140}{13}\mu^2-1120\mu^4,\quad b_1=0.$$

We further substitute the value of b_1, b_2, b_3 and a_i with $1\le i\le 4$ into the coefficients of Y^{-4}, Y^{-2} and Y^0, respectively, to obtain

$$1568\mu^4+\frac{560}{13}\mu^2-\left(3v^2+6a_0-\frac{476}{169}\right)=0, \tag{4.1}$$

$$3968\mu^6+\frac{1904}{13}\mu^4+\left(-8v^2-16a_0+\frac{112}{13}\right)\mu^2+\left(-\frac{1}{13}v^2-\frac{2}{13}a_0+\frac{1}{13}\right)=0, \tag{4.2}$$

$$38080\mu^8+\frac{31360}{13}\mu^6+\frac{280}{13}\mu^4-a_0^2-a_0v^2+a_0=0. \tag{4.3}$$

Equation (4.1) and (4.2) lead to

$$v^2+2a_0=\frac{3968\mu^6+\frac{1904}{13}\mu^4+\frac{112}{13}\mu^2+\frac{1}{13}}{8\mu^2+\frac{1}{13}}=\frac{1568\mu^4+\frac{560}{13}\mu^2+\frac{476}{169}}{3}. \tag{4.4}$$

Thus we get the following equation about μ:

$$640\mu^6+\frac{336}{13}\mu^4-\frac{31}{2197}=\left(\mu^2-\frac{13}{676}\right)\left(640\mu^4+\frac{496}{13}\mu^2+\frac{124}{169}\right)=0.$$

The roots of the equation above are $\mu_1=-\frac{\sqrt{13}}{26}$, $\mu_2=\frac{\sqrt{13}}{26}$, $\mu_3=\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$, $\mu_4=\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$, $\mu_5=-\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$ and $\mu_6=-\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$.

4.1.1. For $\mu=\mu_1=-\frac{\sqrt{13}}{26}$

We substitute the value of μ into equation (4.4) to get

$$v^2+2a_0=\frac{238}{169}. \tag{4.5}$$

We then substitute the value of μ into equation (4.3), and obtain

$$a_0^2+a_0v^2-a_0=\frac{3465}{114244}. \tag{4.6}$$

Solving (4.5) and (4.6) leads to

$$(1)\,a_0=\frac{105}{338},v=\frac{\sqrt{133}}{13};\quad (2)\,a_0=\frac{105}{338},v=-\frac{\sqrt{133}}{13};$$
$$(3)\,a_0=\frac{33}{338},v=\frac{\sqrt{205}}{13};\quad (4)\,a_0=\frac{33}{338},v=-\frac{\sqrt{205}}{13}.$$

In addition, we can calculate that

$$b_4=\frac{105}{338},\quad b_3=0,\quad b_2=-\frac{105}{169},\quad b_1=0.$$

Based on the discussion above, we can obtain four traveling wave solutions:

$$\begin{aligned}
u_1(x,t) &= \frac{105}{338}-\frac{105}{169}\coth^2(-\frac{\sqrt{13}}{26}(x-\frac{\sqrt{133}}{13}t))+\frac{105}{338}\coth^4(-\frac{\sqrt{13}}{26}(x-\frac{\sqrt{133}}{13}t)).\\
u_2(x,t) &= \frac{105}{338}-\frac{105}{169}\coth^2(-\frac{\sqrt{13}}{26}(x+\frac{\sqrt{133}}{13}t))+\frac{105}{338}\coth^4(-\frac{\sqrt{13}}{26}(x+\frac{\sqrt{133}}{13}t)).\\
u_3(x,t) &= \frac{33}{338}-\frac{105}{169}\coth^2(-\frac{\sqrt{13}}{26}(x-\frac{\sqrt{205}}{13}t))+\frac{105}{338}\coth^4(-\frac{\sqrt{13}}{26}(x-\frac{\sqrt{205}}{13}t)).\\
u_4(x,t) &= \frac{33}{338}-\frac{105}{169}\coth^2(-\frac{\sqrt{13}}{26}(x+\frac{\sqrt{205}}{13}t))+\frac{105}{338}\coth^4(-\frac{\sqrt{13}}{26}(x+\frac{\sqrt{205}}{13}t)).
\end{aligned}$$

