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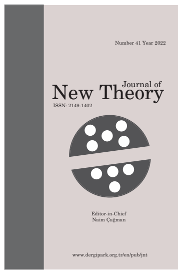

A New Perspective on k-Ideals of a Semiring via Soft Intersection Ideals

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Abstract — In recent years, soft sets have become popular in various fields. For this reason, many studies have been carried out in the field of algebra. In this study, soft intersection k-ideals are defined with the help of a semiring, and some algebraic structures are examined. Moreover, the quotient rings are defined by k-semiring. Isomorphism theorems are examined by quotient rings. Finally, some algebraic properties are investigated by defining soft intersection maximal k-ideals.

Keywords — *Soft sets, soft k-ideals, soft k-semirings, soft maximal k-ideals, quotient k-semirings*

Mathematics Subject Classification (2020) — 03E99, 03E75

1. Introduction

The world is growing and developing rapidly. During the growth and development, we encounter many problems involving uncertainty. Scientists are working quickly to solve such problems. Firstly, Zadeh introduced a different approach to uncertainty by Fuzzy Set Theory [1]. Zadeh defined a function by taking the codomain as $[0, 1]$. With the help of this function, he made an approach to uncertainty by creating a fuzzy set. Rosenfeld dealing with uncertainty developed fuzzy group theory in order to form the algebraic structure of fuzzy set [2]. Then, Pawlak suggested the rough set theory in 1982 [3]. Pawlak made a different approach to uncertainty by defining a rough set with the help of lower and upper approximation. Biswas and Nanda studied algebraic properties of rough sets [4]. After, soft set theory was proposed by Molodtsov as a different approach to uncertainty [5]. Maji et al. applied the theory of soft sets to solve a decision making problem and defined some basic operations on soft sets [6, 7]. Later, Çağman and Enginoğlu redefined operations of soft sets due to some difficulties [8]. Thus, soft sets began to be used by many researchers in various fields like economics, engineering, medical sciences, etc. In addition, many studies have been made by combining soft sets with fuzzy set theory and rough set theory such as fuzzy soft sets and rough soft sets [9–13].

Aktaş and Çağman carried soft sets on a new algebraic structure for the first time. They investigated some algebraic properties by defining soft group [14]. This work paved the way for many studies in algebra. Sun et al. defined soft modules and investigated some algebraic properties [15]. Feng et al. studied soft semirings and soft ideals on soft semirings [16]. Jun et al. applied the soft sets to the theory of BCK/BCI-algebras. Soft BCK/BCI-algebras, soft subalgebras and soft p-ideals of soft BCI-algebras are introduced and their basic properties are derived [17, 18]. Many studies have been added to the literature such as soft relation, soft function, soft mapping, soft BCH-algebras, soft Bl-algebras by soft sets [19–23]. Shabir and Naz introduced soft topological spaces which are defined

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over an initial universe with a fixed set of parameters [25]. Ali et al. defined new operations on soft sets and investigated some important properties associated with these new operations. [24]. In 2012, Çağman et al. studied soft intersection group using the intersection operation on sets [26]. Çıtak and Çağman researched its algebraic properties by defining soft intersection rings [27]. Moreover, Mahmood and Tariq studied generalized k-ideals in semirings using soft intersectional sets [28]. Mahmood et al. carried out concept of soft intersectional ideal on ternary semirings and also discussed some basic results [29]. Studies on soft sets have been increasing in recent years [30–40].

In this paper, we give brief information about semiring and k-semiring and remind some basic concepts in the soft set. Moreover, we present to some algebraic structures on soft sets such as soft intersection group and soft intersection ring. Next, we define soft intersection k-ideals on a semiring. Then, we investigate some algebraic properties of soft intersection k-ideals. In which cases the image and the inverse image of a soft set are soft intersection ideals are investigated. Coset of soft intersection ideal is defined with the help of the extended soft set and its properties were examined. Furthermore, isomorphism theorems are introduced by describing quotient rings with the help of k-semirings. Finally, we define soft intersection maximal k-ideal and research its algebraic properties.

2. Preliminary

Throughout this paper, U is a universal set, E is a set of parameters, $A, B, C \subseteq E$ and $P(U)$ is the power set of U.

Definition 2.1. [41] A nonempty set S together with a binary operation "." is a semigroup if "." is associative in S, that is, $\forall a, b, c \in S, a(bc) = (ab)c$. A semigroup is commutative if "." is commutative in S, that is, for all $a, b \in S, ab = ba$.

Definition 2.2. [41] Let S be a nonempty set together with two binary operations addition and multiplication denoted by "+", "." respectively. S is called a semiring if

i. $(S, +)$ is a commutative semigroup,

ii. $(S, \cdot)$ is a semigroup,

iii. Distributive law holds, that is, $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$, for all $a, b, c \in S$,

iv. There exists $0 \in S$ such that $a + 0 = 0 + a = a$ and $a0 = 0a = 0$, for all $a \in S$.

Definition 2.3. [42] Let S be a semiring. S is called a k-semiring if there exists only $c \in S$ such that $b = a + c$ or $a = b + c$, for all $a, b \in S$.

Definition 2.4. [42] Let S be a semiring and I be a nonempyt subset of S. I is called an ideal of S if

i. $a + b \in I$, for all $a, b \in I$

ii. $ba \in I$ and $ab \in I$, for all $a \in I$ and $b \in S$.

Definition 2.5. [43] Let S be a semiring and I be an ideal of S. I is called a k-ideal of S if $a+r \in I$, for all $a \in I$ and $r \in R$ implies $r \in I$.

Definition 2.6. [44] Let S be a k-semiring and S' be a set of the same cardinality with $S - \{0\}$ such that $S \cap S' = \emptyset$ and $S \cup S' = \overline{S}$. The image of $a \in S - \{0\}$ under a given bijection denoted by a'. Addition and multiplication on $\overline{S}$ are denoted by $\oplus$ and $\odot$, respectively, and are defined as follows:

$$a \oplus b = \begin{cases} a + b, & a, b \in S \\ (x + y)', & a = x', b = y' \in S' \\ c, & a \in S, b = y' \in S', a = y + c \\ c', & a \in S, b = y' \in S', a + c = y \end{cases}$$

and

$$a \odot b = \begin{cases} ab, & a, b \in S \\ xy, & a = x', b = y' \in S' \\ (ay)', & a \in S, b = y' \in S' \\ (xb)', & a = x' \in S', b = \in S \end{cases}$$

Theorem 2.7. [44] Let S be a semiring. If S is a k-semiring, then $(\overline{S}, \oplus, \odot)$ is a ring and it is called the extended ring of S.

Theorem 2.8. [44] Let $\varphi : S \to M$ be a k-semiring homomorphism and $\overline{S}$ and $\overline{M}$ be extended rings of S and M, respectively. Let $\overline{\varphi} : \overline{S} \to \overline{M}$ be a function such that

$$\overline{\varphi}(a) = \begin{cases} \varphi(a), & a \in S \\ \varphi(a)', & a \in S - \{0\} \end{cases}$$

$\overline{\varphi}$ is a ring homomorphism. Then $\overline{\varphi}$ is called an extended ring homomorphism.

