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Metallic Riemannian Structures on the Tangent Bundles of Riemannian Manifolds with $g-$Natural Metrics

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

Abstract

Let (M, g) be a Riemannian manifold and $(TM, \tilde{g})$ be its tangent bundle with the $g-$natural metric. In this paper, a family of metallic Riemannian structures J is constructed on TM, found conditions under which these structures are integrable. It is proved that $(TM, \tilde{g}, J)$ is decomposable if and only if (M, g) is flat.

Keywords: *g-natural metric, metallic Riemannian structure, tangent bundle.*
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1. Introduction

Let (M, g) be a Riemannian manifold and TM be its tangent bundle. In [4], Abbassi and Sarih defined $g-$natural metrics on TM as metrics which arise from g through first order natural operators defined between the natural bundle of Riemannian metrics on M and the natural bundle of $(0, 2)-$tensor fields on TM. Some well-known examples of $g-$natural metrics are the Sasaki metric ([8],[20]), Sasaki type metrics ([9]), the Cheeger-Gromoll metric ([19], [21]), Cheeger-Gromoll type metrics ([7],[10]) and the Kaluza-Klein metric ([6]). Abbassi *et al.* have been studied geometric properties of tangent bundles with respect to $g-$natural metrics (see [1],[2],[3], for instance).

On the other hand, consider the general quadratic equation $x^2 - ax - b = 0$, where a and b are positive integers. The set of positive solutions of this equation $\sigma_{a,b} = \frac{a+\sqrt{a^2+4b}}{2}$ are referred to as the Metallic Means Family. These numbers were introduced by Spinadel in [22] and can be seen as generalizations of the golden number $\phi = \frac{1+\sqrt{5}}{2} = 1.618...$ Inspiring these numbers, Hreţcanu and Crasmareanu introduced metallic structures on Riemannian manifolds in [12]. Investigating metallic structures and their subclasses (such as golden, silver, bronze etc. structures) on Riemannian manifolds is an actual subject in differential geometry (see for example [5],[11],[13, 17]).

In this paper, we introduce a family of metallic structures J on tangent bundles TM with $g-$natural metrics $\tilde{g}$. We study integrability of these structures and prove that locally flatness of the base manifold M is necessary and sufficient for the locally decomposability of the tangent bundle $(TM, \tilde{g}, J)$.

2. Preliminaries

2.1. Tangent bundle

Let M be an $n-$dimensional Riemannian manifold and ∇ be the Levi-Civita connection of g. The tangent bundle TM of the manifold M is a $2n-$dimensional smooth manifold and it is defined by disjoint tangent spaces at distinct points on M. If $\{U, x^i\}$ is a local coordinate system in M, then $\{\pi^{-1}(U), x^i, y^i\}_{i=1,...,n}$ is a local

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coordinate system in TM, where π is the natural projection defined by $\pi : TM \to M$. We have a direct sum decomposition

$$TTM = VTM \oplus HTM$$

for the tangent bundle TM, where $VTM = Ker\pi_*$ is the vertical subspace and HTM is the horizontal subspace defined by ∇. Given a vector field X on M, the horizontal lift $X^h \in HTM$ of X is defined by $\pi_* X^h = X$ and the vertical lift $X^v \in VTM$ of X is defined by $X^v(df) = Xf$, for every smooth functions f on M. Notice that 1-forms df on M are supposed to be functions on TM. Furthermore, the vector field $y^h = y^i(\frac{\partial}{\partial x^i})^h$ yields the geodesic spray on TM. Any tangent vector $Z \in TM$ can be expressed as $Z = X^h + Y^v$, where X and Y are uniquely written vector fields on M.

From [4], it is known that the $g-$natural metric $\tilde{g}$ on the tangent bundle TM of the Riemannian manifold (M, g) is completely determined as follows:

$$\begin{cases} \tilde{g}(X^h, Y^h) = (\alpha_1 + \alpha_3)(w^2)g(X,Y) + (\beta_1 + \beta_3)(w^2)g(X,y)g(Y,y), \\ \tilde{g}(X^h, Y^v) = \tilde{g}(X^v, Y^h) = \alpha_2(w^2)g(X,Y) + \beta_2(w^2)g(X,y)g(Y,y), \\ \tilde{g}(X^v, Y^v) = \alpha_1(w^2)g(X,Y) + \beta_1(w^2)g(X,y)g(Y,y), \end{cases} \tag{2.1}$$

where $w^2 = g(y,y), \;\; \alpha_i, \beta_i : R^+ \to R, \;\; i = 1,2,3$ are six smooth functions and y, X, Y are vector fields on M. Remark that the $g-$natural metric $\tilde{g}$ is Riemannian if and only if

$$\alpha_1(t) > 0, \; \varphi_1(t) > 0, \; \alpha(t) > 0, \; \varphi(t) > 0,$$

for all $t \in R^+$, where

$$\varphi_i(t) = \alpha_i(t) + t\beta_i(t), \; \alpha(t) = \alpha_1(t)(\alpha_1(t) + \alpha_3(t)) - \alpha_2^2(t), \; \varphi(t) = \varphi_1(t)(\varphi_1(t) + \varphi_3(t)) - \varphi_2^2(t).$$

