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Research Article

Certain Weighted Fractional Integral Inequalities for Convex Functions

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Abstract: In this study, by using the monotonicity properties of functions, several inequalities for convex functions are obtained with the help of a weighted fractional integral operator which provides a function f to be integrated in fractional order with respect to another function. It is also seen that the results obtained were generalizations of the previous results presented.

Keywords: Convex functions, weighted fractional operators, fractional integral inequality.

1. Introduction

Fractional calculus plays an important role in the field of inequality theory with its rich content and new fractional operators have been added day by day, especially in recent years. Some of these operators have certain algebraic properties such as semigroup property while some do not. Also, some of them have a singularity problem at some points while some of them do not. Therefore, the application areas of the operators can also differ. Convex analysis has become one of the important application areas of fractional analysis [1–3].

In addition, severel mathematicians have studied certain inequalities for convex functions using different type (for example; R-L fractional integral operator, tempered fractional integral operators, generalized proportional integral operators, generalized proportional Hadamard integral operators) of integral operators. These studies have helped to develop different aspects of operator analysis [9–12].

At first, we recall the elementary notation in convex analysis:

Definition 1.1 *A set $F \subset \mathbb{R}$ is said to be convex if*

$$\varphi a + (1-\varphi)b \in F$$

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for each $a, b \in F$ *and* $\varphi \in [0, 1]$.

Definition 1.2 *The mapping* $f_1 : F \to \mathbb{R}$, *is said to be convex if the following inequality holds:*

$$f_1(\varphi a + (1-\varphi)b) \leq \varphi f_1(a) + (1-\varphi) f_1(b)$$

for all $a, b \in F$ *and* $\varphi \in [0, 1]$. *We say that* f_1 *is concave if* $(-f_1)$ *is convex.*

The properties and definitions of the convex functions have recently ascribed a significant role to its theory and practice in the field of fractional integral operators.

In [7], Ngo et al. established the following inequalities:

$$\int_0^1 g_1^{\zeta+1}(\rho)d\rho \geq \int_0^1 \rho^{\zeta} g_1^{\zeta}(\rho)d\rho$$

and

$$\int_0^1 g_1^{\zeta+1}(\rho)d\rho \geq \int_0^1 \rho g_1^{\zeta}(\rho)d\rho,$$

where $\zeta > 0$ and the positive continuous function g_1 on $[0, 1]$ such that

$$\int_x^1 g_1(\rho)d\rho \geq \int_x^1 \rho d\rho, \quad x \in [0, 1].$$

Then, in [8], Liu et al. established the following inequalities:

$$\int_a^b g_1^{\zeta+\vartheta}(\rho)d\rho \geq \int_a^b (\rho - a)^{\zeta} g_1^{\vartheta}(\rho)d\rho,$$

where $\zeta > 0$, $\vartheta > 0$, and the positive continuous g_1 on $[a, b]$ is such that

$$\int_a^b g_1^{\xi}(\rho)d\rho \geq \int_0^1 (\rho - a)^{\xi} d\rho, \quad \xi = \min(1, \vartheta), \ \rho \in [0, 1].$$

The following two theorems are obtained by Liu in [1]:

Theorem 1.3 *Let* $\hbar_1$ *and* $\hbar_2$ *be continuous and positive functions with* $\hbar_1 \leq \hbar_2$ *on* $[a, b]$ *such that* $\hbar_1$ *is increasing and* $\frac{\hbar_1}{\hbar_2}$ *(*$\hbar_2 \neq 0$*) is decreasing. If* ϕ *is a convex function, then the inequality*

$$\frac{\int_a^b \hbar_1(t)dt}{\int_a^b \hbar_2(t)dt} \geq \frac{\int_a^b \phi\left(\hbar_1(t)\right)dt}{\int_a^b \phi\left(\hbar_2(t)\right)dt}$$

holds, where $\phi(0) = 0$.