Theorem 2.9. [44] Let $\varphi : S \to M$ be a k-semiring homomorphism and $\overline{\varphi} : \overline{S} \to \overline{M}$ be an extended ring homomorphism. Then, $Ker(\overline{\varphi}) = \overline{Ker(\varphi)}$.

Definition 2.10. [5] A soft set (F, A) over U is a function defined by $F : E \to P(U)$ such that $F(e) = \emptyset$ if $e \notin A$. A soft set (F, A) over U can be represented by the set of ordered pairs

$$(F, A) = \{(e, F(e)) : e \in E\}$$

Definition 2.11. [7] Let (F, A) be a soft set over U. If $F(e) = \emptyset$, for all $e \in E$, then (F, A) is called an empty soft set, denoted by $\widetilde{\Phi}$.

If $F(e) = U$, for all $e \in E$, then (F, A) is called universal soft set, denoted by $\widetilde{E}$.

Definition 2.12. [8] Let (F, A) and (G, B) be soft sets over U. Then, (F, A) is a soft subset of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, if $F(e) \subseteq G(e)$, for all $e \in E$.

(F, A) is called a soft proper subset of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B)$, if $F(e) \subseteq G(e)$ for all $e \in E$ and $F(e) \neq G(e)$, for at least one $e \in E$.

(F, A) and (G, B) are equal, denoted by $(F, A) = (G, B)$ if $F(e) = G(e)$, for all $e \in E$.

Definition 2.13. [8] Let (F, A) and (G, B) be two soft sets over U. Then, union $(F, A)\widetilde{\cup}(G, B)$ and intersection $(F, A)\widetilde{\cap}(G, B)$ of (F, A) and (G, B) are defined by

$$(F \cup G)(e) = F(e) \cup G(e) \text{ and } (F \cap G)(e) = F(e) \cap G(e)$$

for all $e \in E$, respectively.

Definition 2.14. [8] Let (F, A) be a soft set over U. Then, complement $(F^{\tilde{c}}, E)$ of (F, A) is defined by

$$F^c(e) = U \setminus F(e)$$

for all $e \in E$.

It is easy to see that $((F, A)^{\tilde{c}})^{\tilde{c}} = (F, A)$ and $\widetilde{\Phi}^{\tilde{c}} = \widetilde{E}$.

Proposition 2.15. [8] Let (F, A) be a soft set over U. Then,

i. $(F, A)\widetilde{\cup}(F, A) = (F, A)$, $(F, A)\widetilde{\cap}(F, A) = (F, A)$

ii. $(F, A)\widetilde{\cup}\widetilde{\Phi} = (F, A)$, $(F, A)\widetilde{\cap}\widetilde{\Phi} = \widetilde{\Phi}$

iii. $(F, A)\widetilde{\cup}\widetilde{E} = \widetilde{E}$, $(F, A)\widetilde{\cap}\widetilde{E} = (F, A)$

iv. $(F, A)\widetilde{\cup}(F, A)^{\tilde{c}} = \widetilde{E}$, $(F, A)\widetilde{\cap}(F, A)^{\tilde{c}} = \widetilde{\Phi}$

Proposition 2.16. [8] Let $(F, A), (G, B)$ and (H, C) be soft sets over U. Then,

i. $(F, A)\widetilde{\cup}(G, B) = (G, B)\widetilde{\cup}(F, A)$, $(F, A)\widetilde{\cap}(G, B) = (G, B)\widetilde{\cap}(F, A)$

ii. $((F, A)\widetilde{\cup}(G, B))^{\tilde{c}} = (G, B)^{\tilde{c}}\widetilde{\cap}(F, A)^{\tilde{c}}$,
$((F, A)\widetilde{\cap}(G, B))^{\tilde{c}} = (G, B)^{\tilde{c}}\widetilde{\cup}(F, A)^{\tilde{c}}$

iii. $((F, A)\widetilde{\cup}(G, B))\widetilde{\cup}(H, C) = (F, A)\widetilde{\cup}((G, B)\widetilde{\cup}(H, C))$,
$((F, A)\widetilde{\cap}(G, B))\widetilde{\cap}(H, C) = (F, A)\widetilde{\cap}((G, B)\widetilde{\cap}(H, C))$

iv. $(F, A)\widetilde{\cup}((G, B)\widetilde{\cap}(H, C)) = ((F, A)\widetilde{\cup}(G, B))\widetilde{\cap}((F, A)\widetilde{\cup}(H, C))$,
$(F, A)\widetilde{\cap}((G, B)\widetilde{\cup}(H, C)) = ((F, A)\widetilde{\cap}(G, B))\widetilde{\cup}((F, A)\widetilde{\cap}(H, C))$

Definition 2.17. [26] Let G be a group and (F, G) be a soft set over U. Then, (F, G) is called a soft intersection group over U iff

i. $F(a - b) \supseteq F(a) \cap F(b)$

ii. $F(ab) \supseteq F(a) \cap F(b)$

for all $a, b \in G$.

Proposition 2.18. [27] If (F, G) is a soft intersection group over U, then $F(e_G) \supseteq F(a)$, for all $a \in G$.

Definition 2.19. [27] Let R be a ring and (F, R) be a soft set over U. Then, (F, R) is called a soft intersection ring over U iff

i. $F(a - b) \supseteq F(a) \cap F(b)$

ii. $F(ab) \supseteq F(a) \cap F(b)$

for all $a, b \in R$.

Definition 2.20. [27] Let R be a ring and (F, R) be a soft set over U. Then, (F, R) is called a soft intersection ideal over U iff

i. $F(a - b) \supseteq F(a) \cap F(b)$

ii. $F(ab) \supseteq F(a) \cup F(b)$

for all $a, b \in R$.

Proposition 2.21. [27] Let R be a ring with identity. If (F, R) is a soft intersection ring/ideal over U, then $F(a) \supseteq F(1_R)$, for all $a \in R$.

3. Soft Intersection k-Ideals on a Semiring

In this section, we define soft intersection k-ideals on a semiring and give some basic theory of soft intersection k-ideals on a semiring. Let U be a universal set and E be a set of parameters where $(S, +, \cdot)$ is a semiring.

Definition 3.1. [45] Let (F, S) be a soft set over U. (F, S) is called a soft intersection semiring over U if

i. $F(a + b) \supseteq F(a) \cap F(b)$

ii. $F(ab) \supseteq F(a) \cap F(b)$

for all $a, b \in S$.

Definition 3.2. Let (F, S) be a soft intersection semiring over U. For all $a, b \in S$,

i. (F, S) is called soft intersection left ideal over U if $F(ab) \supseteq F(b)$.

ii. (F, S) is called soft intersection right ideal over U if $F(ab) \supseteq F(a)$.

If (F, S) is soft intersection left ideal and soft intersection ideal over U, it is called soft intersection ideal over U.

Definition 3.3. Let (F, S) be a soft intersection ideal over U. (F, S) is called soft intersection k-ideal if $F(a) = F(0_S)$ while $F(a + b) = F(0_S)$ and $F(b) = F(0_S)$, for all $a, b \in S$.

