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The Characterizations of the Curve Generated by a Curve with Constant Torsion

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Abstract

In this article, a curve generated by the curve which has a constant torsion is examined in 3-dimensional Euclidean space. And, the characterizations of this curve have been made and some important theorems have been given. It is seen that the $\overline{\alpha}$ is a curve with constant curvature and the relationships between the two curves α and $\overline{\alpha}$ are revealed. In addition, the conditions for this obtained curve to be helix, slant helix, Bertrand and Salkowski curves are given.

Keywords: *Constant curvature; Bertrand curve; Darboux vector; Salkowski curve; slant helix.*
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1. Introduction

While characterizing a regular curve in Euclidean 3−space, curvatures of a curve are utilized. The curvature functions κ and τ of a curve provide information about the shape of a curve. It is well known that if a curve has a constant curvature $\kappa \neq 0$ and a torsion $\tau = 0$, then the curve is a circle with the radius $\frac{1}{\kappa}$. If $\kappa \neq 0$ (constant) and $\tau \neq 0$ (constant), then the curve is a helix [4]. A curve with constant slope is well-known in the differential geometry and is defined by the property that the tangent vector field has a constant angle with a fixed straight line. Izumiya and Takeuchi [8] have introduced the concept of slant helix in Euclidean 3−space with the property that the normal lines makes a constant angle with a fixed direction. A family of curves with constant curvature but non-constant torsion was defined by Salkowski [11]. Curves with constant curvature but non constant torsion are called Salkowski curves [11, 10]. Moreover, curves with constant torsion but non constant curvature are called anti-Salkowski curves. Monterde[10] characterized them as space curves with constant curvature and whose normal vector makes a constant angle with a fixed line. Senturk and Eren studied on Smarandache curves of space-like anti-Salkowski curve ans Salkowski curve in Minkowski 3-space [12, 13, 14]. Constant torsion curves and their main properties have been known since the 1890s. Some studies on these curves are mentioned below. Griffiths formula is applied to find curves with constant torsion under certain conditions. T. Ivey[6] showed that these critical curves are flexible bars of constant torsion without certain boundary conditions. J. Weiner[15] showed that there are closed regular curves with constant τ torsion with $\kappa > 0$ and arbitrarily small $\int_\alpha \tau ds$ total torsion. In addition, Bates and Melko[2] gave an open structure of a closed curve with constant torsion and positive curvature everywhere. Kazaras and Sterlingan[9] gave the explicit formula of curves with constant τ torsion on the sphere S^2 with the help of hypergeometric functions. Some other important elements for recognizing curves are the Frenet vectors of the curve. Curves determined by Frenet vectors have an important place in the literature. Bertrand curves, which were first defined by Bertrand(1850), are an example of this. A Bertrand curve is a curve in Euclidean 3-space whose principal normal is the principal normal of another curve [3]. In this study, a curve with constant curvature is obtained from a curve with unit speed and constant torsion in Euclidean 3−space. The Frenet vectors and curvature functions of that generated curve are calculated. The condition for the generated curve to be a helix, slant helix is determined. It was also obtained that this curve is the Salkowski curve and the Bertrand curve.

2. Preliminaries

Let $\mathbf{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\langle x,y\rangle = \sum_{i=1}^{3} x_i y_i$ where $x,y \in \mathbf{R}^3$. The norm of x is given by $\|x\| = \sqrt{\langle x,x\rangle}$. Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a curve such that I is an open interval. If $\|\alpha'(s)\| = 1$, the α is called the unit speed curve and s is

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called the arclenght parameter of the curve α [4]. If we take the arc-length parameter s, that is,$\|\alpha\| = 1$ for all s, then the tangent vector, the principal normal vector and the bi-normal vector are given by

$$T(s) = \alpha'(s), \quad N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad B(s) = N(s) \wedge T(s)$$

where $\alpha'(s) = \frac{d\alpha}{ds}$. Thus Frenet formulas are given by

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where

$$\kappa(s) = \|T'(s)\|, \quad \tau(s) = \langle N'(s), B(s) \rangle$$

and $\sigma = \frac{ds}{dt}$ [1]. If we take a general parameter t, then the tangent vector, the principal normal vector and the bi-normal vector are given by

