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# Some Spectrum Estimates of the $\alpha q$-Cesàro Matrices with $0<\alpha,q<1$ on $c_0$

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**Abstract**

The main purpose of the this paper is to investigate the spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum, and the compression spectrum of the genaralized $\alpha q$-Cesàro matrix $C_q^\alpha$ with $\alpha,q\in(0,1)$ on the sequence space $c_0$.

***Keywords:*** *q-Hausdorff matrices; Lower bound problem;, q-Cesaro matrices; spectrum; fine spectrum.*
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## 1. Introduction

The history of $q$-mathematics, which has many applications in mathematics and engineering, is so old that it goes back to the time of Euler. With $q\neq 1$, the $q$-analogue of the integer $n$ is given by the following expression;

$$[n]_q=\frac{1-q^n}{1-q}.$$

If $\lim\limits_{\alpha\to 1^-} a_{nk}^{\alpha}(q)=a_{nk}(q)$, a matrix $A^{\alpha}(q)=\left(a_{nk}^{\alpha}(q)\right)$ is called an $\alpha$-generalization of the matrix $A(q)=(a_{nk}(q))$. Also, if $\lim\limits_{q\to 1^-} a_{nk}(q)=a_{nk}$, a matrix $A^{\alpha}(q)=\left(a_{nk}^{\alpha}(q)\right)$ is called an $q$-generalization of the matrix $A=(a_{nk})$.
For $0<\alpha,q<1$, the genaralized $\alpha q$-Cesàro matrix $C_q^\alpha$ is defines, as follows;

$$c_{nk}^{\alpha}(q)=\begin{cases}\frac{(\alpha q)^{n-k}}{1+q+\cdots+q^n} & ,\ 0\le k\le n\\ 0 & ,\ n<k.\end{cases} \tag{1.1}$$

In this case, $\alpha\to 1^-$ in this matrix,

$$c_{nk}(q)=\begin{cases}\frac{q^{n-k}}{1+q+\cdots+q^n} & ,\ 0\le k\le n\\ 0 & ,\ n<k.\end{cases}$$

$q$-Cesàro matrix is obtained.
If $q\to 1^-$ is written in the $q$-Cesàro matrix,

$$c_{nk}=\begin{cases}\frac{1}{n+1} & ,\ 0\le k\le n\\ 0 & ,\ n<k.\end{cases} \tag{1.2}$$

the well-known (1.2) Cesàro matrix of order one, $C_1$ is obtained. The spectrum and spectral decomposition of this matrix over various spaces are discussed in [24], [29] and [51] studies. More information about the $q$-Cesàro matrix can be found in [29, 48]. Spectra and spectral decompositions of Cesàro operators over various spaces have been discussed by many authors with different techniques (See [2]-[6], [13]-[16], [40]-[44], [49], [50]).
This paper is about the spectrum and spectral decompositions of the $\alpha q$-Cesàro operator on Banach space $c_0$. Here $c_0$ is the space of complex sequences converging to zero, which is considered with the supremum norm.
We will make a brief reminder that the reader has a basic knowledge of the spectrum of a bounded linear operator $T$ on an infinite dimensional Banach space $X$.

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Let $\mathbb{N}_0 := \{0,1,2,\ldots\}$ and $\mathbb{C}$ denote the set of complex numbers.
If there is an inverse operator $(T-\lambda I)^{-1}$ for $\lambda \in \mathbb{C}$, then $R(\lambda;T) := T_\lambda^{-1} := (T-\lambda I)^{-1}$ is called the resolvent operator of $T$ and thus the spectrum and the usual discrete spectrum of the operator $T$ can be summarized with the following Table:

| | | 1 | 2 | 3 |
|---|---|---|---|---|
| | | $R(\lambda;T)$ exists<br>and is bounded | $R(\lambda;T)$ exists<br>and is unbounded | $R(\lambda;T)$<br>does not exists |
| I | $R(T-\lambda I)=X$ | $\lambda \in \rho(T,X)$ | – | $\lambda \in \sigma_p(T,X)$ |
| II | $\overline{R(T-\lambda I)}=X$ | $\lambda \in \rho(T,X)$ | $\lambda \in \sigma_c(T,X)$ | $\lambda \in \sigma_p(T,X)$ |
| III | $\overline{R(T-\lambda I)}\neq X$ | $\lambda \in \sigma_r(T,X)$ | $\lambda \in \sigma_r(T,X)$ | $\lambda \in \sigma_p(T,X)$ |