Lemma 2.1. *[8] Let (M, g) be a Riemannian manifold on TM be its tangent bundle. The Lie bracket of vertical and horizontal vector fields on TM is given by*

$$\begin{aligned} [X^h, Y^h] &= [X,Y]^h - (R(X,Y)u)^v, \\ [X^h, Y^v] &= (\nabla_X Y)^v, \\ [X, Y] &= 0, \end{aligned}$$

where X, Y are vector fields on M, ∇ is the Levi-Civita connection of g and R is the Riemannian curvature of ∇.

2.2. Metallic Riemannian structures on tangent bundles

Definition 2.1. [12] Let (M, g) be an $n-$dimensional Riemannian manifold. A metallic structure on M is a $(1,1)-$tensor field J which satisfy the following relations

$$J^2 = aJ + bI, \tag{2.2}$$

$$g(JX, JY) = ag(JX, Y) + bg(X, Y), \tag{2.3}$$

where a, b are positive integers and X, Y are vector fields on M.

The Riemannian metric satisfying (2.3) is referred to as $J-$compatible and the triple (M, g, J) is said to be a metallic Riemannian manifold. When the Nijenhuis tensor N_J of J is zero, it is said that the metallic Riemannian structure is integrable. A metallic Riemannian manifold (M, g, J) with an integrable metallic structure J is called locally decomposable metallic Riemannian manifold. The following proposition characterizes the locally decomposibility of metallic Riemannian manifolds.

Proposition 2.1. *[11] Let (M, g, J) be a metallic Riemannian manifold. Then (M, g, J) is locally decomposable if and only if $\Phi_J g = 0$, where Φ_J is the Tachibana operator defined by*

$$\Phi_J g(X, Y, Z) = (JX)(g(Y, Z)) - X(g(JY, Z)) + g((L_Y J)X, Z) + g(Y, (L_Z J)X).$$

Here, $(L_X J)Y = [X, JY] - J[X, Y]$.

3. Metallic Riemannian structures on tangent bundles with g-natural metrics

In this section we construct a metallic structure on the tangent bundle TM which is equipped with $g-$natural metrics as $J-$compatible metrics.

Theorem 3.1. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle with a $g-$natural metric $\tilde{g}$ described by (2.1). The tensor field J defined by*

$$\begin{aligned} J(X^h) &= pX^h + qX^h, \\ J(X^v) &= rX^h + sX^v, \end{aligned} \tag{3.1}$$

is a metallic Riemannian structure if and only if

$$\begin{cases} q = -\frac{p^2-ap-b}{r}, \ s = a-p, \ \alpha_2(w^2) = \beta_2(w^2) = 0, \\ \alpha_3(w^2) = -\frac{\alpha_1(w^2)(p^2+r^2-ap-b)}{r^2}, \ \beta_3(w^2) = -\frac{\beta_1(w^2)(p^2+r^2-ap-b)}{r^2}, \end{cases} \tag{3.2}$$

where a, b are positive integers, p, q, r, s are non-zero constants and X is a vector field on M.

Proof. The metric $\tilde{g}$ described by (2.1) is J-compatible with the tensor J in (3.1) if and only if (2.2) and (2.3) are valid. Putting (3.1) and (2.1) into (2.2) and (2.3) gives us