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad N(t) = B(t) \wedge T(t), \quad B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|}. \tag{2.1}$$

Then $\{T(t), N(t), B(t)\}$ is a moving frame of $\alpha(t)$ and we have the Frenet-Serret formula:

$$\begin{bmatrix} \dot{T}(t) \\ \dot{N}(t) \\ \dot{B}(t) \end{bmatrix} = \begin{bmatrix} 0 & |\sigma|\kappa & 0 \\ -|\sigma|\kappa(t) & 0 & |\sigma|\tau(t) \\ 0 & -|\sigma|\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}$$

where

$$\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|^3}, \quad \tau(t) = \frac{\langle \alpha'(t) \wedge \alpha''(t), \alpha'''(t) \rangle}{\|\alpha'(t) \wedge \alpha''(t)\|^2}. \tag{2.2}$$

Definition 2.1. *A curve α with non-zero curvature is said to be a Bertrand curve if there exists another curve β and a one-to-one correspondence between α and β such that both curves have common principal normal geodesics at corresponding points. We will say that β is a Bertrand mate of α; the curves α and β are called a pair of Bertrand curves [4].*

Definition 2.2. *If the normal vector of any curve in space makes a constant angle in a certain direction, that is, the curves with a constant first curvature are called Salkowski curves. If the first curvature of the curve is not constant and the second curvature is constant, these curves are also called the anti-Salkowski curves [10].*

The parametric equation of the anti-Salkowski curve $\alpha_m(t)$ in [10] is given by

$$\begin{aligned} \alpha_m(t) \quad = \quad & (\frac{n}{2m(4n^2-1)}(n((1-4n^2)+3\cos(2nt))\cos(t)+(2n^2+1)\sin(t)\sin(2nt)), \\ & \frac{n}{2m(4n^2-1)}(n((1-4n^2)+3\cos(2nt))\sin(t)-(2n^2+1)\cos(t)\sin(2nt)), \\ & \frac{n^2-1}{4n}(2nt+\sin(2nt))) \end{aligned} \tag{2.3}$$

where $n = \frac{m}{\sqrt{m^2+1}}$, $m \in R$ with $m \neq \pm\frac{1}{\sqrt{3}}$. Also, the Frenet vectors of anti-Salkowski curve $\alpha_m(t)$ are expressed by

$$\begin{aligned} T_\alpha &= (n\cos(nt)\sin(t)-\sin(nt)\cos(t), -n\cos(nt)\cos(t)-\sin(nt)\sin(t), \frac{-n}{m}\cos(nt)), \qquad (2.4) \\ N_\alpha &= (\frac{-n}{m}\sin(t), \frac{n}{m}\cos(t), -n), \\ B_\alpha &= (\cos(nt)\cos(t)+n\sin(nt)\sin(t), \cos(nt)\sin(t)-n\sin(nt)\cos(t), \frac{-n}{m}\sin(nt)) \end{aligned}$$

and curvatures of anti-Salkowski curve $\alpha_m(t)$ are given by

$$\begin{aligned} \kappa_\alpha &= -\tan(nt), \\ \tau_\alpha &= -1. \end{aligned} \tag{2.5}$$

Definition 2.3. *A unit speed curve $\alpha : I \longrightarrow E^n$ is called a slant helix if its unit principal normal vector makes a constant angle with a fixed direction d [7].*

Theorem 2.4. *Let α be a unit speed curve with the curvature $\kappa(s) \neq 0$. Then α is a slant helix if and only if*

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \right)(s)$$

is a constant function [7].

3. The curves generated by a curve with constant torsion

Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with an arc length parameter s in 3-dimensional Euclidean space such that I is an open interval in $\mathbb{R}$. The Frenet vectors of the curve α are $\{T_\alpha, N_\alpha, B_\alpha\}$, its curvature is κ_α, and its torsion τ_α is constant. Let the curve $\overline{\alpha}$ which is generated by the curve α be defined as follows:

$$\overline{\alpha}(t) = \frac{1}{\tau_\alpha} N_\alpha(s) - \int_0^s B_\alpha(u)du. \tag{3.1}$$

Hereinafter, unless otherwise stated, the curve α is a unit-speed curve with an arc length parameter s and the constant torsion τ_α. The curve $\overline{\alpha}$ is a curve expressed by equation (3.1), which is generated by the curve α in 3-dimensional Euclidean space.