Theorem 1.4 *Let $\hbar_1$, $\hbar_2$ and $\hbar_3$ be continuous and positive functions with $\hbar_1 \leq \hbar_2$ on $[a,b]$ such that $\hbar_1$ and $\hbar_3$ are increasing and $\frac{\hbar_1}{\hbar_2}$ ($\hbar_2 \neq 0$) is decreasing. If ϕ is a convex function, then the inequality*

$$\frac{\int_a^b \hbar_1(t)dt}{\int_a^b \hbar_2(t)dt} \geq \frac{\int_a^b \phi\left(\hbar_1(t)\right)\hbar_3(t)dt}{\int_a^b \phi\left(\hbar_2(t)\right)\hbar_3(t)dt}$$

holds, where $\phi(0) = 0$.

Now some fractional integral operators used to obtain integral inequalities will be given. First of them is Riemann-Liouville fractional integral operators (see [6]) which is widely used in fractional calculus.

Definition 1.5 *Let $\hbar \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}\hbar$ and $J_{b-}^{\alpha}\hbar$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}\hbar(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\hbar(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}\hbar(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\hbar(t)dt, \quad x < b$$

where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$, respectively. Here is $J_{a+}^{0}\hbar(x) = J_{b-}^{0}\hbar(x) = \hbar(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 1.6 *Let $(a,b) \subseteq \mathbb{R}$ and $\sigma(x)$ be an increasing positive and monotonic function on the interval $(a,b]$ with a continuous derivative $\sigma'(x)$ on the interval (a,b) with $\sigma(0) = 0,\ 0 \in [a,b]$. Then, the left-side and right-side of the weighted fractional integrals of a function $\hbar$ with respect to another function $\sigma(x)$ on $[a,b]$ are defined by [3]*

$$\begin{aligned} \left({}_{a+}\Im_w^{\ell:\sigma}\hbar\right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar(t)w(t)dt, \qquad (1) \\ \left({}_{w}\Im_{b-}^{\ell:\sigma}\hbar\right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)}\int_x^b \sigma'(t)\left[\sigma(t)-\sigma(x)\right]^{\ell-1}\hbar(t)w(t)dt, \quad \ell > 0 \end{aligned}$$

where $w^{-1}(x) = \frac{1}{w(x)}$, $w(x) \neq 0$ ($w(x) > 0$).

Remark 1.7 *In Definition 1.6,*

- *To obtain Riemann-Liouville fractional integral operator, one can choose $w(x) = 1$ and $\sigma(x) = x$ in definition of the weighted fractional integral operators (1).*

- *To obtain the following version of fractional integral operator which is defined in [4, 5], one can choose* $w(x) = 1$ *in (1):*

$$\begin{aligned} \left({}_{a+}\Im^{\ell:\sigma}\hbar\right)(x) &= \frac{1}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar(t)dt, \\ \left(\Im_{b-}^{\ell:\sigma}\hbar\right)(x) &= \frac{1}{\Gamma(\ell)}\int_x^b \sigma'(t)\left[\sigma(t)-\sigma(x)\right]^{\ell-1}\hbar(t)dt, \quad \ell>0. \end{aligned}$$

2. Main Results

In this section, inequalities for convex functions by utilizing weighted fractional operators presented.

Theorem 2.1 *Let* $\hbar_1$ *and* $\hbar_2$ *be two positive continuous functions on the interval* $[a,b]$ *and* $\hbar_1 \leq \hbar_2$ *on* $[a,b]$*. If* $\frac{\hbar_1}{\hbar_2}$ *is decreasing and* $\hbar_1$ *is increasing on* $[a,b]$*, then for a convex function* ϕ *with* $\phi(0)=0$*, the weighted fractional operator given by (1) satisfies the following inequality*

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)} \geq \frac{\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_2\right)(x)}, \tag{2}$$

where $x>a>0,\ \ell\in\mathbb{C},\ Re(\ell)>0.$

Proof $\frac{\phi(x)}{x}$ is increasing since ϕ is defined as convex function satisfying $\phi(0)=0$. Besides the function $\frac{\phi(\hbar_1(x))}{\hbar_1(x)}$ is also increasing as $\hbar_1$ is increasing. Obviously, the function $\frac{\hbar_1(x)}{\hbar_2(x)}$ is decreasing.