Example 3.4. Let Z be a universal set and $(Z_6, +, \cdot)$ be a set of parameters. Let the soft set (F, Z_6) soft set be defined as in the following way:

$$F(\bar{0}) = \{0, 1\}, F(\bar{1}) = \{0\}, F(\bar{2}) = \{0, 1\}, F(\bar{3}) = \{0\}, F(\bar{4}) = \{0, 1\}, \text{ and } F(\bar{5}) = \{0\}$$

Then, (F, Z_6) is a soft intersection k-ideal over Z.

Definition 3.5. Let (F, S) be a soft set over U. $F_K = \{a \in S : F(a) \supseteq K, K \in P(U)\}$ is called K-level set of the soft set (F, S).

Theorem 3.6. Let (F, S) be a soft intersection ideal over U. Then, K-level set F_K is an ideal of S where $F(0_S) \supseteq K$.

Proof. Since $F(0_S) \supseteq K$, then $0_S \in F_K$. Thus $F_K \neq \emptyset$ and $F_K \subseteq S$. Now, we provide $a + b \in F_K$, for all $a, b \in F_K$. Since $a, b \in F_K$ then $F(a) \supseteq K$ and $F(b) \supseteq K$. $F(a + b) \supseteq F(a) \cap F(b) \supseteq K$ so $a + b \in F_K$. Now, will provide $as \in F_K$ and $sa \in F_K$, for all $a \in F_K$ and $s \in S$. Since $a \in F_K$, then $F(a) \supseteq K$. Moreover $F(as) \supseteq F(a) \supseteq K$ and $F(sa) \supseteq F(a) \supseteq K$. Then , $as \in F_K$ and $sa \in F_K$. □

Theorem 3.7. Let (F, S) be a soft intersection ideal over U and $F(0_S) = K$. If K-level set F_K is a k-ideal of S, then (F, S) is a soft intersection k-ideal over U.

Proof. Suppose that F_K is a k-ideal of S. Let $F(a + b) = F(0_S)$ and $F(b) = F(0_S)$, for all $a, b \in S$. Therefore, $a + b \in F_K$ and $b \in F_K$. Since F_K is a k-ideal of S, then $a \in F_K$. Thus, $F(a) \supseteq K$ Therefore, (F, S) is a soft intersection k-ideal over U. □

Definition 3.8. Let I be an ideal of semiring S. λ_I is called a soft characteristic function iff λ_I is a mapping of S into $P(U)$ where

$$\lambda_I(a) = \begin{cases} U, & a \in I \\ \emptyset, & a \notin I \end{cases}$$

for all $a \in S$.

Theorem 3.9. Let I be a k-ideal of semiring S. Soft caharacteristic function λ_I is a soft intersection k-ideal over U.

Proof. Firstly, we provide that soft characteristic function λ_I is a soft intersection ideal over U. Since I is an ideal, then $a + b \in I$, for all $a, b \in I$. Morever, $\lambda_I(a) = U$ and $\lambda_I(b) = U$ and $\lambda_I(a + b) = U$. Therefore, $\lambda_I(a + b) \supseteq \lambda_I(a) \cap \lambda_I(b)$. In addition, $\lambda_I(a) = \emptyset$ and $\lambda_I(b) = \emptyset$, for all $a, b \notin I$. Thus, $\lambda_I(a) \cap \lambda_I(b) = \emptyset$ and then, $\lambda_I(a + b) \supseteq \lambda_I(a) \cap \lambda_I(b)$. On the other hands, $\lambda_I(a) = U$ and $\lambda_I(b) = \emptyset$, for all $a \in I, b \notin I$. Thus, $\lambda_I(a) \cap \lambda_I(b) = \emptyset$. It follows that $\lambda_I(a + b) \supseteq \lambda_I(a) \cap \lambda_I(b)$. Now, we will provide that $\lambda_I(ab) \supseteq \lambda_I(a)$ and $\lambda_I(ab) \supseteq \lambda_I(b)$, for all $a, b \in S$. Since I is an ideal of S, then $ab \in I$, for all $a, b \in I$. Thus, $\lambda_I(a) = U$ and $\lambda_I(b) = U$ and $\lambda_I(ab) = U$. Therefore, $\lambda_I(ab) \supseteq \lambda_I(a)$ and $\lambda_I(ab) \supseteq \lambda_I(b)$. Since I is an ideal of S, then $ab \in I$ and $ba \in I$, for all $a \in I, b \notin I$. Thus, $\lambda_I(ab) = U$ and $\lambda_I(a) = U$ and$\lambda_I(b) = \emptyset$. Therefore, $\lambda_I(ab) \supseteq \lambda_I(a)$ and $\lambda_I(ab) \supseteq \lambda_I(b)$. Lastly, we will provide that characteristic function λ_I is a soft intersection k-ideal of S. Suppose that $\lambda_I(a + b) = \lambda_I(0_s)$ and $\lambda_I(b) = \lambda_I(0_s)$, for all $a, b \in S$. It follows that $a + b \in I$ and $b \in I$ by definition of soft caharacteristic function. Since I is a k-ideal of S, then $a + b \in I$ and $b \in I$, $a \in I$. Therefore, $\lambda_I(a) = U$. Thus, $\lambda_I(a) = \lambda_I(0_s)$. Consequently, λ_I is a soft intersection k-ideal of S over U. □

Definition 3.10. Let $\varphi : S \to M$ be a semiring homomorphism and (G, M) be a soft set over U. A soft set $(\varphi^{-1}(F), S)$ is defined by $\varphi^{-1}(F(a)) = F(\varphi(a))$, for all $a \in S$. The soft set is called a soft inverse image of (G, M).

Theorem 3.11. Let $\varphi : S \to M$ be a semiring homomorphism and (G, M) be a soft set over U. (G, M) is a soft intersection k-ideal over U iff $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U.

PROOF. Suppose that (G, M) is a soft intersection k-ideal over U. Firstly, we provide that $(\varphi^{-1}(G), S)$ is a soft intersection ideal,

$$\begin{aligned}\varphi^{-1}(G)(a+b) &= G(\varphi(a+b))\\ &= G(\varphi(a)+\varphi(b))\\ &\supseteq G(\varphi(a)) \cap G(\varphi(b))\\ &= \varphi^{-1}(G)(a) \cap \varphi^{-1}(G)(b)\end{aligned}$$

for all $a, b \in S$. Thus, $\varphi^{-1}(G)(a+b) \supseteq \varphi^{-1}(G)(a) \cap \varphi^{-1}(G)(b)$. For all $a, b \in S$,

$$\begin{aligned}\varphi^{-1}(G)(ab) &= G(\varphi(ab))\\ &= G(\varphi(a)\varphi(b))\\ &\supseteq G(\varphi(a))\\ &= \varphi^{-1}(G)(a)\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(G)(ab) &= G(\varphi(ab))\\ &= G(\varphi(a)\varphi(b))\\ &\supseteq G(\varphi(b))\\ &= \varphi^{-1}(G)(b)\end{aligned}$$

Then, $\varphi^{-1}(G)(ab) \supseteq \varphi^{-1}(G)(a)$ and $\varphi^{-1}(G)(ab) \supseteq \varphi^{-1}(G)(b)$. Therefore, $(\varphi^{-1}(G), S)$ is a soft intersection ideal over U.