The traveling wave solution $u_1(x,t)$ at $T=1$ and $T=3$ is given in Figure 4.1. The figure is generated using MATLAB 2019a. Note that $u_1(x,t)$ is defined for $x\neq\frac{\sqrt{133}}{13}t$, thus we only plot part of the spatial domain such that $x-\frac{\sqrt{133}}{13}t$ is large enough. The formulation of $u_1(x,t)$ indicates that the wave travels from left to right, and it is consistent with the observation from Figure 4.1. The behavior of u_2, u_3 and u_4 are very similar to that of u_1. Therefore, we skip the plots of these solutions.

4.1.2. For $\mu=\mu_2=\frac{\sqrt{13}}{26}$

It is easy to show that the values of b_i $(1\le i\le 4)$, a_j $(0\le j\le 4)$ and v_0 are the same as their values in the case when $\mu=\mu_1=-\frac{\sqrt{13}}{26}$. Also note that $\coth^2(-\xi)=\coth^2(\xi)$ and $\coth^4(-\xi)=\coth^4(\xi)$. Therefore, the traveling wave solutions for this case are exactly the same as $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$ and $u_4(x,t)$ in the previous section.

4.1.3. For $\mu=\mu_3=\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$

We substitute the value of μ into equation (4.4) to get

$$v^2+2a_0=\frac{14203-819\sqrt{31}i}{16900}. \tag{4.7}$$

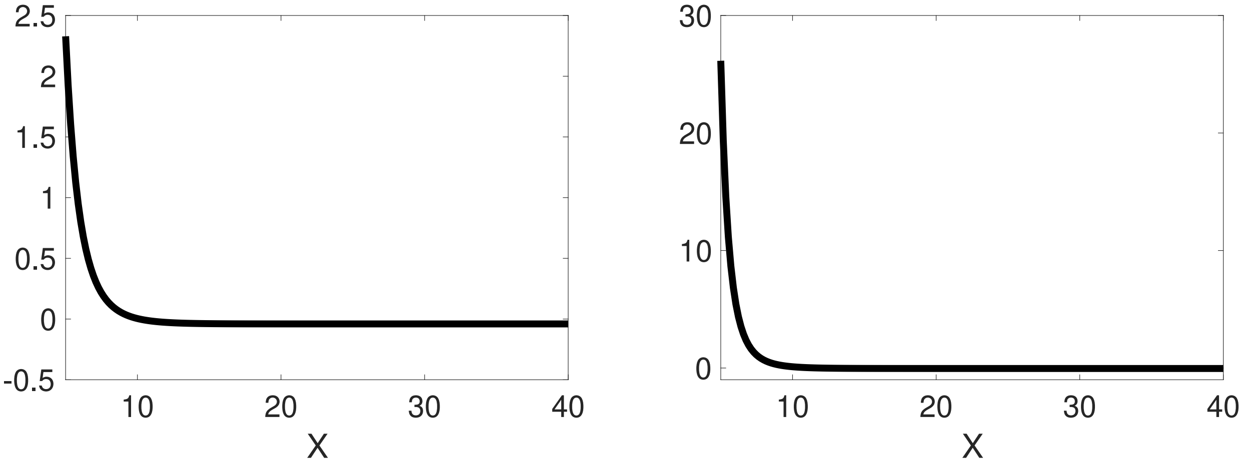

Figure 4.1: The traveling wave solution $u_1(x,t)$ at $T=1$ (the left figure) and $T=3$ (the right figure).

We then substitute the value of μ into equation (4.3) to get

$$a_0^2+a_0v^2-a_0=\frac{-6595281+2189313\sqrt{31}i}{456976000}. \tag{4.8}$$

(4.7) and (4.8) lead to

$$a_0^2+\frac{2697+819\sqrt{31}i}{16900}a_0+\frac{-6595281+2189313\sqrt{31}i}{456976000}=0.$$

Thus $a_0=\frac{-5394-1638\sqrt{31}i\pm\sqrt{11873682-4222386\sqrt{31}i}}{67600}$ and v can be solved using (4.7).