Thus, for all $[a,x],\ a<x\leq b,$ it can be written $\varphi\leq t$

$$\left(\frac{\phi(\hbar_1(t))}{\hbar_1(t)}-\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\right)\left(\frac{\hbar_1(\varphi)}{\hbar_2(\varphi)}-\frac{\hbar_1(t)}{\hbar_2(t)}\right)\geq 0.$$

It follows that

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\frac{\hbar_1(\varphi)}{\hbar_2(\varphi)}+\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\frac{\hbar_1(t)}{\hbar_2(t)}-\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\frac{\hbar_1(\varphi)}{\hbar_2(\varphi)}-\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\frac{\hbar_1(t)}{\hbar_2(t)}\geq 0. \tag{3}$$

Multiplying (3) by $\hbar_2(t)\hbar_2(\varphi)$, we have

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t)+\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)-\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)-\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)\geq 0. \tag{4}$$

Now, multiplying both sides of (4) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t)w(t)dt$$

$$+\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)w(t)dt$$

$$-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)w(t)dt$$

$$-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)w(t)dt \geq 0.$$

Then, it follows that

$$\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)+\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)$$

$$-\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)-\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_1\right)(x)\geq 0. \tag{5}$$

Again, multiplying both sides of (5) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\ell-1}w(\varphi)$ and then integrating with respect to φ from a to x, we obtain

$$\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x) \tag{6}$$

$$\geq\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x).$$

It follows that

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)}\geq\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)}. \tag{7}$$

Now, since $\frac{\phi(x)}{x}$ is an increasing function and $\hbar_1\leq\hbar_2$ on $[a,b]$, we get

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\leq\frac{\phi(\hbar_2(t))}{\hbar_2(t)} \tag{8}$$

for $t\in[a,x]$.

Multiplying both sides of (8) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar_2(t)w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_2(t)w(t)dt$$

$$\leq\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_2(t))}{\hbar_2(t)}\hbar_2(t)w(t)dt,$$

which yields

$$\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\leq\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_2\right)(x).\tag{9}$$

Hence from (7) and (9), we have (2). □

Remark 2.2 *In Theorem 2.1, if we choose $w(x)=1$ and $\sigma(x)=x$, then we obtain Theorem 3.1 in [9].*

Remark 2.3 *In Theorem 2.1, if we choose $w(x)=1=\ell$, $\sigma(x)=x$ and $x=b$, then we obtain Theorem 1.3.*

Theorem 2.4 *Let $\hbar_1$ and $\hbar_2$ be two positive continuous functions and $\hbar_1\leq\hbar_2$ on $[a,b]$. If $\frac{\hbar_1}{\hbar_2}$ is decreasing and $\hbar_1$ is increasing on $[a,b]$, then for a convex function ϕ with $\phi(0)=0$, the weighted fractional operator given by (1) satisfies the following inequality*

$$\frac{\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)}\geq 1,$$

where $x>a>0$, $\ell,\rho\in\mathbb{C}$, $Re(\ell)>0$ and $Re(\rho)>0$.

Proof $\frac{\phi(x)}{x}$ is increasing since ϕ is defined as convex function satisfying $\phi(0)=0$. Besides the function $\frac{\phi(\hbar_1(x))}{\hbar_1(x)}$ is also increasing as $\hbar_1$ is increasing. Obviously, the function $\frac{\hbar_1(x)}{\hbar_2(x)}$ is decreasing for all $[a,x]$, $a<x\leq b$. Multiplying both sides of (5) by $\frac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\rho-1}w(\varphi)$ and then integrating the resulting identity from a to x, we obtain

$$\begin{aligned}&\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\\ \geq\;&\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x).\end{aligned}\tag{10}$$

Similar to the (9) inequality, multiplying both sides of (8) by

$$\frac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\rho-1}\hbar_2(t)w(t)$$

and then integrating with respect to the variable t from a to x, we have

$$\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\leq\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_2\right)(x).\tag{11}$$

Hence, from (9), (11) and (10), we have the needful result. □

Remark 2.5 *If we choose $\ell = \rho$, then Theorem 2.4 will lead to Theorem 2.1.*

Remark 2.6 *In Theorem 2.4, if we choose $w(x) = 1$ and $\sigma(x) = x$, then we obtain Theorem 3.3 in [9].*