Now, we provide that $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U. Suppose that $\varphi^{-1}(G)(a+b) = \varphi^{-1}(G)(0_S)$ and $\varphi^{-1}(G)(b) = \varphi^{-1}(G)(0_S)$, for all $a, b \in S$.

Thus, it follows that $G(\varphi(a+b)) = G(0_M)$ and $G(\varphi(b)) = G(0_M)$. Since φ is a homomorphism then $G(\varphi(a)+\varphi(b)) = G(0_M)$, $G(\varphi(b)) = G(0_M)$ and since (G, M) is a soft intersection k-ideal over U, then $G(\varphi(a)) = G(0_{S'}) = \varphi(0_S)$.

Therefore, $\varphi^{-1}(G)(a) = \varphi^{-1}(G)(0_S)$. Consequently, $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U.

Conversely, suppose that $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U. Firstly, we provide that (G, M) is a soft intersection k-ideal over U. Since φ is an onto homomorphism then there exist $a, b \in S$ such that $x = \varphi(a)$ and $y = \varphi(b)$, for all $x, y \in M$. Then,

$$\begin{aligned}G(x+y) &= G(\varphi(a)+\varphi(b))\\ &= G(\varphi(a+b))\\ &= \varphi^{-1}(G)(a+b)\\ &\supseteq \varphi^{-1}(G)(a) \cap \varphi^{-1}(G)(b)\\ &= G(\varphi(a)) \cap G(\varphi(b))\\ &= G(x) \cap G(y)\end{aligned}$$

Thus, it follows that $G(x+y) \supseteq G(x) \cap G(y)$. Moreover,

$$\begin{aligned} G(xy) &= G(\varphi(a)\varphi(b)) \\ &= G(\varphi(ab)) \\ &= \varphi^{-1}(G)(ab) \\ &\supseteq \varphi^{-1}(G)(a) \\ &= G(\varphi(a)) \\ &= G(x) \end{aligned}$$

and

$$\begin{aligned} G(xy) &= G(\varphi(a)\varphi(b)) \\ &= G(\varphi(ab)) \\ &= \varphi^{-1}(G)(ab) \\ &\supseteq \varphi^{-1}(G)(b) \\ &= G(\varphi(b)) \\ &= G(y) \end{aligned}$$

Therefore, $G(xy) \supseteq G(x)$ and $G(xy) \supseteq G(y)$. Consequently, (G, M) is a soft intersection ideal over U. Now, we provide that (G, M) is a soft intersection k-ideal over U. Suppose that $G(x+y) = G(0_M)$ and $G(y) = G(0_M)$, for all $x, y \in M$. Since φ is an onto homomorphism then there exist $a, b \in S$ such that $x = \varphi(a)$ and $y = \varphi(b)$, for all $x, y \in M$. Thus,

$$\begin{aligned} G(x+y) &= G(\varphi(a)+\varphi(b)) \\ &= G(\varphi(a+b)) \\ &= \varphi^{-1}(G)(a+b) \end{aligned}$$

and

$$\begin{aligned} G(0_M) &= G(\varphi(0_S)) \\ &= \varphi^{-1}(G)(0_S) \end{aligned}$$

Therefore, it follows $\varphi^{-1}(G)(a+b) = \varphi^{-1}(G)(0_S)$. Moreover,

$$G(y) = G(\varphi(b)) = \varphi^{-1}(G)(b)$$

and thus,

$$\varphi^{-1}(G)(b) = \varphi^{-1}(G)(0_S)$$

Since $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U, then $\varphi^{-1}(G)(a) = \varphi^{-1}(G)(0_S)$ while $\varphi^{-1}(G)(a+b) = \varphi^{-1}(G)(0_S)$ and $\varphi^{-1}(G)(b) = \varphi^{-1}(G)(0_S)$. Therefore,

$$\varphi^{-1}(G)(a) = \varphi^{-1}(G)(0_S)$$

$$G(\varphi(a)) = G(\varphi(0_S))$$

and

$$G(x) = G(0_{S'})$$

Hence, (G, M) is a soft intersection k-ideal over U. □

Definition 3.12. Let $\varphi : S \to M$ be a semiring homomorphism and (F, S) be a soft set over U. A soft set $(\varphi(F), M)$ is called soft image set of (F, S) defined by

$$\varphi(F)(x) = \begin{cases} \cup\{F(a) : a \in S, \varphi(a) = x\}, & \varphi^{-1}(x) \neq \emptyset \\ \emptyset, & otherwise \end{cases}$$

Definition 3.13. Let A, B be two sets and $\varphi : A \to B$ be a function. Let (F, A) be a soft set over U. (F, A) is called φ-invariant if $F(a) = F(b)$ while $\varphi(a) = \varphi(b)$, for all $a, b \in A$.

Theorem 3.14. Let $\varphi : S \to M$ be a semiring epimorphism and (F, S) be a φ-invariant soft intersection ideal over U. Then, $(\varphi(F), M)$ is a soft intersection ideal over U.

Proof. Since φ is an onto homomorphism, then there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$, for all $x, y \in M$. Thus, $x + y = \varphi(a) + \varphi(b) = \varphi(a + b)$ and $xy = \varphi(a)\varphi(b) = \varphi(ab)$. Since (F, S) is a φ-invariant then

$$\begin{aligned} x + y = \varphi(a+b) &\Rightarrow \varphi^{-1}(x+y) = a + b \\ &\Rightarrow \varphi(\varphi^{-1}(x+y)) = \varphi(a+b) \\ &\Rightarrow F(\varphi^{-1}(x+y)) = F(a+b) \\ &\Rightarrow \varphi(F)(x+y) = F(a+b) \end{aligned}$$

$\varphi(F)(x) = F(a)$ and $\varphi(F)(y) = F(b)$ and

$$\begin{aligned} xy = \varphi(ab) &\Rightarrow \varphi^{-1}(xy) = ab \\ &\Rightarrow \varphi(\varphi^{-1}(xy)) = \varphi(ab) \\ &\Rightarrow F(\varphi^{-1}(xy)) = F(ab) \\ &\Rightarrow \varphi(F)(xy) = F(ab) \end{aligned}$$

$\varphi(F)(x) = F(a)$ and $\varphi(F)(y) = F(b)$. Therefore,

$$\varphi(F)(x+y) = F(a+b) \supseteq F(a) \cap F(b) = \varphi(F)(x) \cap \varphi(F)(y)$$

and $\varphi(F)(xy) = F(ab) \supseteq F(a) = \varphi(F)(x)$ and $\varphi(F)(xy) = F(ab) \supseteq F(b) = \varphi(F)(y)$.

Consequently, $(\varphi(F), M)$ is a soft intersection ideal over U. □

Theorem 3.15. Let $\varphi : S \to M$ be a semiring epimorphism and (F, S) be a φ-invariant soft intersection ideal over U. (F, S) is a soft intersection k-ideal over U iff $(\varphi(F), M)$ is soft intersection k-ideal over U.