4.2. When $a_4\neq 0$ and $b_4=0$

Note that the coefficients of $O(Y^i)$ and $O(Y^{-i})$ for $i=1,\ldots,8$ are symmetric in the sense that if we interchange a_i, b_i in the formulations of $O(Y^i)$, we can obtain the formulations of $O(Y^{-i})$. Thus, we can show that

$$b_4=b_3=b_2=b_1=0,$$

and

$$a_4=840\mu^4,\quad a_3=0,\quad a_2=-\frac{140}{13}\mu^2-1120\mu^4,\quad a_1=0.$$

In addition, it is also easy to verify that equations (4.1), (4.2) and (4.3) are also satisfied. Therefore, the solution of μ is the same as that in the case when $a_4=0$, i.e., $\mu_1=-\frac{\sqrt{13}}{26}$, $\mu_2=\frac{\sqrt{13}}{26}$, $\mu_3=\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$, $\mu_4=\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$, $\mu_5=-\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$ and $\mu_6=-\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$. So we can obtain another four traveling wave solutions for $\mu=\mu_1$ and $\mu=\mu_2$:

$$\begin{aligned}
u_5(x,t) &= \frac{105}{338}-\frac{105}{169}\tanh^2(\frac{\sqrt{13}}{26}(x-\frac{\sqrt{133}}{13}t))+\frac{105}{338}\tanh^4(\frac{\sqrt{13}}{26}(x-\frac{\sqrt{133}}{13}t)).\\
u_6(x,t) &= \frac{105}{338}-\frac{105}{169}\tanh^2(\frac{\sqrt{13}}{26}(x+\frac{\sqrt{133}}{13}t))+\frac{105}{338}\tanh^4(\frac{\sqrt{13}}{26}(x+\frac{\sqrt{133}}{13}t)).\\
u_7(x,t) &= \frac{33}{338}-\frac{105}{169}\tanh^2(\frac{\sqrt{13}}{26}(x-\frac{\sqrt{205}}{13}t))+\frac{105}{338}\tanh^4(\frac{\sqrt{13}}{26}(x-\frac{\sqrt{205}}{13}t)).\\
u_8(x,t) &= \frac{33}{338}-\frac{105}{169}\tanh^2(\frac{\sqrt{13}}{26}(x+\frac{\sqrt{205}}{13}t))+\frac{105}{338}\tanh^4(\frac{\sqrt{13}}{26}(x+\frac{\sqrt{205}}{13}t)).
\end{aligned}$$

We further use MATLAB 2019a to visualize the traveling wave solutions u_5, u_6, u_7 and u_8 for $t\in[0,30]$. Note that these functions are defined for all real numbers. Figure 4.2 shows that u_5 travels in the positive x-direction and u_6 travels in the negative x-direction. As one can observe in Figure 4.3, the solutions u_7 and u_8 have quite similar behavior as u_5 and y_6, though they have slightly different magnitudes and propagating speeds.