$$\begin{cases} p^2 + qr - ap - b = 0, \ q(p+s-a) = 0, \ r(p+s-a) = 0, \ qr + s^2 - as - b = 0, \\ (\alpha_1+\alpha_3)(w^2)p^2 + (2q\alpha_2(w^2) - a\alpha_1(w^2) - a\alpha_3(w^2))p + q^2\alpha_1(w^2) - b\alpha_1(w^2) - b\alpha_3(w^2) = 0, \\ (\beta_1+\beta_3)(w^2)p^2 + (2q\beta_2(w^2) - a\beta_1(w^2) - a\beta_3(w^2))p + q^2\beta_1(w^2) - b\beta_1(w^2) - b\beta_3(w^2) = 0, \\ pr(\alpha_1+\alpha_3)(w^2) + (p(s-a) + qr - b)\alpha_2(w^2) + q(s-a)\alpha_1(w^2) = 0, \\ pr(\beta_1+\beta_3)(w^2) + (p(s-a) + qr - b)\beta_2(w^2) + q(s-a)\beta_1(w^2) = 0, \\ r(q\alpha_2(w^2) + (p-a)(\alpha_1+\alpha_3)(w^2)) + (ps - as - b)\alpha_2(w^2) + qs\alpha_1(w^2) = 0, \\ r(q\beta_2(w^2) + (p-a)(\beta_1+\beta_3)(w^2)) + (ps - as - b)\beta_2(w^2) + qs\beta_1(w^2) = 0, \\ (r^2 + s^2 - as - b)\alpha_1(w^2) + r^2\alpha_3(w^2) + (2s\alpha_2(w^2) - a\alpha_2(w^2))r = 0, \\ (r^2 + s^2 - as - b)\beta_1(w^2) + r^2\beta_3(w^2) + (2s\beta_2(w^2) - a\beta_2(w^2))r = 0. \end{cases} \tag{3.3}$$

Direct computations prove that system of equations (3.3) is satisfied if and only if (3.2) is valid. Thus, we prove the theorem. □

Particular cases of the $g-$natural metric in (2.1) give some well-known examples of Riemannian metrics on TM. More precisely, we obtain

(1) Sasaki metric g^s, if

$$\alpha_1(t) = 1, \ \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0, \tag{3.4}$$

(2) Cheeger-Gromoll metric g^{cg}, if

$$\alpha_2(t) = \beta_2(t) = 0, \ \alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}, \ \alpha_3(t) = \frac{t}{1+t},$$

(3) Cheeger-Gromoll type metrics g^{ml}, if

$$\alpha_2(t) = \beta_2(t) = 0, \ \alpha_1(t) = \frac{1}{(1+t)^m}, \ \ \alpha_3(t) = 1 - \alpha_1(t), \ \beta_1(t) = -\beta_3(t) = \frac{l}{(1+t)^m},$$

(4) Kaluza-Klein metric g^{kk}, if

$$\alpha_2(t) = \beta_2(t) = (\beta_1+\beta_3)(t) = 0.$$

Now, we can express the following theorems and examples for these metrics.

Theorem 3.2. *Let (TM, g^s) be the tangent bundle of a Riemannian manifold (M, g) with the Sasaki metric g^s. The tensor field J given by*

$$\begin{aligned} J(X^h) &= kX^h + \sqrt{-k^2+ak+b}X^v, \\ J(X^v) &= \sqrt{-k^2+ak+b}X^h + (a-k)X^v, \end{aligned} \tag{3.5}$$

for an arbitrary non-zero constant k satisfying $-k^2+ak+b > 0$ and an arbitrary vector field X on M is a metallic Riemannian structure on TM and (TM, J, g^s) is a metallic Riemannian manifold.

Proof. From Theorem 3.1 and (3.4), we occur that (3.3) is true if and only if

$$p = k,\ q = r = \sqrt{-k^2 + ak + b},\ s = a - k,$$

where k is a non-zero constant satisfying $-k^2 + ak + b > 0$. Thus, the theorem is proved. □

Example 3.1. Let (R^2, g^e) be the Euclidean 2-manifold and (u^1, u^2) be a local coordinate neighbourhood on R^2. In this case, the vectors $\{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\}$ yield a local frame field on R^2. The components of the metric g^e are

$$g_{ij} = \delta_{ij} = \begin{cases} 1,\ if\ i = j, \\ 0,\ if\ i \neq j,\ i, j = 1, 2. \end{cases}$$

Denote the tangent bundle of R^2 by TR^2 and choose a local frame field on TR^2 as $\{\bar{u}^1, \bar{u}^2, v^1, v^2\}$, where $\bar{u}^i = u^i \circ \pi,\ i = 1, 2$. The Sasaki metric g^s on TR^2 is defined by

$$\begin{cases} g^s(\frac{\partial}{\partial \bar{u}^i}, \frac{\partial}{\partial \bar{u}^j}) = g^e(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}), \\ g^s(\frac{\partial}{\partial \bar{u}^i}, \frac{\partial}{\partial v^j}) = 0, \\ g^s(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j}) = g^e(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}), \end{cases}$$

for $i, j = 1, 2$ (see [18]). From (3.5), we occur

$$\begin{aligned} J(\frac{\partial}{\partial \bar{u}^i}) &= k\frac{\partial}{\partial \bar{u}^i} + \sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i}, \\ J(\frac{\partial}{\partial v^i}) &= \sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + (a - k)\frac{\partial}{\partial v^i}, \end{aligned} \tag{3.6}$$