Theorem 3.1. *Let the Frenet vectors of the curve α be denoted by $\{T_\alpha, N_\alpha, B_\alpha\}$; its curvature is $\kappa_\alpha > 0$, and its torsion $\tau_\alpha \neq 0$ is constant. Let the curve $\overline{\alpha}$ be defined by (3.1). There are the following relations between the Frenet vectors of the two curves $\overline{\alpha}$ and α:*

$$\begin{aligned} T_{\overline{\alpha}} &= -T_\alpha, \\ N_{\overline{\alpha}} &= -N_\alpha, \\ B_{\overline{\alpha}} &= B_\alpha. \end{aligned} \tag{3.2}$$

Proof. Assume that the curve α is a unit speed curve with an arc length parameter s and a constant torsion $\tau_\alpha = c$ in 3-dimensional Euclidean space. Let $\overline{\alpha}$ be a curve which is given by equation (3.1) such that it is generated by the curve α. If the curve $\overline{\alpha}$ is differentiated with respect to the parameter s such that $\tau_\alpha \neq 0$, then

$$\overline{\alpha}'(t(s)) = \frac{d\overline{\alpha}}{dt}\frac{dt}{ds} = \frac{1}{\tau_\alpha}(-\kappa_\alpha(s)T_\alpha(s) + \tau_\alpha B_\alpha(s)) - B_\alpha(s) \tag{3.3}$$

is obtained. Equation (3.3) is arranged

$$\overline{\alpha}'(t(s)) = -\sigma\frac{\kappa_\alpha(s)}{\tau_\alpha}T_\alpha(s) \tag{3.4}$$

where $\sigma = \frac{ds}{dt}$. Also, the norm of the curve $\overline{\alpha}$ is

$$\|\overline{\alpha}'(t)\| = \sigma\frac{\kappa_\alpha}{\tau_\alpha}. \tag{3.5}$$

When equations (3.4) and (3.5) are substituted into equation (2.1),

$$T_{\overline{\alpha}} = -T_\alpha$$

is obtained. If we take the derivative of equation (3.3) with respect to the parameter s, then

$$\overline{\alpha}''(t(s)) = \frac{d^2\overline{\alpha}'}{dt^2}\frac{dt}{ds} = -(\sigma\frac{\kappa_\alpha(s)}{\tau_\alpha})'T_\alpha(s) - \sigma\frac{\kappa_\alpha(s)}{\tau_\alpha}T'_\alpha(s) \tag{3.6}$$

where $\sigma = \frac{ds}{dt}$. And equation (3.6) is arranged

$$\overline{\alpha}''(t(s)) = -\frac{\sigma^2}{\tau_\alpha}\left(\kappa'_\alpha(s)T_\alpha(s) + \kappa^2(s)N_\alpha(s)\right)$$

is obtained. A vector product of $\overline{\alpha}'$ and $\overline{\alpha}''$ is

$$\overline{\alpha}' \wedge \overline{\alpha}'' = \sigma^3\frac{\kappa_\alpha^3(s)}{\tau_\alpha^2}B_\alpha(s). \tag{3.7}$$

The norm of the vector $\overline{\alpha}' \wedge \overline{\alpha}''$ is

$$\|\overline{\alpha}' \wedge \overline{\alpha}''\| = \sigma^3\frac{\kappa_\alpha^3(s)}{\tau_\alpha^2}. \tag{3.8}$$