Proof. Suppose that (F, S) be a soft intersection k-ideal over U. $(\varphi(F), M)$ is a soft intersection ideal over U by Theorem 3.14. Now, we provide that $(\varphi(F), M)$ is a soft intersection k-ideal over U. Suppose that $\varphi(F)(x+y) = \varphi(F)(0_M)$ and $\varphi(F)(y) = \varphi(F)(0_M)$, for all $x, y \in M$. Since φ is an onto homomorphism, then there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$, for all $x, y \in M$. Thus,

$$\begin{aligned} \varphi(F)(x+y) &= \varphi(F)(\varphi(a) + \varphi(b)) \\ &= \varphi(F)(\varphi(a+b)) \\ &= F(a+b) \end{aligned}$$

and

$$\begin{aligned} \varphi(F)(y) &= \varphi(F)(\varphi(b)) \\ &= F(b) \end{aligned}$$

Moreover, since (F, S) is a φ-invariant then $\varphi(F)(0_M) = F(0_S)$. Since $\varphi(F)(x + y) = \varphi(F)(0_M)$ then $F(a + b) = F(0_S)$ and $\varphi(F)(y) = \varphi(F)(0_M)$. Therefore, $F(b) = F(0_S)$. Since (F, S) is a soft intersection k-ideal over U, then $F(a) = F(0_S)$ while $F(a + b) = F(0_S)$ and $F(b) = F(0_S)$. Thus,

$$\begin{aligned} \varphi(F)(x) &= \varphi(F)(\varphi(a)) \\ &= F(a) \\ &= F(0_S) \\ &= \varphi(F)(0_M) \end{aligned}$$

Therefore, $(\varphi(F), M)$ is a soft intersection k-ideal over U. Otherwise, suppose that $(\varphi(F), M)$ be a soft intersection k-ideal over U. Let $F(a+b) = F(0_S)$ and $F(b) = F(0_S)$, for all $a, b \in S$. Since φ is an onto homomorphism, then there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$, for all $x, y \in M$. Then,

$$\begin{aligned} \varphi(F)(x+y) &= \varphi(F)(\varphi(a+b)) \\ &= F(a+b) \\ &= F(0_S) \\ &= \varphi(F)(0_M) \end{aligned}$$

and

$$\begin{aligned} \varphi(F)(y) &= \varphi(F)(\varphi(b)) \\ &= F(b) \\ &= F(0_S) \\ &= \varphi(F)(0_M) \end{aligned}$$

Since $(\varphi(F), M)$ is a soft intersection k-ideal, then $\varphi(F)(x+y) = \varphi(F)(0_M)$ and $\varphi(F)(y) = \varphi(F)(0_M)$ while $\varphi(F)(x) = \varphi(F)(0_M)$.

$$\varphi(F)(x) = \varphi(F)(0_M)$$

$$\varphi(F)(\varphi(a)) = \varphi(F)(0_M)$$

$$F(a) = F(0_S)$$

Hence, (F, S) is a soft intersection k-ideal over U. □

4. Quotient Structure of k-Semiring Over Soft Intersection Ideals

In this section, we define quotient structure of k-semiring and give some basic theory of this. Let S be a semiring and $\overline{S}$ be an extended ring of S. We take a soft intersection ideal (F, S) over U which all level subset is a k-ideal of S. Then,

$$S = \bigcup_{K \in ImF} F_K \text{ and } \overline{S} = \bigcup_{K \in ImF} \overline{F_K} \Leftrightarrow T \supset K \Leftrightarrow F_T \subset F_K \Leftrightarrow \overline{F_T} \subset \overline{F_K}$$

Definition 4.1. Let S be a semiring and $\overline{S}$ be an extended ring of S. (F, S) be a soft intersection ideal over U where all level sets of (F, S) are k-ideal of S. $(\overline{F}, \overline{S})$ soft set is defined by $\overline{F}(a) = \bigcup\{K : a \in \overline{F_K}, K \in ImF\}$, for all $a \in \overline{S}$. The soft set $(\overline{F}, \overline{S})$ is called extended soft set over U.

Theorem 4.2. Let $(\overline{F}, \overline{S})$ be an extended soft set over U. Then, $(\overline{F}, \overline{S})$ is a soft intersection ideal over U.

Proof. The proof is clear. □

Theorem 4.3. Let $(\overline{F}, \overline{S})$ be an extended soft set over U. Then, $(\overline{F}, \overline{S})$ is an extended of (F, S).

Proof. Suppose that $a \in S$ and $F(a) = K$. Thus, $a \in F_T$, for all $T \subseteq K$. $F(a) \supseteq T \supset K$, for some $T \subseteq K$. This is contradiction with $F(a) = K$ and so $a \notin F_T$, for all $T \supset K$. Since $a \in F_K \subseteq \overline{F_K}$ and $a \notin F_T$, for all $T \supset K$, then $\overline{F}(a) = K = F(a)$. Suppose that $a = b^{'}$, for $a \in S^{'}$ and some $b \in S$. $\overline{F}(a) = \bigcup\{K : a \in \overline{F_K}, K \in ImF\} = V$. Thus, $a = b^{'} \in \overline{F_K}$, for all $K \subseteq V$ and $a = b^{'} \notin \overline{F_K}$, for all $K \supset V$. Therefore, $b \in \overline{F_K}$, for all $K \subseteq V$ and $b \notin \overline{F_K}$, for $K \supset V$. Hence, $\overline{F}(b) = V$ and so $\overline{F}(a) = \overline{F}(b^{'}) = \overline{F}(b) = F(b)$. Thus, $(\overline{F}, \overline{S})$ is an extended of (F, S). □

Theorem 4.4. Let $(\overline{F}, \overline{S})$ be an extended soft set over U. (F, S) is a soft intersection k-ideal over U iff $(\overline{F}, \overline{S})$ is a soft intersection k-ideal over U.

Proof. Suppose that (F,S) is a soft intersection k-ideal over U. $(\overline{F},\overline{S})$ is a soft intersection ideal over U by Theorem 4.2. Otherwise, suppose that $(\overline{F},\overline{S})$ is a soft intersection ideal over U. Then, let $F(a+b)=F(0_S)$ and $F(b)=F(0_S)$, for all $a,b\in S$. Since (F,S) is a soft intersection ideal, then $(\overline{F},\overline{S})$ is an extended of (F,S). Therefore,

$$\begin{aligned} F(a)=\overline{F}(a) &= \overline{F}(a\oplus 0_S) \\ &= \overline{F}(a\oplus b\oplus b^{'}) \\ &\supseteq \overline{F}(a\oplus b)\cap \overline{F}(b^{'}) \\ &= F(a+b)\cap F(b) \\ &= F(0_S)\cap F(0_S) \\ &= F(0_S) \end{aligned}$$

Thus, it follows that $F(a)\supseteq F(0_S)$. Since $F(0_S)\supseteq F(a)$, for all $a\in S$, then $F(0_S)=F(a)$. So, (F,S) is a soft intersection k-ideal over U. □

Definition 4.5. Let $(\overline{F},\overline{S})$ be an extended soft set over U. Define $x+(F,S):S\rightarrow P(U)$ by $(x+F)(a)=\overline{F}(a\oplus x^{'})$, for all $a,x,x^{'}\in S$. A soft set $x+(F,S)$ is called a coset of soft intersection ideal (F,S).