for an arbitrary non-zero constant k satisfying $-k^2 + ak + b > 0$. To prove the triple (TR^2, J, g^s) is a metallic Riemannian manifold, we should show that the relations (2.2) and (2.3) are fulfilled. Taking $i, j = 1, 2$ and using (3.6) we get

$$\begin{aligned} J^2(\frac{\partial}{\partial \bar{u}^i}) &= J(k\frac{\partial}{\partial \bar{u}^i} + \sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i}) = kJ(\frac{\partial}{\partial \bar{u}^i}) + \sqrt{-k^2 + ak + b}J(\frac{\partial}{\partial v^i}) \\ &= k(k\frac{\partial}{\partial \bar{u}^i} + \sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i}) \\ &\quad + \sqrt{-k^2 + ak + b}(\sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + (a - k)\frac{\partial}{\partial v^i}) \\ &= (ak + b)\frac{\partial}{\partial \bar{u}^i} + a\sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} aJ(\frac{\partial}{\partial \bar{u}^i}) + bI(\frac{\partial}{\partial \bar{u}^i}) &= ak\frac{\partial}{\partial \bar{u}^i} + a\sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i} + b\frac{\partial}{\partial \bar{u}^i} \\ &= (ak + b)\frac{\partial}{\partial \bar{u}^i} + a\sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i}. \end{aligned} \tag{3.8}$$

Equations (3.7) and (3.8) imply that $J^2(\frac{\partial}{\partial \bar{u}^i}) = aJ(\frac{\partial}{\partial \bar{u}^i}) + bI(\frac{\partial}{\partial \bar{u}^i})$. Similarly, taking $i, j = 1, 2$ and using (3.6) we obtain

$$\begin{aligned} J^2(\frac{\partial}{\partial v^i}) &= J(\sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + (a - k)\frac{\partial}{\partial v^i}) = \sqrt{-k^2 + ak + b}J(\frac{\partial}{\partial \bar{u}^i}) + (a - k)J(\frac{\partial}{\partial v^i}) \\ &= \sqrt{-k^2 + ak + b}(k\frac{\partial}{\partial \bar{u}^i} + \sqrt{-k^2 + ak + b}\frac{\partial}{\partial v^i}) \\ &\quad + (a - k)(\sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + (a - k)\frac{\partial}{\partial v^i}) \\ &= a\sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + (a^2 - ak + b)\frac{\partial}{\partial v^i}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} aJ(\frac{\partial}{\partial v^i}) + bI(\frac{\partial}{\partial v^i}) &= a\sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + a(a - k)\frac{\partial}{\partial v^i} + b\frac{\partial}{\partial \bar{u}^i} \\ &= a\sqrt{-k^2 + ak + b}\frac{\partial}{\partial \bar{u}^i} + (a^2 - ak + b)\frac{\partial}{\partial v^i}. \end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) imply that $J^2(\frac{\partial}{\partial v^i}) = aJ(\frac{\partial}{\partial v^i}) + bI(\frac{\partial}{\partial v^i})$. So, the condition (2.2) is fulfilled. Now, we examine the condition (2.3). We have

$$\begin{aligned} g^s(J(\frac{\partial}{\partial \bar{u}^1}), J(\frac{\partial}{\partial \bar{u}^1})) &= g^s((k\frac{\partial}{\partial \bar{u}^1} + \sqrt{-k^2+ak+b}\frac{\partial}{\partial v^1}), k\frac{\partial}{\partial \bar{u}^1} + \sqrt{-k^2+ak+b}\frac{\partial}{\partial v^1})) \\ &= k^2 g^s(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^1}) + (-k^2+ak+b) g^s(\frac{\partial}{\partial v^1}, \frac{\partial}{\partial v^1}) \\ &= k^2 g^e(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}) + (-k^2+ak+b) g^e(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}) = ak+b, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} ag^s(J(\frac{\partial}{\partial \bar{u}^1}), \frac{\partial}{\partial \bar{u}^1}) + bg^s(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^1}) &= ag^s(k\frac{\partial}{\partial \bar{u}^1} + \sqrt{-k^2+ak+b}\frac{\partial}{\partial v^1}, \frac{\partial}{\partial \bar{u}^1}) + bg^s(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^1}) \\ &= ak+b \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we have

$$g^s(J(\frac{\partial}{\partial \bar{u}^1}), J(\frac{\partial}{\partial \bar{u}^1})) = ag^s(J(\frac{\partial}{\partial \bar{u}^1}), \frac{\partial}{\partial \bar{u}^1}) + bg^s(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^1}) = ak+b. \tag{3.13}$$