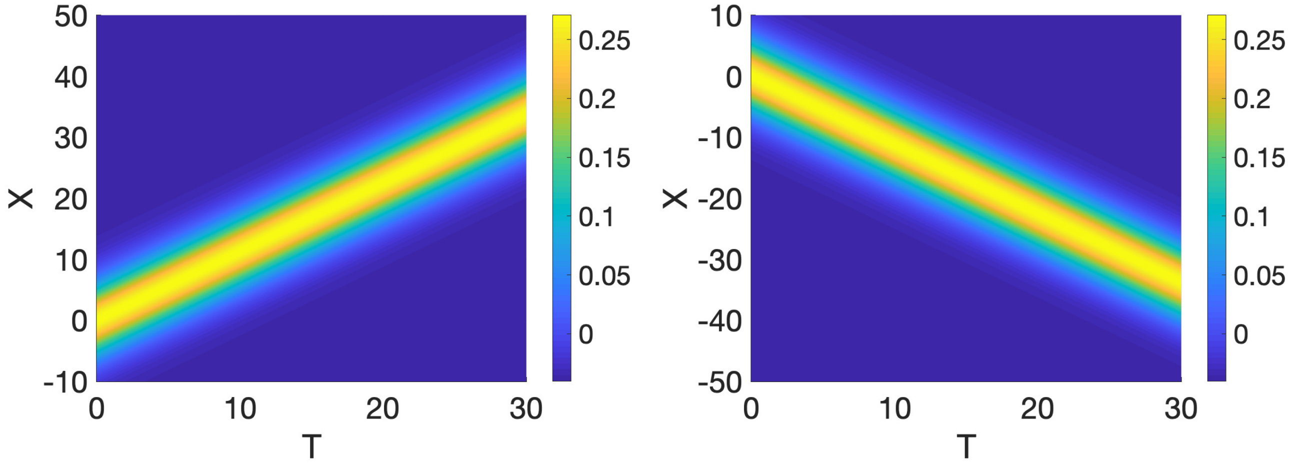

Figure 4.2: The traveling wave solution $u_5(x,t)$ (the left figure) and $u_6(x,t)$ (the right figure).

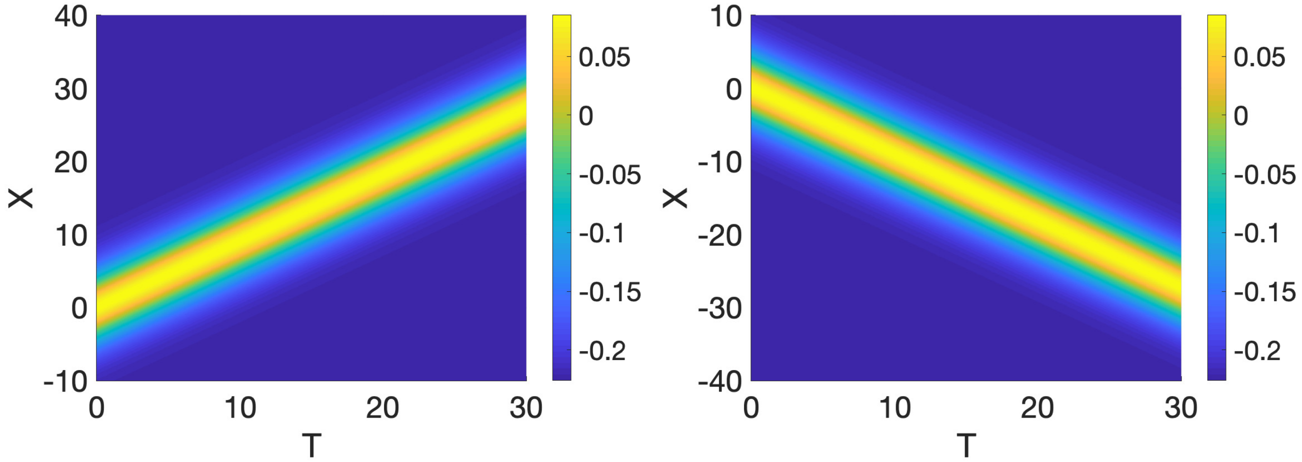

Figure 4.3: The traveling wave solution $u_7(x,t)$ (the left figure) and $u_8(x,t)$ (the right figure).

4.3. When $a_4 \neq 0$ and $b_4 \neq 0$

Similar to the procedure discussed in the previous sections, we can solve that

$$a_4 = b_4 = 840\mu^4, \quad a_3 = b_3 = a_1 = b_1 = 0, \quad a_2 = b_2 = -\frac{140}{13}\mu^2 - 1120\mu^4.$$

We can also show that equations (4.1), (4.2) and (4.4) are also satisfied, and the solutions of μ are $\mu_1 = -\frac{\sqrt{13}}{26}$, $\mu_2 = \frac{\sqrt{13}}{26}$, $\mu_3 = \sqrt{\frac{-31+3\sqrt{31}i}{1040}}$, $\mu_4 = \sqrt{\frac{-31-3\sqrt{31}i}{1040}}$, $\mu_5 = -\sqrt{\frac{-31+3\sqrt{31}i}{1040}}$ and $\mu_6 = -\sqrt{\frac{-31-3\sqrt{31}i}{1040}}$. The coefficient of $O(Y^0)$ leads to

$$-2(2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4) - 2a_2^2 - 2a_4^2 - a_0^2 - a_0 v^2 + a_0 = 0.$$