Remark 2.7 *In Theorem 2.4, if we choose $w(x) = 1 = \ell = \rho$, $\sigma(x) = x$ and $x = b$, then we obtain Theorem 1.3.*

Theorem 2.8 *Let $\hbar_1$, $\hbar_2$ and $\hbar_3$ be positive continuous functions and $\hbar_1 \leq \hbar_2$ on $[a,b]$. If $\frac{\hbar_1}{\hbar_2}$ is decreasing and $\hbar_1$ and $\hbar_3$ are increasing on $[a,b]$, then for a convex function ϕ with $\phi(0) = 0$, then the following inequality holds for the weighted fractional operator (1)*

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)} \geq \frac{\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)},$$

where $x > a > 0$, $\ell \in \mathbb{C}$, $Re(\ell) > 0$.

Proof Since $\hbar_1 \leq \hbar_2$ on $[a,b]$ and $\frac{\phi(x)}{x}$ is increasing for $t,\varphi \in [a,x]$, $a < x \leq b$, we get

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)} \leq \frac{\phi(\hbar_2(t))}{\hbar_2(t)}. \tag{12}$$

Multiplying both sides of (12) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar_2(t)\hbar_3(t)w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\begin{aligned} &\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_2(t)\hbar_3(t)w(t)dt \\ \leq\ &\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_2(t))}{\hbar_2(t)}\hbar_2(t)\hbar_3(t)w(t)dt \end{aligned}$$

which, in view of (1), can be written as

$$\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x) \leq \left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x). \tag{13}$$

Also, since the function ϕ is convex and such that $\phi(0) = 0$, $\frac{\phi(t)}{t}$ is increasing. Since $\hbar_1$ is increasing, so is $\frac{\phi(\hbar_1(t))}{\hbar_1(t)}$. Clearly, the function $\frac{\hbar_1(t)}{\hbar_2(t)}$ is decreasing for $t,\varphi \in [a,x]$, $a < x \leq b$. Thus

$$\left(\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_3(t) - \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_3(\varphi)\right)\left(\hbar_1(\varphi)\hbar_2(t) - \hbar_1(t)\hbar_2(\varphi)\right) \geq 0.$$

It becomes

$$\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t) + \frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)$$

$$-\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)-\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)\geq 0. \tag{14}$$

Multiplying both sides of (14) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}w(t)$ and then integrating with respect to the variable t from a to x, we obtain

$$\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t)w(t)dt$$

$$+\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)w(t)dt$$

$$-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)w(t)dt$$

$$-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)w(t)dt\geq 0.$$

This follows that

$$\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)+\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)$$

$$-\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)-\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\left(\phi\circ\hbar_1\right)\hbar_3\right)(x)\geq 0. \tag{15}$$

Again, multiplying both sides of (15) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\ell-1}w(\varphi)$ and then integrating with respect to the variable φ from a to x, we have

$$\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)$$

$$\geq\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\left(\phi\circ\hbar_1\right)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\left(\phi\circ\hbar_1\right)\hbar_3\right)(x).$$

Therefore, we can write

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)}\geq\frac{\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)}. \tag{16}$$

Hence, from (13) and (16), we obtain the required result. □

Remark 2.9 *In Theorem 2.8, if we choose $w(x)=1$ and $\sigma(x)=x$, then we obtain Theorem 3.5 in [9].*

Remark 2.10 *In Theorem 2.8, if we choose $w(x)=1=\ell$, $\sigma(x)=x$ and $x=b$, then we obtain Theorem 1.4.*

Theorem 2.11 *Let $\hbar_1$, $\hbar_2$ and $\hbar_3$ be positive continuous functions and $\hbar_1 \leq \hbar_2$ on $[a,b]$. If $\frac{\hbar_1}{\hbar_2}$ is decreasing and $\hbar_1$ and $\hbar_3$ are increasing on $[a,b]$, then for a convex function ϕ with $\phi(0)=0$ then the following inequality holds for the weighted fractional operator (1)*

$$\frac{\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}\geq 1,$$

where $x>a>0$, $\ell,\rho\in\mathbb{C}$, $Re(\ell)>0$ and $Re(\rho)>0$.