By similar way, we obtain

$$g^s(J(\frac{\partial}{\partial \bar{u}^2}), J(\frac{\partial}{\partial \bar{u}^2})) = ag^s(J(\frac{\partial}{\partial \bar{u}^2}), \frac{\partial}{\partial \bar{u}^2}) + bg^s(\frac{\partial}{\partial \bar{u}^2}, \frac{\partial}{\partial \bar{u}^2}) = ak+b, \tag{3.14}$$

$$g^s(J(\frac{\partial}{\partial \bar{u}^1}), J(\frac{\partial}{\partial \bar{u}^2})) = ag^s(J(\frac{\partial}{\partial \bar{u}^1}), \frac{\partial}{\partial \bar{u}^2}) + bg^s(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^2}) = 0, \tag{3.15}$$

$$g^s(J(\frac{\partial}{\partial v^1}), J(\frac{\partial}{\partial v^1})) = ag^s(J(\frac{\partial}{\partial v^1}), \frac{\partial}{\partial v^1}) + bg^s(\frac{\partial}{\partial v^1}, \frac{\partial}{\partial v^1}) = a(a-k)+b, \tag{3.16}$$

$$g^s(J(\frac{\partial}{\partial v^2}), J(\frac{\partial}{\partial v^2})) = ag^s(J(\frac{\partial}{\partial v^2}), \frac{\partial}{\partial v^2}) + bg^s(\frac{\partial}{\partial v^2}, \frac{\partial}{\partial v^2}) = a(a-k)+b, \tag{3.17}$$

$$g^s(J(\frac{\partial}{\partial v^1}), J(\frac{\partial}{\partial v^2})) = ag^s(J(\frac{\partial}{\partial v^1}), \frac{\partial}{\partial v^2}) + bg^s(\frac{\partial}{\partial v^1}, \frac{\partial}{\partial v^2}) = 0. \tag{3.18}$$

Equations (3.13)- (3.18) show that the condition (2.3) is fulfilled. Therefore, (TR^2, J, g^s) is a metallic Riemannian manifold.

Theorem 3.3. *There does not exist any metallic Riemannian structure J of the form (3.1) on (TM, g^{cg}).*

Proof. It is clear that taking $\alpha_2(t) = \beta_2(t) = 0,\ \alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t},\ \alpha_3(t) = \frac{t}{1+t}$ in (3.3) does not yield a solution. This completes the proof. □

Theorem 3.4. *Let (TM, g^{ml}) be the tangent bundle of a Riemannian manifold (M, g) with a Cheeger-Gromoll type metric g^{ml}. The tensor field J given by*

$$\begin{aligned} J(X^h) &= (a-k_1)X^h + k_2 X^v, \\ J(X^v) &= -\frac{k_1^2 - ak_1 - b}{k_2} X^h + k_1 X^v, \end{aligned}$$

for arbitrary non-zero constants k_1, k_2 when $k_1^2 - ak_1 - b > 0$ and an arbitrary vector field X on M is a metallic Riemannian structure on TM and (TM, J, g^{ml}) is a metallic Riemannian manifold if and only if $l = 0$ and $m = \frac{\ln(\frac{k_2^2}{k_1^2 - ak_1 - b})}{\ln(1+(g(y,y))^2)} = m_\mu$.

Proof. Taking $\alpha_2(t) = \beta_2(t) = 0,\ \alpha_1(t) = \frac{1}{(1+t)^m},\ \ \alpha_3(t) = 1 - \alpha_1(t),\ \beta_1(t) = -\beta_3(t) = \frac{l}{(1+t)^m}$ in (3.3) yields one solution as

$$s = k_1,\ q = k_2,\ p = a - k_1,\ r = -\frac{k_1^2 - ak_1 - b}{k_2},\ l = 0,\ m = m_\mu = \frac{\ln(\frac{k_2^2}{k_1^2 - ak_1 - b})}{\ln(1+(g(y,y))^2)}$$

for arbitrary non-zero constants k_1, k_2 when $k_1^2 - ak_1 - b > 0$. Thus the theorem is proved. □

Theorem 3.5. *Let (TM, g^{kk}) be the tangent bundle of a Riemannian manifold (M, g) with the Kaluza-Klein metric g^{kk}. The tensor field J given by*

$$\begin{aligned} J(X^h) &= k_1X^h + k_2X^v, \\ J(X^v) &= -\frac{k_1^2 - ak_1 - b}{k_2}X^h + (a-k_1)X^v, \end{aligned} \tag{3.19}$$

for arbitrary non-zero constants k_1, k_2 and an arbitrary vector field X on M is a metallic Riemannian structure on TM and (TM, J, g^{ml}) is a metallic Riemannian manifold if and only if $\alpha_3(w^2) = -\frac{\alpha_1(w^2)(k_1^2+k_2^2-ak_1-b)}{k_1^2-ak_1-b}$ and $\beta_1(w^2) = 0$.