For $\mu = \mu_1$ or $\mu = \mu_2$, the equation above leads to

$$a_0^2 + a_0 v^2 - a_0 = -\frac{23380}{28561}.$$

Since $v^2 + 2a_0 = \frac{238}{169}$, we have

$$a_0^2 - \frac{69}{169}a_0 - \frac{23380}{28561} = 0.$$

Its solution is $a_0 = \frac{69 \pm \sqrt{98281}}{338}$. Thus, we have

$$v = \sqrt{\frac{238}{169} - 2a_0} = \sqrt{1 \pm \frac{\sqrt{98281}}{169}}.$$

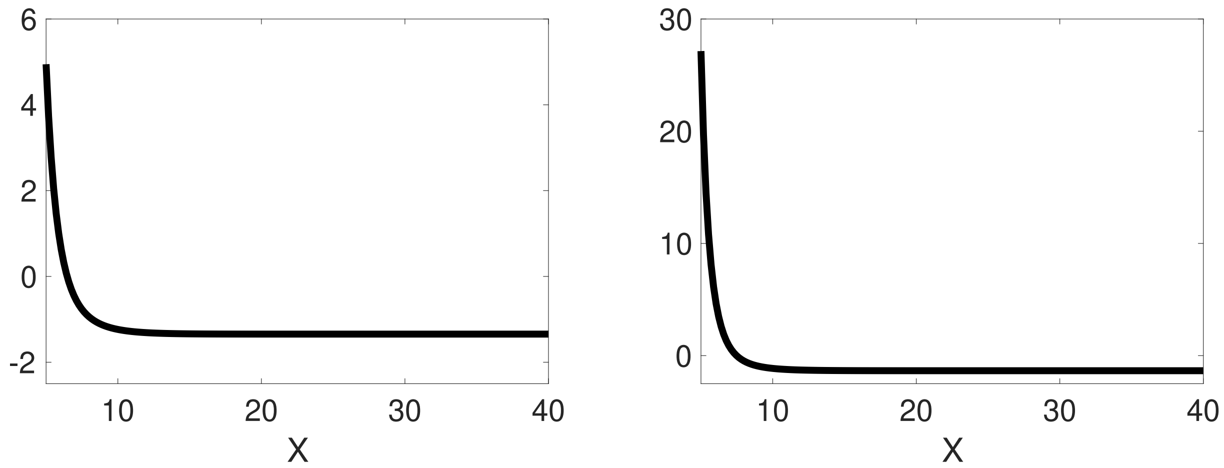

Figure 4.4: The traveling wave solution $u_9(x,t)$ at $T=1$ (the left figure) and $T=1.6$ (the right figure).

Since we assume the constant velocity v of the traveling wave solution is a real number, here we only take $v=\sqrt{1+\frac{\sqrt{98281}}{169}}$, and the corresponding value of a_0 is $a_0=\frac{69-\sqrt{98281}}{338}$. We then use the value of μ to find $a_4=b_4=\frac{105}{338}$, and $a_2=b_2=-\frac{105}{169}$. Note that $\tanh^2(-\xi)=\tanh^2(\xi)$, $\tanh^4(-\xi)=\tanh^4(\xi)$, $\coth^2(-\xi)=\coth^2(\xi)$ and $\coth^4(-\xi)=\coth^4(\xi)$. Therefore, we can obtain the following traveling wave solutions for $\mu=\mu_1$ and μ_2:

$$\begin{aligned} u_9(x,t) &= \frac{69-\sqrt{98281}}{338}-\frac{105}{169}\tanh^2\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right) \\ &\quad -\frac{105}{169}\coth^2\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right)+\frac{105}{338}\tanh^4\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right) \\ &\quad +\frac{105}{338}\coth^4\left(\frac{\sqrt{13}}{26}\left(x-\sqrt{1+\frac{\sqrt{98281}}{169}}t\right)\right). \end{aligned}$$

Due to the coth function in the formulation of $u_9(x,t)$, the domain of u_9 is not the entire real axis. We plot the traveling wave solution $u_9(x,t)$ at $T=1$ and $T=1.6$ in Figure 4.4. The figure shows that the solution is traveling in the positive x-direction. When $\mu=\mu_3$, μ_4, μ_5 or μ_6, there is no real solution for v, thus we do not consider the other cases of μ.