Theorem 4.6. Let $(\overline{F},\overline{S})$ be an extended of (F,S). $x+(F,S)=y+(F,S)$, for all $x,y\in S$ iff $\overline{F}(x\oplus y^{'})=F(0_S)$.

Proof. Suppose that $x+(F,S)=y+(F,S)$, for all $x,y\in S$,

$$\begin{aligned} \overline{F}(x\oplus y^{'}) &= (y+F)(x) \\ &= (x+F)(x) \\ &= \overline{F}(x\oplus x^{'}) \\ &= \overline{F}(0_S) \end{aligned}$$

Thus, it follows that $\overline{F}(x\oplus y^{'})=F(0_S)$. Conversely, suppose that $\overline{F}(x\oplus y^{'})=F(0_S)$, for $x,y\in R$.

$$\begin{aligned} (x+F)(a) &= \overline{F}(a\oplus x^{'}) \\ &\supseteq \overline{F}(a\oplus x^{'}\oplus y\oplus y^{'} \\ &= \overline{F}(a\oplus y^{'})\cap \overline{F}(y\oplus y^{'}) \\ &= \overline{F}(a\oplus y^{'})\cap \overline{F}(0_S) \\ &= \overline{F}(a\oplus Y^{'}) \\ &= (y+F)(a) \end{aligned}$$

for all $a\in S$. Therefore, $x+(F,S)\supseteq y+(F,S)$. Similarly, it follows that $y+(F,S)\supseteq x+(F,S)$. Thus, $x+(F,S)=y+(F,S)$. □

Theorem 4.7. Let $(\overline{F},\overline{S})$ be an extended soft set over U. If $x+(F,S)=a+(F,S)$ and $y+(F,S)=b+(F,S)$, for $x,y,a,b\in S$, then

i. $x+y+(F,S)=a+b+(F,S)$

ii. $xy+(F,S)=ab+(F,S)$

Proof. Suppose that $x+(F,S)=a+(F,S)$ and $y+(F,S)=b+(F,S)$, for $x,y,a,b\in S$.

i. Since $x+(F,S)=a+(F,S)$ and $y+(F,S)=b+(F,S)$ then $\overline{F}(x\oplus a^{'})=F(0_S)$ and $\overline{F}(y\oplus b^{'})=F(0_S)$, for $x,y,a,b\in S$. Then,

$$\begin{aligned} \overline{F}(x\oplus y\oplus a^{'}\oplus b^{'})=F(0_S) &\supseteq \overline{F}(x\oplus a^{'})\cap \overline{F}(y\oplus b^{'}) \\ &= F(0_S)\cap F(0_S) \\ &= F(0_S) \\ &= \overline{F(0_S)} \end{aligned}$$

Therefore, $\overline{F}(x \oplus a^{'}) \cap \overline{F}(y \oplus b^{'}) = \overline{F(0_S)}$ and so $x + y + (F, S) = a + b + (F, S)$

ii.

$$\begin{aligned}
\overline{F}(ab \oplus (xy)^{'}) &= \overline{F}(ab \oplus (ay)^{'} \oplus ay \oplus (xy)^{'}) \\
&= \overline{F}(a \odot (b \oplus y^{'}) \oplus (a \oplus x^{'}) \odot y) \\
&\supseteq \overline{F}(a \odot (b \oplus y^{'}) \cap F((a \oplus x^{'}) \odot y)) \\
&\supseteq \overline{F}(b \oplus y^{'}) \cap F((a \oplus x^{'})) \\
&= F(0_S) \cap F(0_S) \\
&= F(0_S)
\end{aligned}$$

Thus, $\overline{F}(y \oplus b^{'}) = F(0_S)$. And so, $ab + (F, S) = xy + (F, S)$.

□

Define "+" and "·" binary operations on $S/(F, S)$ set of coset of soft intersection ideal (F, S), respectively by

$$[x + (F, S)] + [y + (F, S)] = x + y + (F, S)$$

and

$$[x + (F, S)][y + (F, S)] = xy + (F, S)$$

$S/(F, S)$ is a k-semiring under this operation and identity element of $S/(F, S)$ is $1_S + (F, S)$.

Definition 4.8. $S/(F, S)$ set of coset of soft intersection ideal (F, S) is called a quotient ring.

5. Isomorphism Theorem Over Soft Intersection Ideals

In this section, we investigate isomorphism theorem over soft intersection ideals.

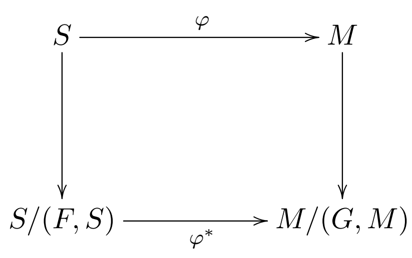
Theorem 5.1. Let $\varphi : S \to M$ be a k-semiring epimorphism. Let$(\overline{F}, \overline{S})$ be extended soft set over U and $F_S \subseteq Ker\varphi$. There is $f : S/(F, S) \to M$ an only epimorphism such that $\varphi = f \circ g$ where $g(x) = x + (F, S)$, for all $x \in S$.

Proof. Define $f : S/(F, S) \to M$ function by $f(x + (F, S)) = \varphi(x)$, for all $x \in S$. Now, we provide that f is well defined. Suppose that $x + (F, S) = y + (F, S)$, for all $x + (F, S), y + (F, S) \in S/(F, S)$. It follows that $\overline{F}(x \oplus y^{'}) = F(0_S) = \overline{F}(0_S)$ by Theorem 4.6. Thus, $x + y^{'} \in \overline{F_{\overline{S}}}$. Since $\overline{F_{\overline{S}}} = \overline{F_S} \subseteq \overline{Ker\varphi} = Ker\overline{\varphi}$ then $\overline{\varphi}(x + y^{'}) = 0_{\overline{S}}$. Thus, $\overline{\varphi}(x) = \overline{\varphi}(y)$. Therefore, $\varphi(x) = \varphi(y)$ and so $\varphi(x+(F, S)) = \varphi(y+(F, S))$. Since φ is onto function then f is onto function, Moreover, it can easily be shown that f is a homomorphism. In addition, $\varphi(x) = f(x + (F, S)) = f(g(x)) = (f \circ g)(x)$, for all $x \in S$. Finally, we will provide that f is a unique. Suppose that $\varphi = h \circ g$ such that $h : S/(F, S) \to M$. Then, $f(x + (F, S)) = \varphi(x) = (h \circ g)(x) = h(x + (F, S))$, for all $x \in S$. It follows that $f = h$. □

Proposition 5.2. Let $\varphi : S \to M$ be a k-semiring epimorphism. Suppose that (F, S) and (G, M) be define two soft intersection ideals over U as level sets of (F, S) and (G, M) be two k-ideals of S and M, respectively. If $(\varphi(F), M)\widetilde{\subset}(G, M)$, then $(\overline{\varphi}(\overline{F})\overline{M})\widetilde{\subseteq}(\overline{G}, \overline{M})$.