Proof. The proof is similar to the proof of the previous theorem. □

Example 3.2. Let (R^2, g^e) be the Euclidean 2-manifold and TR^2 be its tangent bundle as in Example 3.1. The Kaluza-Klein metric g^{kk} associated with (R^2, g_e) is given by

$$\begin{cases} g^{kk}(\frac{\partial}{\partial \bar{u}^i}, \frac{\partial}{\partial \bar{u}^j}) = (\alpha_1 + \alpha_3)(1)g^e(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}), \\ g^{kk}(\frac{\partial}{\partial \bar{u}^i}, \frac{\partial}{\partial v^j}) = 0, \\ g^{kk}(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j}) = \alpha_1(1)g^e(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}),\ i,j = 1,2, \end{cases}$$

where $\alpha_1, \alpha_3 : R^+ \to R$ smooth functions and $\alpha_3(1) = -\frac{\alpha_1(1)(k_1^2+k_2^2-ak_1-b)}{k_1^2-ak_1-b}$. From (3.19), the tensor field J is defined by

$$\begin{aligned} J(\frac{\partial}{\partial \bar{u}^i}) &= k_1\frac{\partial}{\partial \bar{u}^i} + k_2\frac{\partial}{\partial v^j}, \\ J(\frac{\partial}{\partial v^j}) &= -\frac{k_1^2 - ak_1 - b}{k_2}\frac{\partial}{\partial \bar{u}^i} + (a-k_1)\frac{\partial}{\partial v^j}, \end{aligned}$$

for arbitrary non-zero constants k_1, k_2. We have

$$\begin{aligned} J^2(\frac{\partial}{\partial \bar{u}^i}) &= aJ(\frac{\partial}{\partial \bar{u}^i}) + bI(\frac{\partial}{\partial \bar{u}^i}) = (ak_1 + b)\frac{\partial}{\partial \bar{u}^i} + ak_2\frac{\partial}{\partial v^i}, \\ J^2(\frac{\partial}{\partial v^i}) &= aJ(\frac{\partial}{\partial v^{ii}}) + bI(\frac{\partial}{\partial v^i}) = -\frac{k_1^2 - ak_1 - b}{k_2}\frac{\partial}{\partial \bar{u}^i} + (a^2 - ak_1 + b)\frac{\partial}{\partial v^i},\ i = 1,2. \end{aligned}$$

So, the condition (2.2) is fulfilled. We also have

$$g^{kk}(J(\frac{\partial}{\partial \bar{u}^1}), J(\frac{\partial}{\partial \bar{u}^1})) = ag^{kk}(J(\frac{\partial}{\partial \bar{u}^1}), \frac{\partial}{\partial \bar{u}^1}) + bg^{kk}(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^1}) = \frac{-ak_1k_2^2 - bk_2^2}{k_1^2 - ak_1 - b}\alpha_1(1),$$

$$g^{kk}(J(\frac{\partial}{\partial \bar{u}^1}), J(\frac{\partial}{\partial \bar{u}^2})) = ag^{kk}(J(\frac{\partial}{\partial \bar{u}^1}), \frac{\partial}{\partial \bar{u}^2}) + bg^{kk}(\frac{\partial}{\partial \bar{u}^1}, \frac{\partial}{\partial \bar{u}^2}) = 0,$$

$$g^{kk}(J(\frac{\partial}{\partial v^1}), J(\frac{\partial}{\partial v^1})) = ag^{kk}(J(\frac{\partial}{\partial v^1}), \frac{\partial}{\partial v^1}) + bg^{kk}(\frac{\partial}{\partial v^1}, \frac{\partial}{\partial v^1}) = (a(a-k) + b)\alpha_1(1),$$

$$g^{kk}(J(\frac{\partial}{\partial v^1}), J(\frac{\partial}{\partial v^2})) = ag^{kk}(J(\frac{\partial}{\partial v^1}), \frac{\partial}{\partial v^2}) + bg^{kk}(\frac{\partial}{\partial v^1}, \frac{\partial}{\partial v^2}) = 0.$$

Thus, the the condition (2.3) is fulfilled. This shows that the triple (TR^2, J, g^{kk}) is a metallic Riemannian manifold.