5. The Boussinesq equation with $\beta=-1$

For the case of $\beta=-1$, we get the following system

$$\begin{aligned} O(Y^8): &\quad a_4^2-840\mu^4a_4=0, \\ O(Y^7): &\quad -360a_3\mu^4+2a_3a_4=0, \\ O(Y^6): &\quad 2a_2a_4-20\mu^2a_4-120\mu^4a_2+2080\mu^4a_4+a_3^2=0, \\ O(Y^5): &\quad 2a_1a_4+2a_2a_3-12\mu^2a_3-24\mu^4a_1+816\mu^4a_3=0, \\ O(Y^4): &\quad 240\mu^4a_2-1696a_4\mu^4-6\mu^2a_2+32a_4\mu^2+a_2^2+a_4v^2-a_4+2a_0a_4+2a_1a_3=0, \\ O(Y^3): &\quad 2a_0a_3-a_3+2a_1a_2+2a_4b_1-2\mu^2a_1+18\mu^2a_3+40\mu^4a_1-576\mu^4a_3+a_3v^2=0, \\ O(Y^2): &\quad 2a_0a_2-a_2+2a_3b_1+2a_4b_2+8\mu^2a_2-12\mu^2a_4-136\mu^4a_2+480\mu^4a_4+a_2v^2+a_1^2=0, \\ O(Y): &\quad 2a_0a_1-a_1+2a_2b_1+2a_3b_2+2a_4b_3+2\mu^2a_1-6\mu^2a_3-16\mu^4a_1+120\mu^4a_3+a_1v^2=0, \end{aligned}$$

$$\begin{aligned}
O(Y^0): &\quad 2a_1b_1 - a_0 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4 - 2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4 - 2\mu^2 b_2 + 16\mu^4 b_2 \\
&\qquad -24\mu^4 b_4 + a_0 v^2 + a_0^2 = 0, \\
O(Y^{-1}): &\quad 2a_0b_1 - b_1 + 2a_1b_2 + 2a_2b_3 + 2a_3b_4 + 2\mu^2 b_1 - 6\mu^2 b_3 - 16\mu^4 b_1 + 120\mu^4 b_3 + b_1 v^2 = 0, \\
O(Y^{-2}): &\quad 2a_0b_2 - b_2 + 2a_1b_3 + 2a_2b_4 + 8\mu^2 b_2 - 12\mu^2 b_4 - 136\mu^4 b_2 + 480\mu^4 b_4 + b_2 v^2 + b_1^2 = 0, \\
O(Y^{-3}): &\quad 2a_0b_3 - b_3 + 2a_1b_4 + 2b_1b_2 - 2\mu^2 b_1 + 18\mu^2 b_3 + 40\mu^4 b_1 - 576\mu^4 b_3 + b_3 v^2 = 0, \\
O(Y^{-4}): &\quad 240\mu^4 b_2 - 1696 b_4 \mu^4 - 6\mu^2 b_2 + 32 b_4 \mu^2 + b_2^2 + b_4 v^2 - b_4 + 2a_0 b_4 + 2b_1 b_3 = 0, \\
O(Y^{-5}): &\quad 2b_1b_4 + 2b_2b_3 - 12\mu^2 b_3 - 24\mu^4 b_1 + 816\mu^4 b_3 = 0, \\
O(Y^{-6}): &\quad 2b_2b_4 - 20\mu^2 b_4 - 120\mu^4 b_2 + 2080\mu^4 b_4 + b_3^2 = 0, \\
O(Y^{-7}): &\quad -360 b_3 \mu^4 + 2b_3 b_4 = 0, \\
O(Y^{-8}): &\quad -840\mu^4 b_4 + b_4^2 = 0.
\end{aligned}$$