Proof. This proof is clear. □

Theorem 5.3. Let $\varphi : S \to M$ be a k-semiring homomorphism. Suppose that (F, S) and (G, M) be define two soft intersection ideals over U as level sets of (F, S) and (G, M) be two k-ideals of S and M, respectively. If $F(0_S) = G(0_M)$ then there exist $\varphi^* : S/(F, S) \to M/(G, M)$ homomorphism such that



Proof. Suppose that $\varphi^* : S/(F,S) \to M/(G,M)$ function is defined by $\varphi^*(x+(F,S)) = \varphi(x) + (G,M)$. If $x+(F,S) = y+(F,S)$, then by Theorem 4.6 $\overline{F}(x \oplus y') = F(0_S)$. Thus,

$$\begin{aligned}\overline{G}(\overline{\varphi}(x) \oplus \overline{(}\varphi(y))') &= \overline{G}(\overline{\varphi}(x \oplus y'))\\ &= \overline{\varphi}^{-1}(\overline{G})(x \oplus y')\\ &= \overline{F}(x \oplus y')\\ &= F(0_S)\\ &= G(0_M)\end{aligned}$$

Therefore,

$$\overline{G}(\overline{\varphi}(x) \oplus \overline{(}\varphi(y))') = \overline{G}(\varphi(x) \oplus \varphi(y)') = G(0_M)$$

Hence, $\varphi(x)+(G,M) = \varphi(y)+(G,M)$. So, φ^* is well defined. Since

$$\begin{aligned}\varphi^*((x+(F,S)+y+(F,S)) &= \varphi^*((x+y+(F,S))\\ &= \varphi(x+y)+(G,M)\\ &= \varphi(x)+\varphi(x)+(G,M)\\ &= \varphi(x)+(G,M)+\varphi(y)+(G,M)\\ &= \varphi^*(x+(F,S))+\varphi^*(y+(F,S))\end{aligned}$$

and

$$\begin{aligned}\varphi^*([(x+(F,S)][y+(F,S)]) &= \varphi^*((xy+(F,S))\\ &= \varphi(xy)+(G,M)\\ &= \varphi(x)\varphi(x)+(G,M)\\ &= \varphi(x)+(G,M)+\varphi(y)+(G,M)\\ &= \varphi^*(x+(F,S))\varphi^*(y+(F,S))\end{aligned}$$

for all $x+(F,S), y+(F,S) \in S/(F,S)$, then φ^* is a homomorphism. □

Corollary 5.4. Let $\varphi : S \to M$ be a k-semiring epimorphism. φ^* is an isomorphism iff (F,S) is a $\mu \circ \varphi$-invariant where $\mu : M \to M/(G,M)$, $\mu(z) = z+(G,M)$.

Proposition 5.5. Let $\varphi : S \to M$ be a k-semiring epimorphism and (G,M) be a soft intersection ideal over U. Suppose that $(\varphi^{-1}(G),S) = (F,S)$. All level set of (G,M) is a k-ideal iff all level set of $(\varphi^{-1}(G),S)$ is a k-ideal.

Proof. Suppose that all level set of (G,M) soft intersection ideal is a k-ideal. Then, $a+b \in F_K$ and $b \in F_K$, for all $a,b \in S$ such that $K \in P(U)$. Thus, $F(a+b) \supseteq K$ and $F(b) \supseteq K$. Since $(\varphi^{-1}(G),S) = (F,S)$ then $G(\varphi(a+b)) \supseteq K$ and $G(\varphi(b)) \supseteq K$. Since all level set of (G,M) is a k-ideal, then $G(\varphi(a)) \supseteq K$. Hence, $F(a) \supseteq K$ and $a \in F_K$. Therefore, F_K is a k-ideal of S. Conversely, suppose that all level set of soft intersection ideal $(\varphi^{-1}(G),S)$ is a k-ideal. $G(a+b) \supseteq K$ and $G(b) \supseteq K$, for all $K \in P(U)$. Since φ is onto homomorphism, then there are $a,b \in S$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Thus,

$$\begin{aligned}G(a+b) &= G(\varphi(x)+\varphi(y))\\ &= G(\varphi(x+y))\\ &= \varphi^{-1}(G)(x+y)\\ &\supseteq K\end{aligned}$$

and

$$\begin{aligned}G(b) &= G(\varphi(y))\\ &= \varphi^{-1}(G)(y)\\ &\supseteq K\end{aligned}$$

Therefore, it follows that $\varphi^{-1}(G)(x+y) \supseteq K$ and $\varphi^{-1}(G)(y) \supseteq K$. Hence, $\varphi^{-1}(G)(x) \supseteq K$. Moreover,

$$\begin{aligned} G(a) &= G(\varphi(x) \\ &= \varphi^{-1}(G)(x) \\ &\supseteq K \end{aligned}$$

Consequently G_K is a k-ideal of M. □

Proposition 5.6. Let $\varphi : S \to M$ be a k-semiring epimorphism and (G, M) be a soft intersection ideal over U. Suppose that $(\varphi^{-1}(G), S) = (F, S)$. Hence, $(\overline{F}, \overline{S}) = (\overline{\varphi}^{-1}(\overline{G}), \overline{S})$.

Proof. It is clear. □

Theorem 5.7. Let $\varphi : S \to M$ be a k-semiring epimorphism and (G, M) be a soft intersection ideal over U. Suppose that $(\varphi^{-1}(G), S) = (F, S)$. Hence, $S/(F, S) \cong M/(G, M)$.

Proof. We know that $(\varphi(F), M) = (\varphi(\varphi^{-1}(G)), M) = (G, M)$ and $F(0_S) = G(0_M)$. Moreover, let $(\mu \circ \varphi)(a) = (\mu \circ \varphi)(b)$, for all $a, b \in S$ such that $\mu(z) = z + (G, M)$, for all $z \in M$. By Proposition 5.6,

$$\begin{aligned} \varphi(a) + (G, M) = \varphi(b) + (G, M) &\Rightarrow \overline{G}(\varphi(a) \oplus \varphi(b)') = G(0_M) = F(0_S) \\ &\Rightarrow \overline{G}(\overline{\varphi}(a) \oplus \overline{\varphi}(b)') = G(0_M = F(0_S) \\ &\Rightarrow \overline{G}(\overline{\varphi}(a + b')) = G(0_M) = F(0_S) \\ &\Rightarrow (\overline{\varphi})^{-1}(\overline{G})(a + b') = G(0_M) = F(0_S) \\ &\Rightarrow \overline{F}(a + b') = F(0_S) \\ &\Rightarrow a + (F, S) = b + (F, S) \end{aligned}$$

Therefore, (F, S) is a $\mu \circ \varphi$-invariant. By Corollary 5.4, it follows that $S/(F, S) \cong M/(G, M)$. □

6. Soft Intersection Maximal k-Ideals

In this section, firstly we define soft intersection maximal k-ideal over a k-semiring. And, we investigate some properties.