4. Integrable metallic Riemannian structures on tangent bundles with g-natural metrics

In this last section, we study the integrability of the metallic structure J on tangent bundles with g-natural metrics. From Proposition 2.1, we know that a metallic structure J on the tangent bundle TM with a $g-$natural metric $\tilde{g}$ is integrable if and only if $\Phi_J\tilde{g} = 0$. In this case, $(TM, \tilde{g}, J)$ is called locally decomposable metallic Riemannian manifold. We express the proposition below.

Proposition 4.1. *Let (TM, g^s) be the tangent bundle of a Riemannian manifold (M, g) with the Sasaki metric g^s. The metric g^s is pure with respect to the metallic Riemannian structure J introduced in Theorem 3.2 as*

$$\begin{aligned} J(X^h) &= kX^h + \sqrt{-k^2 + ak + b}X^v, \\ J(X^v) &= \sqrt{-k^2 + ak + b}X^h + (a - k)X^v, \end{aligned}$$

for an arbitrary non-zero constant k satisfying $-k^2 + ak + b > 0$ and an arbitrary vector field X on M.

Proof. The purity condition is given by $g^s(J\widetilde{X}, \widetilde{Y}) - g^s(\widetilde{X}, J\widetilde{Y}) = 0$, for all vector fields $X^h,\ X^v,\ Y^h,\ Y^v$ on TM. We have

$$\begin{aligned} g^s(JX^h, Y^h) - g^s(X^h, JY^h) &= kg(X, Y) - kg(X, Y) = 0, \\ g^s(JX^h, Y^v) - g^s(X^h, JY^v) &= \sqrt{-k^2 + ak + b}g(X, Y) - \sqrt{-k^2 + ak + b}g(X, Y) = 0, \\ g^s(JX^v, Y^v) - g^s(X^v, JY^v) &= (a - k)g(X, Y) - (a - k)g(X, Y) = 0. \end{aligned}$$

So, the metric g^s is pure with respect to the metallic Riemannian structure J. □

In the following theorem, we examine the conditions under which (TM, g^s, J) is locally decomposable metallic Riemannian manifold.

Theorem 4.1. *Let (TM, g^s) be the tangent bundle of a Riemannian manifold (M, g) with the Sasaki metric g^s. Then (TM, g^s, J) is a locally decomposable metallic Riemannian manifold if and only if (M, g) is flat.*

Proof. Having in mind Proposition 2.1 and Proposition 4.1 and using the relations

$$X^h(g(Y, Z))^v = (Xg(Y, Z))^v,\ X^v(g(Y, Z))^v = 0,$$

for all vector fields on M, we have

$$\Phi_J g^s(\tilde{X}, \tilde{Y}, \tilde{Z}) = (J\widetilde{X})(g^s(\tilde{Y}, \widetilde{Z})) - \widetilde{X}(g^s(J\widetilde{Y}, \widetilde{Z})) + g^s((L_{\widetilde{Y}}J)\widetilde{X}, \widetilde{Z}) + g^s(\widetilde{Y}, (L_{\widetilde{Z}}J)\widetilde{X}),$$

for all vector fields on TM. It follows that

$$\begin{aligned} \Phi_J g^s(X^h, Y^h, Z^h) &= \sqrt{-k^2 + ak + b}g^s((R(Y, X)u - R(u, Y)X)^h, Z^h), \\ \Phi_J g^s(X^v, Y^v, Z^h) &= \sqrt{-k^2 + ak + b}g^s((R(u, Y)Z)^h, Z^h), \\ \Phi_J g^s(X^v, Y^h, Z^v) &= \sqrt{-k^2 + ak + b}g^s((R(X, Y)u)^v, Z^v), \\ \Phi_J g^s(X^h, Y^h, Z^v) &= \Phi_J g^s(X^h, Y^v, Z^v) = \Phi_J g^s(X^h, Y^v, Z^h) = 0, \\ \Phi_J g^s(X^v, Y^v, Z^v) &= \Phi_J g^s(X^v, Y^h, Z^h) = 0, \end{aligned}$$

where R is the Riemannian curvature of g. So, it is clear that (TM, g^s, J) is a locally decomposable metallic Riemannian manifold if and only if (M, g) is flat. □

Proposition 4.2. *Let (TM, g^{ml}) be the tangent bundle of a Riemannian manifold (M, g) with a Cheeger-Gromoll type metric g^{ml} with $m = m_\mu$ and $l = 0$. The metric g^{ml} is pure with respect to the metallic Riemannian structure J introduced in Theorem 3.4 as*

$$\begin{aligned} J(X^h) &= (a - k_1)X^h + k_2X^v, \\ J(X^v) &= -\frac{k_1^2 - ak_1 - b}{k_2}X^h + k_1X^v, \end{aligned}$$

for arbitrary non-zero constants k_1, k_2 and an arbitrary vector field X on M.