5.1. When $a_4 = 0$ and $b_4 \neq 0$

By considering the coefficients for Y^7, Y^6 and Y^5, we can show that $a_4 = a_3 = a_2 = a_1 = 0$. Then we use the coefficient for Y^0 terms to get

$$-a_0 - 2\mu^2 b_2 + 16\mu^4 b_2 - 24\mu^4 b_4 + a_0 v^2 + a_0^2 = 0.$$

Since $b_4 \neq 0$, we have $b_4 = 840\mu^4$ using the coefficient for Y^{-8}. Similarly, we can calculate the values of b_1, b_2 and b_3, i.e.,

$$b_1 = b_3 = 0, \quad b_2 = \frac{140}{13}\mu^2 - 1120\mu^4.$$

Similar to equations (4.1)-(4.3), we can use the coefficients of Y^{-4}, Y^{-2} and Y^0 to derive the following equalities:

$$1568\mu^4 - \frac{560}{13}\mu^2 - \left(3v^2 + 6a_0 - \frac{476}{169}\right) = 0, \tag{5.1}$$

$$3968\mu^6 - \frac{1904}{13}\mu^4 + \left(-8v^2 - 16a_0 + \frac{112}{13}\right)\mu^2 + \left(-\frac{1}{13}v^2 - \frac{2}{13}a_0 + \frac{1}{13}\right) = 0, \tag{5.2}$$

$$38080\mu^8 - \frac{31360}{13}\mu^6 + \frac{280}{13}\mu^4 - a_0^2 - a_0 v^2 + a_0 = 0. \tag{5.3}$$

Equation (5.1) and (5.2) lead to

$$v^2 + 2a_0 = \frac{3968\mu^6 - \frac{1904}{13}\mu^4 + \frac{112}{13}\mu^2 + \frac{1}{13}}{8\mu^2 + \frac{1}{13}} = \frac{1568\mu^4 - \frac{560}{13}\mu^2 + \frac{476}{169}}{3}. \tag{5.4}$$

Eventually, we can obtain the equation about μ. That is,

$$640\mu^6 + \frac{2800}{13}\mu^4 - \frac{1120\mu^2}{169} - \frac{31}{2197} = 0.$$

There are two real roots and four pure imaginary roots to the equation above, but here we only consider the two real roots, i.e.,

$$\mu = \pm\frac{6263491387804093}{36028797018963968} \approx \pm 0.1738468. \tag{5.5}$$

For either two values of μ, we can substitute it into equation (5.4) to get

$$v^2 + 2a_0 = \frac{8847763345396973}{9007199254740992} \approx 0.9822991. \tag{5.6}$$

We then use the value of μ in equation (5.3) to get

$$a_0^2 + a_0 v^2 - a_0 = -\frac{4366459107829337}{288230376151711744} \approx -0.0151492. \tag{5.7}$$

(5.6) and (5.7) lead to

$$a_0^2 + \frac{159435909344019}{9007199254740992}a_0 - \frac{4366459107829337}{288230376151711744} = 0.$$

Therefore, we have

$$a_0 = \frac{\pm\sqrt{4941615711925531876692800332649} - 159435909344019}{18014398509481984} \approx -0.1322503 \text{ or } 0.1145494.$$

Next, we use the values of a_0 to calculate the value of v so that we can eventually get

$$v = \frac{\sqrt{2}\sqrt{11230173773696513}}{134217728} \approx 1.1166090 \text{ or } \frac{\sqrt{2}\sqrt{27136898943141883}}{268435456} \approx 0.8678711.$$

We then further calculate

$$b_4 = 840\mu^4 \approx 0.7672664 \text{ and } b_2 = \frac{140}{13}\mu^2 - 1120\mu^4 \approx -0.6975465.$$

Therefore, the four traveling wave solutions for this case are of the following form $u(x,t) = a_0 + b_2\coth^2(\mu(x-vt)) + b_4\coth^4(\mu(x-vt))$, where a_0, b_2, b_4, μ and v are given in the previous calculations. Since there exist two distinct values for μ and v, there are four traveling wave solutions in such a form.