Definition 6.1. Let S be a k-semiring and (F, S) be a soft intersection k-ideal over U where all subsets of S are k-ideal. A soft intersection ideal (F, S) is called soft intersection maximal k-ideal if

i. $F(0_S) = U$

ii. $F(1_S) \subset F(0_S)$

iii. $\overline{F}(1_S \oplus (sa)') = F(0_S)$, for some $s \in S$, while $F(a) \subset F(0_S)$, for some $a \in S$.

Theorem 6.2. Let (F, S) be a soft intersection k-ideal over U where all subssets of semiring S be k-ideals. (F, S) is a soft intersection maximal k-ideal over U iff $(\overline{F}, \overline{S})$ is a soft intersection maximal k-ideal over U.

Proof. Suppose that (F, S) is a soft intersection k-ideal over U. Hence

i. $\overline{F}(0_S) = F(0_S) = U$

ii. $\overline{F}(1_S) = F(1_S) \subset F(0_S) = \overline{F}(0_S)$

Let $\overline{F}(a) \subset \overline{F}(0_S)$. Since $a \in \overline{S}$, then $a \in S$ or $a \in S'$. Since $\overline{F}(a) = F(a)$, and $\overline{F}(a) \subset \overline{F}(0_S) = F(0_S)$ and then $F(a) \subset F(0_S)$. Hence, we obtain that $\overline{F}(1_S \oplus (sa)') = F(0_S) = \overline{F}(0_S)$, for some $s \in S$.If $a \in S'$ then $a = x'$ such that there exist $x \in S$ and so $\overline{F}(a) = \overline{F}(x') = \overline{F}(x) = F(x) \subset F(0_S)$. Therefore, $\overline{F}(1_S \oplus (sa)') = F(0_S) = \overline{F}(0_S)$, for some $s \in S$. Hence, $(\overline{F}, \overline{S})$ is a soft intersection maximal k-ideal over U.

Conversely, suppose that $(\overline{F}, \overline{S})$ is a soft intersection maximal k-ideal over U. Hence,

i. $\overline{F}(0_S) = F(0_S) = U$

ii. $\overline{F}(1_S) = F(1_S) \subset F(0_S) = \overline{F}(0_S)$

Let $F(a) \subset F(0_S)$. Hence, $F(a) = \overline{F}(a) \subset F(0_S) = \overline{F}(0_S)$. Then, we obtain that $\overline{F}(a) \subset \overline{F}(0_S)$. Therefore, $\overline{F}(1_S \oplus (sa)^{'}) = F(0_S)$. Consequently, (F, S) is a soft intersection k-ideal over U. □

Theorem 6.3. Let $\varphi : S \to M$ be a k-semiring endomorphism and (G, M) be a soft intersection ideal over U. (G, M) is a soft intersection k-ideal over U iff $(\varphi^{-1}(G), S)$ is a soft intersection maximal k-ideal over U.

Proof. Suppose that (G, M) is a soft intersection k-ideal over U. By Theorem 3.11, we know that $(\varphi^{-1}(G), S)$ is a soft intersection k-ideal over U. Now, we indicate that $(\varphi^{-1}(G), S)$ is a soft intersection maximal k-ideal. Then,

i. $\varphi^{-1}(G)(0_S) = G(\varphi(0_S)) = G(0_M) = U$

ii. $\varphi^{-1}(G)(1_S) = G(\varphi(1_S)) = G(1_M) \subset G(0_M) = G(\varphi(0_S)) = \varphi^{-1}(G)(0_S)$. Hence, $\varphi^{-1}(G)(1_S) \subset \varphi^{-1}(G)(0_S)$.

iii. Let $\varphi^{-1}(G)(a) \subset \varphi^{-1}(G)(0_S)$, for some $a \in S$. Hence, $G(\varphi(a)) \subset G(\varphi(0_S)) = G(0_M)$. Therefore, $G(\varphi(a)) \subset G(0_M)$.

Since (G, M) is a soft intersection k-ideal, then

$$\overline{G}(1_M \oplus (\varphi(s)\varphi(a))^{'}) = G(0_M)$$

$$\overline{G}(\varphi(1_S) \oplus \overline{\varphi}(sa)^{'}) = G(0_M)$$

$$\overline{G}(\overline{\varphi}(1_S) \oplus \overline{\varphi}(sa)^{'}) = G(\varphi(0_S))$$

and

$$(\overline{\varphi})^{-1}(\overline{G})(1_S \oplus (sa)^{'}) = \varphi^{-1}(G)(0_S)$$

Therefore, $(\varphi^{-1}(G), S)$ is a soft intersection maximal k-ideal over U. Conversely, $(\varphi^{-1}(G), S)$ be a soft intersection maximal k-ideal over U. By Theorem 3.11, (G, M) is a soft intersection k-ideal over U.

i. $G(0_M) = G(\varphi(0_S)) = \varphi^{-1}(G)(0_S) = U$

ii. $G(1_M) = G(\varphi(1_S)) = \varphi^{-1}(G)(1_S) \subset \varphi^{-1}(G)(0_S) = G(\varphi(0_S)) = G(0_M)$ Hence, it follows that $G(1_{S'}) \subset G(0_M)$.

iii. Let $G(x) \subset G(0_M)$, for $x \in M$. There exists $a \in S$ such that $\varphi(a) = x$ Therefore, since $G(x) = G(\varphi(a)) = \varphi^{-1}(G)(a) \subset G(0_S)$ then $\varphi^{-1}(G)(1_S \oplus (as)^{'}) = \varphi^{-1}(G)(0_s) = G(\varphi(0_S)) = G(0_M)$

Hence, by Theorem 4.6, it follows that

$$\begin{aligned} G(0_M) &= (\overline{\varphi})^{-1}(\overline{G}(1_S \oplus (as)^{'}) \\ &= \overline{G}(\overline{\varphi}(1_S \oplus (as)^{'}) \\ &= \overline{G}(\overline{\varphi}(1_S) \oplus \overline{\varphi}(as)^{'}) \\ &= \overline{G}(\varphi(1_S) \oplus (\varphi(a)\varphi(s))^{'}) \\ &= \overline{G}(1_M \oplus (xk)^{'})) \end{aligned}$$

where $\varphi(s) = k$. Consequently, (G, M) is a soft intersection maximal k-ideal over U. □

7. Conclusion

In this study, we defined soft intersection k-ideals on a semiring and then investigated some algebraic properties of soft intersection k-ideals. Moreover, isomorphism theorems are presented by describing quotient rings with the help of k-semiring, defined soft intersection maximal k-ideal, soft intersection maximal k-ideals are defined and their algebraic properties are investigated. Some other algebraic structures, such as prime rings [46, 47] and semi prime rings [48, 49], are worth studying in future studies.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript. This paper is derived from the first author's master's thesis, supervised by the second author.

Conflicts of Interest

The authors declare no conflict of interest.

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