Hence, the tangent vector $T_{\overline{\alpha}}$ of the curve $\overline{\alpha}$ is equal to T_α. Equations (3.7) and (3.8) are substituted into equation (2.1). Therefore, the binormal vector of the curve $\overline{\alpha}$ is equal to the below equation

$$B_{\overline{\alpha}} = B_\alpha.$$

By the vector product of the binormal vector $B_{\overline{\alpha}}$ and the tangent vector $T_{\overline{\alpha}}$ of the curve $\overline{\alpha}$, the principal normal vector of the curve $\overline{\alpha}$

$$N_{\overline{\alpha}} = -N_\alpha$$

is attained. □

Theorem 3.2. *Let the Frenet vectors of the curve α be $\{T_\alpha, N_\alpha, B_\alpha\}$; its curvature is κ_α, and its torsion $\tau_\alpha \neq 0$ is constant. Let the curve $\overline{\alpha}$ be defined by (3.1). Then there are relations between the curvatures of the two curves $\overline{\alpha}$ and α are as follows:*

$$\kappa_{\overline{\alpha}} = \tau_\alpha = constant, \tag{3.9}$$

$$\tau_{\overline{\alpha}} = -\frac{\tau_\alpha^2}{\kappa_\alpha(s)}. \tag{3.10}$$

Proof. Assume that the curve α is a unit speed curve with an arc length parameter of s and a constant torsion $\tau_\alpha \neq 0$ in 3-dimensional Euclidean space. Let $\overline{\alpha}$ be a curve which is given by Equation (3.1) such that it is generated by the curve α. Equations (3.8) and (3.5) are substituted in equation (2.2). Therefore, the equation is

$$\kappa_{\overline{\alpha}} = \tau_\alpha$$

accured. If equation (3.6) is derivatized again from both sides with respect to the parameter s, and it is arranged, then,

$$\overline{\alpha}'''(t(s)) = -\frac{\sigma^3}{\tau_\alpha}\left(\left(\kappa_\alpha'' - \kappa_\alpha^3\right)T_\alpha + 3\kappa_\alpha\kappa_\alpha' N_\alpha + \kappa_\alpha^2\tau_\alpha B_\alpha\right)$$

is obtained. From the inner product of the two vectors $\overline{\alpha}' \wedge \overline{\alpha}''$ and $\overline{\alpha}'''$

$$< \overline{\alpha}' \wedge \overline{\alpha}'', \overline{\alpha}''' > = -\frac{\sigma^6\kappa_\alpha^5(s)}{\tau_\alpha^2} \tag{3.11}$$

is obtained. If equations (3.11), (3.8) are substituted in equation (2.2), then the equation

$$\tau_{\overline{\alpha}} = -\frac{\tau_\alpha^2}{\kappa_\alpha(s)}$$

is reached. □

Theorem 3.3. *Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a constant torsion curve in 3-dimensional Euclidean space where I is an open interval and let $\overline{\alpha}$ be a curve with the equation (3.1) generated from the constant torsion curve α. Then there is a relation between the Darboux vectors of the two curves $\alpha, \overline{\alpha}$ as follows:*

$$W_{\overline{\alpha}} = \frac{\tau_\alpha}{\kappa_\alpha} W_\alpha$$

where κ_α, τ_α are curvatures of the curve α.

Proof. Assume that the curve α is given and the curve $\overline{\alpha}$ is a curve which is generated by the curve α. Let $W_\alpha, W_{\overline{\alpha}}$ be the Darboux vectors of the two curves $\alpha, \overline{\alpha}$ respectively. Then there are equations as below respectively

$$W_\alpha = \tau_\alpha T_\alpha + \kappa_\alpha B_\alpha$$

and

$$W_{\overline{\alpha}} = \tau_{\overline{\alpha}} T_{\overline{\alpha}} + \kappa_{\overline{\alpha}} B_{\overline{\alpha}}. \tag{3.12}$$

where $\kappa_\alpha, \tau_\alpha$ are curvatures of the curve α and $\kappa_{\overline{\alpha}}, \tau_{\overline{\alpha}}$ are curvatures of the curve $\overline{\alpha}$ respectively. If equations (3.2), (3.9) and (3.10) substituted in equation (3.12), then

$$W_{\overline{\alpha}} = -\frac{\tau_\alpha^2}{\kappa_\alpha}(-T_\alpha) + \tau_\alpha B_\alpha \tag{3.13}$$

is obtained. If we arrange equation (3.13), then the equation

$$W_{\overline{\alpha}} = \frac{\tau_\alpha}{\kappa_\alpha}(\tau_\alpha T_\alpha + \kappa_\alpha B_\alpha)$$

is reached. Hence, the above equation gives the relation.