Proof. Following the same way in the proof of Proposition 4.1, one can easily show the purity of the metric g^{ml} with respect to the metallic Riemannian structure J. We omit here. □

Theorem 4.2. *Let (TM, g^{ml}) be the tangent bundle of a Riemannian manifold (M, g) with the Cheeger-Gromoll type metric g^{ml}. Then (TM, g^{ml}, J) is a locally decomposable metallic Riemannian manifold if and only if (M, g) is flat.*

Proof. For $l=0$ and $m=m_\mu$, the Cheeger-Gromoll type metric g_{ml} is given by

$$\begin{cases} g^{ml}(X^h,Y^h)=g(X,Y), \\ g^{ml}(X^h,Y^v)=g^{ml}(X^v,Y^h)=0, \\ g^{ml}(X^v,Y^v)=\alpha_1(w^2)g(X,Y), \end{cases}$$

where $w^2=g(y,y)$ and X,Y are vector fields on M. Taking into account Proposition 2.1 and Proposition 4.2 and using the relations

$$X^h(g(Y,Z))^v=(Xg(Y,Z))^v,\ X^v(g(Y,Z))^v=0,$$

for all vector fields on M, we have

$$\Phi_J g^{ml}(\tilde{X},\tilde{Y},\tilde{Z})=(J\widetilde{X})(g^{ml}(\tilde{Y},\widetilde{Z}))-\widetilde{X}(g^{ml}(J\widetilde{Y},\widetilde{Z}))+g^{ml}((L_{\widetilde{Y}}J)\widetilde{X},\widetilde{Z})+g^{ml}(\widetilde{Y},(L_{\widetilde{Z}}J)\widetilde{X}),$$

for all vector fields on TM. By direct computations, we have

$$\begin{aligned} \Phi_J g^{ml}(X^h,Y^h,Z^h) &= k_2 g^{ml}((R(Y,X)u-R(u,Y)X)^h,Z^h), \\ \Phi_J g^{ml}(X^v,Y^v,Z^h) &= -\frac{k_1^2-ak_1-b}{k_2}g^{ml}((R(u,Y)Z)^h,Z^h), \\ \Phi_J g^{ml}(X^v,Y^h,Z^v) &= -\frac{k_1^2-ak_1-b}{k_2}g^{ml}((R(X,Y)u)^v,Z^v), \\ \Phi_J g^{ml}(X^h,Y^h,Z^v) &= \Phi_J g^{ml}(X^h,Y^v,Z^v)=\Phi_J g^{ml}(X^h,Y^v,Z^h)=0, \\ \Phi_J g^{ml}(X^v,Y^v,Z^v) &= \Phi_J g^{ml}(X^v,Y^h,Z^h)=0, \end{aligned}$$

where R is the Riemannian curvature of g. Hence, it is obvious that (TM,g^{ml},J) is a locally decomposable metallic Riemannian manifold if and only if (M,g) is flat. □

Theorem 4.3. *Let (TM,g^{kk}) be the tangent bundle of a Riemannian manifold (M,g) with the Kaluza-Klein metric g^{kk}. The metric g^{kk} is pure with respect to the metallic Riemannian structure J introduced in Theorem 3.5 as*

$$\begin{aligned} J(X^h) &= k_1X^h+k_2X^v, \\ J(X^v) &= -\frac{k_1^2-ak_1-b}{k_2}X^h+(a-k_1)X^v, \end{aligned}$$

for arbitrary non-zero constants k_1,k_2 and an arbitrary vector field X on M.

Proof. Direct calculations show that $g^{kk}(J\widetilde{X},\widetilde{Y})-g^{kk}(\widetilde{X},J\widetilde{Y})=0$ for $\tilde{X}=X^h,Y^v$ and $\tilde{Y}=Y^h,Y^v$, where X,Y are vector fields on M. So, the metric g^{kk} is pure with respect to the metallic Riemannian structure J defined by (3.19). □

Following the same method in Theorem 4.1 or 4.2, one can easily prove the final theorem of the paper below.

Theorem 4.4. *Let (TM,g^{kk}) be the tangent bundle of a Riemannian manifold (M,g) with the Kaluza-Klein metric g^{kk}. Then (TM,g^{kk},J) is a locally decomposable metallic Riemannian manifold if and only if (M,g) is flat.*

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