Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 11 NO: 4 PAGE: 1014-1017 YEAR: 2022
DOI:10.17798/bitlisfen.1150947

On Closed Subspaces of Grand Lebesgue Spaces

Yasin KAYA[1*]

[1]*Dicle University, Department of Mathematics*
(ORCID:0000-0002-7779-6903)

Keywords: Grand Lebesgue spaces, Embeddings, Hilbert spaces.

Abstract
We prove a generalized version of a theorem of Grothendieck over finite measure space. We prove a closed subspace of grand Lebesgue space that consist of functions of L^{∞} must be finite dimensional. By using embeddings of Banach spaces $L^{p}(\Omega), L^{p)}(\Omega)$ and $L^{2}(\Omega)$ we work inside space $L^{2}(\Omega)$. Then we take advantage of many useful properties of Hilbert space.

1. Introduction

In [1] Iwaniec and Sbordone aimed to tackle problem of integrability of the Jacobian under minimal assumptions, introduced grand Lebesgue spaces $L^{p)}(\Omega)$. Grand Lebesgue spaces $L^{p)}(\Omega)$ are a generalization of classical Lebesgue spaces $L^{p}(\Omega)$. Among many applications of $L^{p)}(\Omega)$ one is this: $L^{p)}(\Omega)$ spaces gave rise to be the convenient spaces in which several nonlinear equations in the theory of partial differential equations have to be considered [2,3]. For a detailed discussions of properties and some generalization of grand Lebesgue spaces we refer the readers to book [4]. We recall here briefly some basic definitions and general facts about grand Lebesgue spaces. Let $\Omega \subset \mathbb{R}^{n}$ be a Lebesgue measurable set with $|\Omega| < \infty$ and $1 < p < \infty$. Grand Lebesgue space $L^{p)}(\Omega)$ is defined as the spaces of the measurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_{\Omega} |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty. \tag{1}$$

Grand Lebesgue spaces $L^{p)}(\Omega)$ are Banach spaces. Let X and Y be two normed linear spaces such that $X \subset Y$. If the operator $I : X \to Y$ defined by $I(x) = x$ is continuous we say the embedding $X \subset Y$ continuous. Getting back to the spaces $L^{p)}(\Omega)$, the following continuous embeddings holds:

$$L^{p}(\Omega) \subset L^{p)}(\Omega) \subset L^{p-\varepsilon}(\Omega) \quad \text{for} \quad 0 < \varepsilon < p-1. \tag{2}$$

Furthermore, since $g(x) = x^{-\frac{1}{p}} \in L^{p)}(0,1)$ and $g(x) = x^{-\frac{1}{p}} \notin L^{p}(0,1)$ hold. $L^{p)}(\Omega)$ space is strictly larger than Lebesgue space $L^{p}(\Omega)$. The subspace $C_0^{\infty}(\Omega)$ is not dense in $L^{p)}(\Omega)$. Its closure $\left[L^{p}\right]_{p)}$ consists of all those functions $f \in L^{p)}(\Omega)$ which satisfy

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Omega} |f|^{p-\varepsilon} dx = 0. \tag{3}$$

2. Material and Method

There are many advantages of Hilbert spaces comparing with unspecific Banach spaces. One of them Hilbert spaces have bases with fine adequate properties and notion of orthogonality of vectors. One of the most useful examples of a Hilbert space is the

*Corresponding author: *ykaya@dicle.edu.tr*

Received: 29.07.2022, Accepted: 31.10.2022

space $L^2(\Omega)$. The space $L^2(\Omega)$ equipped with the following inner product space

$$\langle f,g\rangle = \int_\Omega f(x)\overline{g(x)}d\mu(x) \,. \tag{4}$$

Proof of the following lemma is to be found in any book on functional analysis.

Lemma 2.1. (Generalized Pythagorean Theorem) Let $(D,\langle.\rangle,\|.\|)$ be an inner product space over field F. If the vectors $g_1,g_2,...,g_n \in D$ are orthogonal, then

$$\|g_1+g_2+...+g_n\|^2 = \|g_1\|^2+\|g_2\|^2+...+\|g_n\|^2 \tag{5}$$

3. Results and Discussion

Our result is the following theorem. Our proof is an adaptation of the proof given in [5]. But actually the clasical result of this theorem due to Grothendieck [6]. In [7] this theorem was proved for variable exponent Lebesgue space $L^{p(x)}(\Omega)$, which is an another generalization of classical Lebesgue space $L^p(\Omega)$. For a discussion of $L^{p(x)}(\Omega)$ spaces see [8] and references therein.

Theorem 3.1. Let $\Omega \subset \mathbb{R}^n$ have finite Lebesgue measure. Let D be a closed subspace of grand Lebesgue space $L^{p)}(\Omega)$ such that $D \subset L^\infty(\Omega)$. Then the space D is finite dimensional.

Before proceeding with the proof the theorem we need the following lemma whose proof can be found in [5, lemma 4.3 page 175]. (Actually we state a part of the lemma).