□

Corollary 3.4. *The torsion $\tau_{\overline{\alpha}}$ of the curve $\overline{\alpha}$ is constant if and only if the curvature κ_α of the curve α is constant.*

Proof. It is trivial from (3.10). □

Theorem 3.5. *The curve $\overline{\alpha}$ is a helix if and only if the curve α is a helix.*

Proof. Assume that the curve $\overline{\alpha}$ is a helix. Then, the ratio of curvatures of the curve $\overline{\alpha}$ is as follows:

$$\frac{\kappa_{\overline{\alpha}}}{\tau_{\overline{\alpha}}} = c$$

where c is constant. By Theorem (3.2), the equation

$$\frac{\kappa_{\overline{\alpha}}}{\tau_{\overline{\alpha}}} = -\frac{\kappa_{\alpha}}{\tau_{\alpha}}$$

is achieved. Since τ_{α} is constant so κ_{α} is. The curve α is a helix. Assume that α is a helix. It is trivial that $\overline{\alpha}$ is a helix. □

Theorem 3.6. *Let the curve α be denoted by $\kappa_{\alpha} > 0$ τ_{α} in 3-dimensional Euclidean space. Let the curve $\overline{\alpha}$ be a curve which is generated by α such that given by (3.1). Then the curve $\overline{\alpha}$ is a slant helix if and only if the curve α is a slant helix such that*

$$\kappa_{\alpha} = \sqrt{\frac{1}{(cs+c_1)^2} - \tau_{\alpha}^2}.$$

Proof. Suppose that the curve α is a unit speed curve which has a curvature $\kappa_{\alpha} > 0$ and constant torsion τ_{α} in 3-dimensional Euclidean space. Let the curve $\overline{\alpha}$ be a curve which is generated by α as specified by equation (3.1). Assume that the curve $\overline{\alpha}$ is a slant helix. Therefore, it holds by Theorem 2.4 that c is constant

$$\frac{\kappa_{\overline{\alpha}}^2}{(\kappa_{\overline{\alpha}}^2 + \tau_{\overline{\alpha}}^2)^{3/2}} \left(\frac{\tau_{\overline{\alpha}}}{\kappa_{\overline{\alpha}}}\right)' = c = constant \tag{3.14}$$

where $\kappa_{\overline{\alpha}}$, $\tau_{\overline{\alpha}}$ are the curvature and torsion of the curve $\overline{\alpha}$ respectively. If Theorem 3.2 is used in equation (3.14), the equation

$$\frac{\kappa_{\alpha}' \kappa_{\alpha}}{(\kappa_{\alpha}^2 + \tau_{\alpha}^2)^{\frac{3}{2}}} = c \tag{3.15}$$

is obtained. Then, equation (3.15) is rearranged and a differential equation

$$\kappa_{\alpha}' \kappa_{\alpha} = c(\kappa_{\alpha}^2 + \tau_{\alpha}^2)^{\frac{3}{2}}$$

is acquired. If it is solved, we get the ...
Now assume that the curve α is a unit speed slant helix that has a curvature $\kappa_{\alpha} > 0$ and a constant torsion τ_{α} in 3-dimensional Euclidean space such that

$$\langle N_{\alpha}, \overrightarrow{d} \rangle = constant$$

where $\overrightarrow{d}$ is a fixed vector. By equation (3.2),

$$\langle N_{\overline{\alpha}}, \overrightarrow{d} \rangle = constant$$

is obtained. Therefore, the curve $\overline{\alpha}$ is a slant helix. □

Corollary 3.7. $(\alpha, \overline{\alpha})$ *is a Bertrand pair.*

Corollary 3.8. *The curve $\overline{\alpha}$ which is generated by the curve α is a Salkowski curve.*

Proof. Since the curvature $\kappa_{\overline{\alpha}}$ of the curve $\overline{\alpha}$ is constant and the torsion $\tau_{\overline{\alpha}}$ of the curve $\overline{\alpha}$ is non constant, thus the curve $\overline{\alpha}$ is a Salkowski curve by the definition of the Salkowski curve in (2.2). □