Proof By the assumption of Theorem 2.11, multiplying both sides of (15) by

$$\tfrac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\rho-1}w(\varphi)$$

and then integrating with respect to the variable φ from a to x, we have

$$\begin{aligned}&\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x) \quad (17)\\ \geq\ &\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\rho:\sigma}\left(\phi\circ\hbar_1\right)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\left(\phi\circ\hbar_1\right)\hbar_3\right)(x).\end{aligned}$$

Since $\hbar_1\leq\hbar_2$ on $[a,b]$ and $\frac{\phi(x)}{x}$ is increasing for $t,\varphi\in[a,x]$, $a<x\leq b$, we get

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\leq\frac{\phi(\hbar_2(t))}{\hbar_2(t)}. \quad (18)$$

Multiplying both sides of (18) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar_2(t)\hbar_3(t)w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\leq\left({}_{a+}\Im_w^{\ell:\sigma}\left(\phi\circ\hbar_2\right)\hbar_3\right)(x). \quad (19)$$

Similarly, multiplying both sides of (18) by $\frac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\rho-1}\hbar_2(t)\hbar_3(t)w(t)$ and then integrating with respect to the variable t from a to x, we can write

$$\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\leq\left({}_{a+}\Im_w^{\rho:\sigma}\left(\phi\circ\hbar_2\right)\hbar_3\right)(x). \quad (20)$$

So, from (17), (19) and (20) we have

$$\frac{\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}\geq 1.$$

□

Remark 2.12 *If we choose* $\ell = \rho$*, then Theorem 2.11 will lead to Theorem 2.8.*

Remark 2.13 *In Theorem 2.11, if we choose* $w(x) = 1$ *and* $\sigma(x) = x$*, then we obtain Theorem 3.7 in [9].*

3. Conclusion

In this paper, first we gave different definitions of fractional integral operators and then we introduced some inequalities using the monotonicity properties of functions for weighted fractional operators. The obtained results are an extension of some known results in the literature. Especially, we would like to emphasize that different types of integral inequalities can be obtained using this operators.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Çetin Yıldız]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Mustafa Gürbüz]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Liu W.J., Ngo Q.A., Huy V.N., *Several interesting integral inequalities*, Journal of Mathematical Inequalities, 3, 201-212, 2009.

[2] Mitrinović D.S., Pečarić J.E., Fink A.M., Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.

[3] Jarad F., Abdeljawad T., Shah K., *On the weighted fractional operators of a function with respect to another function*, Fractals, 28(08), 20040011, 2020.

[4] Osler T.J., *The fractional derivative of a composite function*, SIAM Journal on Mathematical Analysis, 1, 288-293, 1970.

[5] Almeira R., *A Caputo fractional derivative of a function with respect to another function*, Communications in Nonlinear Science and Numerical Simulation, 44, 460-481, 2017.

[6] Kilbas A.A., *Hadamard-type fractional calculus*, Journal of the Korean Mathematical Society, 38, 1191-1204, 2001.

[7] Ngo Q.A., Thang D.D., Dat T.T., Tuan D.A., *Notes on an integral inequality*, Journal of Inequalities in Pure and Applied Mathematics, 7(4), 120, 2006.

[8] Liu W.J., Cheng G.S., Li C.C., *Further development of an open problem concerning an integral inequality*, Journal of Inequalities in Pure and Applied Mathematics, 9(1), 14, 2008.

[9] Dahmani Z., *A note on some new fractional results involving convex functions*, Acta Mathematica Universitatis Comenianae, LXXXI, 241-246, 2012.

[10] Rahman G., Nisar K.S., Abdeljawad T., Ullah S., *Certain fractional proportional integral inequalities via convex functions*, Mathematics, 8, 222, 2020.

[11] Rahman G., Nisar K.S., Abdeljawad T., *Tempered fractional integral inequalities for convex functions*, Mathematics, 8(4), 500, 2020.

[12] Rahman G., Abdeljawad T., Jarad F., Khan A., Nisar K.S., *Certain inequalities via generalized proportional Hadamard fractional integral operators*, Advances in Differential Equations, 2019:454, 2019.