Lemma 3.2. With the same set of hypoteses of theorem 3.1 there exists $\beta > 0$ such that

$$\|f\|_{L^p} \le \beta \|f\|_{L^2} \text{ for all } f \in D. \tag{6}$$

Proof of the theorem 3.1. Since D is a closed subspace of the Banach space $L^{p)}(\Omega)$ then D is itself a Banach space. Let define an identity mapping by

$$I : D \to L^\infty(\Omega) \text{ and } I(f) = f.$$

The identity mapping is obviously linear, now we show it also has a closed graph. Let $f_n \in D$ and $g \in L^\infty(\Omega)$ such that $f_n \to f$ in D and $I(f_n) = f_n \to g$ in $L^\infty(\Omega)$. Since

$$\|f_n - g\|_{p)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{|\Omega|}\int_\Omega |f_n-g|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} \le \|f_n-g\|_{L^\infty}\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{|\Omega|}\int_\Omega dx\right)^{\frac{1}{p-\varepsilon}} \tag{7}$$

holds we have $f = g$ almost everywhere, so by the closed graph theorem I is continuous, and so there is a constant $c > 0$ such that

$$\|f\|_{L^\infty} \le c\|f\|_{L^{p)}} \text{ for all } f \in D. \tag{8}$$

From (8), $L^p(\Omega) \subset L^{p)}(\Omega)$ embedding and Lemma 3. there exists $K > 0$ such that

$$\|f\|_{L^\infty} \le K\|f\|_{L^2} \text{ for all } f \in D. \tag{9}$$

Now we are aware that we placed D inside $L^2(\Omega)$. We know $L^2(\Omega)$ is a Hilbert space. There are many advantages to be gained from working with $L^2(\Omega)$ space. Let $\{g_1,g_2,...,g_n\}$ be an orthonormal set in $L^2(\Omega)$ of functions in D. Define

$$B = \left\{c = (c_1,c_2,...,c_n) \in \mathbb{R}^n : \sum_{j=1}^n |c_j|^2 \le 1\right\}. \tag{10}$$

For each $c = (c_1,c_2,...,c_n) \in B$ let us define a function

$$g_c = \sum_{j=1}^n c_j g_j(x) \tag{11}$$

Moreover $g_c(x) \in D$ and by the generalized Pythagorean theorem (Lemma 2.1.)

$$\|g_c(x)\|_{L^2} = \left\|\sum_{j=1}^n c_j g_j(x)\right\|_{L^2} = \left(\sum_{j=1}^n \|c_j g_j(x)\|_{L^2}^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^n |c_j|^2\right)^{\frac{1}{2}} \le 1. \tag{12}$$

Consequently from (9) we have

$$\|g_c\|_{L^\infty} \le K \tag{13}$$

Hence for each c there exists a Lebesgue measurable set Ω_c of full measure in Ω i.e., $|\Omega_c| = |\Omega|$, such that

$$|g_c(x)| \le K \text{ for all } x \in \Omega_c \tag{14}$$

By first using a countable dense subset of points in B, then applying the continuity of the mapping $c \to g_c(x)$, we notice that (14) gives

$$|g_c(x)| \le K \text{ for all } x \in \Omega' \text{ and all } c \in B \tag{15}$$

Where Ω' is a set of full measure in Ω. Furthermore, we claim that for every $x \in \Omega'$ we have

$$\sum_{j=1}^{n} |g_j(x)|^2 \le K^2 \tag{16}$$

It is sufficies to prove this inequality under the assumption that the left- hand side not zero. Let us define

$$\theta = \left(\sum_{j=1}^{n} |g_j(x)|^2 \right)^{\frac{1}{2}} \tag{17}$$

and set $c_j = \overline{g_j(x)} / \theta$ then by (15) we obtain for each $x \in \Omega'$

$$\frac{1}{\theta} \sum_{j=1}^{n} |g_j(x)|^2 \le K, \tag{18}$$

that means $\theta \le K$, as we claimed.
Finally, integrating (16) and being aware that $\int_\Omega |g_j(x)|^2 dx = 1$ we have $n \le K^2 |\Omega|$. Since $K^2 |\Omega|$ is a constant number the dimension of D must be finite.

4. Conclusion and Suggestions

We proved the theorem for Lebesgue measure, also the same proof methods can be applied to case when (M, Σ, μ) is a given measure space provided that $\mu(M) < \infty$. It will be interesting to search whether theorem 3.1 remains valid under weaker hypotheses.

Acknowledgment

If necessary, the people, institutions and organizations that helped in the study should be thanked for their help and support.

Contributions of the authors

The contributions of each author to the article should be indicated.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics

References

[1] T. Iwaniec and C. Sbordone, "On the integrability of Jacobian under minimal hypotheses," *Archive for Rational Mechanics and Analysis*, vol. 119, no. 2, pp.129-143,1992.

[2] L. Greco, T. Iwaniec, and C. Sbordone, "Inverting the $p-$harmonic operator,"*manuscripta mathematica*, vol.92, pp. 249-258,1997.

[3] A. Fiorenza and C. Sbordone, "Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1, " *Studia Mathematica*, vol. 3, no.127, pp. 223-231,1998.

[4] R. Castillo and H. Rafeiro, *An introductory course in Lebesgue Spaces*. Cham, Switzerland: Springer international publishing, 2018.

[5] E.M. Stein and R. Shakarchi, *Functional analysis: Introduction to further topics in analysis*. Princeton, NJ: Princeton University Press, 2011.

[6] A. Grothendieck, "Sur certains sous-espaces vectoriels de L^p, " *Canadian Journal of Mathematics*, vol. 6, pp.158-160,1954.

[7] Y. Kaya, "On a closed subspace of $L^{p(.)}(\Omega)$," *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi,* vol. 9, no. 2, pp. 682-688, 2020.

[8] D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces: Foundations and Harmonic Analysis.* Springer Science & Business Media, 2013.