Theorem 3.9. *Let α be a curve which has a constant torsion $\tau_{\alpha} = 1$ in 3-dimensional Euclidean space and let $\overline{\alpha}$ be a curve with the equation (3.1) generated by α which has the constant torsion. The sum of two curves $\overline{\alpha}$ and α is a Bertrand curve.*

Proof. Suppose that the curve α be a curve which has a constant torsion $\tau_{\alpha} = 1$ in 3-dimensional Euclidean space. Let the curve $\overline{\alpha}$ be a curve with equation (3.1) generated by α which has a constant torsion. Since $(\alpha, \overline{\alpha})$ is a Bertrand pair there is an equation

$$a\kappa_{\alpha} + b\tau_{\alpha} = 1 \tag{3.16}$$

where $a, b \in \mathbb{R}$. Suppose that γ is a curve which is the sum of the two curves α and $\overline{\alpha}$ such that

$$\gamma = a\alpha + b\overline{\alpha} \tag{3.17}$$

where $a, b \in \mathbb{R}$. If we derivate of equation (3.17) and arranged

$$\gamma' = (a - b\sigma\kappa_{\alpha}) T_{\alpha} \tag{3.18}$$

is obtained. If we derivate of equation (3.18) two times

$$\gamma'' = (a - b\sigma\kappa_{\alpha})' T_{\alpha} + (a - b\sigma\kappa_{\alpha}) \kappa_{\alpha} N_{\alpha}$$

and

$\gamma''' = \left[(a-b\sigma\kappa_\alpha)'' - (a-b\sigma\kappa_\alpha)\kappa_\alpha^2\right]T_\alpha + \left[(a-b\sigma\kappa_\alpha)'\kappa_\alpha + (a-b\sigma\kappa_\alpha)\kappa_\alpha'\right]N_\alpha + (a-b\sigma\kappa_\alpha)\kappa_\alpha B_\alpha$

is obtained. The vector product of γ' and γ'' is

$\gamma' \wedge \gamma'' = (a-b\sigma\kappa_\alpha)^2\kappa_\alpha.$

By equation(2.1), the Frenet vectors of the curve γ are denoted as below:

$$\begin{aligned} T_\gamma &= T_\alpha = -T_{\overline{\alpha}}, \\ N_\gamma &= N_\alpha = -N_{\overline{\alpha}}, \\ B_\gamma &= B_\alpha = B_{\overline{\alpha}}. \end{aligned}$$

The curvatures of the curve γ which are determined by equation (2.2)

$$\begin{aligned} \tau_\gamma &= \frac{1}{(a-b\sigma\kappa_\alpha)}, \\ \kappa_\gamma &= \frac{\kappa_\alpha}{(a-b\sigma\kappa_\alpha)}. \end{aligned}$$

are given as above. Therefore,

$$c\kappa_\gamma + d\tau_\gamma = c\left(\frac{\kappa_\alpha}{(a-b\sigma\kappa_\alpha)}\right) + d\left(\frac{1}{(a-b\sigma\kappa_\alpha)}\right) \tag{3.19}$$

is obtained. If equation (3.19) is substituted for equation (3.16), then

$c\kappa_\gamma + d\tau_\gamma = 1$

is obtained in the case where $c, d \in \mathbb{R}$. □

Corollary 3.10. *Let α be a unit speed curve which has a constant torsion, $\overline{\alpha}$ be a curve which is generated by α, and γ be a curve which is the sum of the two curves α and $\overline{\alpha}$ in 3-dimensional Euclidean space. Then $(\alpha,\overline{\alpha})$, (α,γ), and $(\overline{\alpha},\gamma)$ are Bertrand pairs.*