5.2. When $a_4 \neq 0$ and $b_4 = 0$

Since the coefficients for Y^i and Y^{-i} terms (for $i = 1,2,\ldots,8$) are symmetric, the calculations from the previous sections can be directly applied here. Therefore, the four traveling wave solutions for this case are of the form $u(x,t) = a_0 + a_2\tanh^2(\mu(x-vt)) + a_4\tanh^4(\mu(x-vt))$. Here, the values of a_0, μ and v are the same as that in the previous section, and the values of a_2 and a_4 are equal to the values of b_2 and b_4 in the previous section, respectively.

5.3. When $a_4 \neq 0$ and $b_4 \neq 0$

Using the algebraic equations for the coefficients of Y^i and Y^{-i} for $i = 1,2,\ldots,8$, we can find the values of a_i and b_j for $i, j = 1,2,3,4$:

$$a_4 = b_4 = 840\mu^4, \quad a_1 = a_3 = b_1 = b_3 = 0, \quad a_2 = b_2 = \frac{140}{13}\mu^2 - 1120\mu^4. \tag{5.8}$$

One can also show that the value of μ is the same as in (5.5). That is, $\mu = \pm\frac{6263491387804093}{36028797018963968}$. In addition, we can show that equation (5.6) also holds. So we have $v^2 + 2a_0 = \frac{8847763345396973}{9007199254740992}$. We further use the equation about $O(Y^0)$ to get

$$-2(2\mu^2 a_2 + 16\mu^4 a_2 - 24\mu^4 a_4) + 2a_2^2 + 2a_4^2 + a_0^2 + a_0 v^2 - a_0 = 0.$$

We then substitute the values of a_4 and a_2 from (5.8) into the equation above to get

$$a_0^2 + a_0 v^2 - a_0 + 3996160\mu^8 - \frac{573440}{13}\mu^6 + \frac{31920}{160}\mu^4 = 0,$$

which can be further reduced to

$$a_0^2 + a_0 v^2 - a_0 + \frac{5154129393924667}{2251799813685248} = 0$$

using the value of μ. Eventually, we can use the equation about $v^2 + 2a_0$ to derive a quadratic equation with respect to a_0:

$$a_0^2 + \frac{159435909344019}{9007199254740992}a_0 - \frac{5154129393924667}{2251799813685248} = 0.$$

The two roots to the equation above are

$$a_0 = \frac{\pm\sqrt{742813746781938776364458008666985} - 15943590934019}{18014398509481984} \approx -1.5217852 \text{ or } 1.5040843.$$

Since $v^2 + 2a_0 \approx 0.9822991$ and we look for a real number v, here we only choose the negative number for a_0, i.e.,

$$a_0 = -\frac{\sqrt{742813746781938776364458008666985} + 15943590934019}{18014398509481984} \approx -1.5217852.$$

Finally, we can calculate the two values of v, i.e., $v \approx \pm 2.0064570$, and we have $a_2 \approx -0.6975465$ and $a_4 \approx 0.7672664$. Therefore, the two traveling wave solutions for this case are in the following form:

$$u(x,t) = a_0 + a_2\tanh^2(\mu(x-vt)) + a_4\tanh^4(\mu(x-vt)) + a_2\coth^2(\mu(x-vt)) + a_4\coth^4(\mu(x-vt)),$$

where the values of μ, v, a_0, a_2 and a_4 can be found in the discussion above.

6. Conclusion

In this paper, we apply the tanh-coth method to obtain several new traveling wave solutions for the sixth-order Boussinesq equation with $\beta = 1$ or $\beta = -1$. By balancing the nonlinear quadratic term and the sixth-order derivative term in the equation, we are able to determine the number of terms in the expansion solution. By further solving the algebraic system about the unknown parameters, we obtain new solutions for the equation. These new exact solutions can also be used to assess the performance of various numerical methods for the sixth-order Boussinesq equation.

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