Example 3.11. *If the Anti-Salkowski curve $\alpha_m(t)$ given by (2.3) is replaced by the Frenet vectors of the curve $\alpha_m(t)$ are given in (2.4) and torsion of the curve $\alpha_m(t)$ is given in (2.5),then the curve $\overline{\alpha}_m(t)$ which is generated by the curve $\alpha_m(t)$, then the Salkowski curve*

$$\begin{aligned} \overline{\alpha}_m(t) = {} & \left(-\frac{n}{m}\sin(t), \frac{n}{m}\cos(t), n\right) - \int_0^t (-\cos(nt)\cos(t) - n\sin(nt)\sin(t), \\ & -\cos(nt)\sin(t) + n\sin(nt)\cos(t), \frac{n}{m}\sin(nt))dt \end{aligned}$$

is obtained. In Fig. 1, the graph of the Salkowski curve $\overline{\alpha}$ is given for $m = 1$ and in Fig. 2, the graph of the Anti-Salkowski curve α_m is given for $m = 1$.

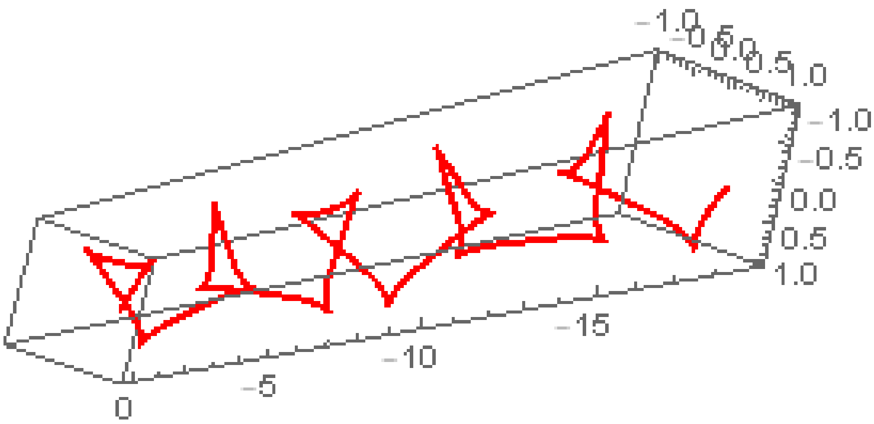

Figure 3.1: The Salkowski curve $\overline{\alpha}$ for $m = 1$

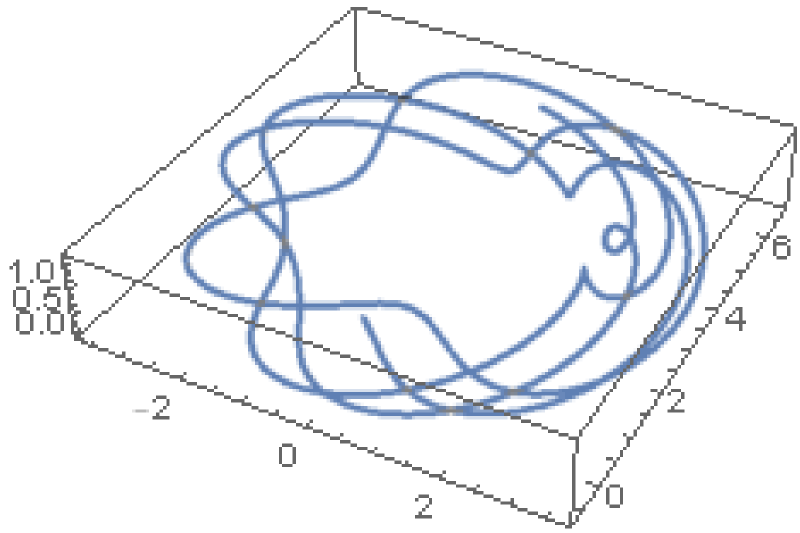

Figure 3.2: The Anti-Salkowski curve α for $m = 1$

4. Conclusion

The Frenet vectors, curvatures and Darboux vector of a curve $\overline{\alpha}$ obtained from a curve α with constant torsion are determined. It is described how the curve $\overline{\alpha}$ behaves, and its connections to other unique curves are made clear. It is discovered that the curve $\overline{\alpha}$ is a Salkowski and Bertrand curve. It is discovered the condition for the curve $\overline{\alpha}$ to be a slant helix. Additionally, it is demonstrated that the sum of the α and $\overline{\alpha}$ curves is a Bertrand curve.

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Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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