Table 1. Goldberg's and fine decomposition of the spectrum

If $R(\lambda;T)$ exists, $R(\lambda;T)$ is bounded and $R(\lambda;T)$ is defined on a dense set in $X$, we write $\lambda \in \rho(T,X)$ and $\rho(T,X)$ is called the resolvent set of $T$. It corresponds to $I_1 \cup II_1$ in this Table. From the Closed Graph Theorem it is always $I_2 = \emptyset$. $\sigma(T,X) = \mathbb{C}\backslash\rho(T,X)$ is the spectrum of $T$ and from the Table the set $\sigma(T,X)$ consists of the union of sets $I_3$, $II_2$, $II_3$, $III_1$, $III_2$, $III_3$. The set $I_3 \cup II_3 \cup III_3$ is called the point spectrum of $T$ and is denoted by $\sigma_p(T,X)$. Essentially, $\sigma_p(T,X)$ is the set of eigenvalues of $T$. The set $III_1 \cup III_2$ is called the residual spectrum of $T$ and is denoted by $\sigma_r(T)$. As read from the Table 1, $\sigma_r(T,X)$ is the set of $\lambda$'s such that "$R(\lambda;T)$ exists as bounded or unbounded, $R(\lambda;T)$ is not defined on a dense in $X$". The set $II_2$ is called the continouous spectrum of $T$ and is denoted by $\sigma_c(T)$. The set $\sigma_c(T)$ can also be interpreted from the Table 1. The spectrum is a union of point spectrum, continuous spectrum, and residual spectrum, and the sets that make up the union are disjoint two-by-two. If $X$ is a Banach space then $\sigma(T^*,X^*) = \sigma(T,X)$. (A more detailed review for these can be found in [37]. ) A fine separation of the spectrum is also done. This is called the Goldberg classification of the spectrum. Since $I_3$, $II_2$, $II_3$, $III_1$, $III_2$, $III_3$ are subsets of the spectrum, these are replaced by the notations $I_3\sigma(T,X)$, $II_3\sigma(T,X)$, $III_3\sigma(T,X)$, $II_2\sigma(T,X)$, $III_1\sigma(T,X)$, $III_2\sigma(T,X)$. Of course, since their union forms the spectrum, the point spectrum is finer than the residual spectrum and continuous spectrum decomposition. Therefore, $I_3$, $II_2$, $II_3$, $III_1$, $III_2$, $III_3$ are called fine decomposition of the spectrum. (A more detailed review for these can be found in [32].)
The fine spectrum of some bounded linear operators over various spaces has been specified by many authors (See, [1], [19], [25], [27], [28], [30], [31], [33]-[36], [39], [45], [46], [47], [52]-[56]).
Let $X$ be Banach space on a field $\mathbb{K}$ and $T \in B(X)$. If $(x_n) \subset X$ is a sequence such that $\|Tx_n\| \to 0$ while $n \to \infty$ and $\|x\| = 1$, then $(x_n)$ is called a Weyl sequence for $T$. The set

$$\sigma_{ap}(T,X) := \{\lambda \in \mathbb{K} : \text{ there is a Weyl sequence for } \lambda I - T\}$$

is called the approximate point spectrum of $T$. The set

$$\sigma_\delta(T,X) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not onto}\},$$

which is a subset of the spectrum, is called the defect spectrum of $T$. The set

$$\sigma_{co}(T,X) := \left\{\lambda \in \mathbb{K} : \overline{R(\lambda I - T)} \neq X\right\}$$

is called the compression spectrum of $T$. These three sets make up the non-discrete spectrum of the spectrum. In order to obtain this non-discrete decomposition for a finite linear operator $T$, the following Table created in the articles [9]-[11], [17] and [18] is used, considering the proposition and Theorems in [8].

| | | (1) | (2) | (3) |
|---|---|---|---|---|
| | | $R(\lambda;T)$ exists<br>and is bounded | $R(\lambda;T)$ exists<br>and is unbounded | $R(\lambda;T)$<br>does not exists |
| $(I)$ | $R(\lambda I-T)=X$ | $\lambda \in \rho(T,X)$ | – | $\lambda \in \sigma_{ap}(T,X)$ |
| $(II)$ | $R(\lambda I-T)\neq X$<br>$\overline{R(\lambda I-T)}=X$ | $\lambda \in \rho(T,X)$ | $\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$ | $\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$ |
| $(III)$ | $\overline{R(\lambda I-T)}\neq X$ | $\lambda \in \sigma_\delta(T,X)$<br>$\lambda \in \sigma_{co}(T,X)$ | $\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$<br>$\lambda \in \sigma_{co}(T,X)$ | $\lambda \in \sigma_{ap}(T,X)$<br>$\lambda \in \sigma_\delta(T,X)$<br>$\lambda \in \sigma_{co}(T,X)$ |

Table 2. Non-discrete decomposition of the spectrum

Using Table 2, the non-discrete spectrum of some bounded linear operators on various spaces is determined (See, [7], [12], [20]-[23] ).

## 2. Boundedness and Spectra of $C_q^{\alpha}$

Our purpose in this section is to first show that $C_q^{\alpha}$ is a bounded linear operator on $c_0$, and then determines its spectrum. The following Lemma gives the constraint condition on the sequence space $c_0$ of an operator given by an infinite matrix.

**Lemma 2.1.** *[38, p.163] $A=(a_{nk})\in B(c_0)$ if and only if $\|A\|=\sup_n\sum_k|a_{nk}|<\infty$ and for each $k$, $\lim_{n\to\infty}a_{nk}=0$.*

**Lemma 2.2.** *Let $0<q<1$, $0<\alpha<1$. Then $C_q^{\alpha}\in B(c_0)$ and $\left\|C_q^{\alpha}\right\|=1$.*

*Proof.* For each $k$,

$$\lim_{n\to\infty}c_{nk}=\lim_{n\to\infty}\frac{(\alpha q)^{n-k}}{1+q+\cdots+q^n}=\lim_{n\to\infty}\frac{(\alpha q)^{n-k}}{\frac{1-q^{n+1}}{1-q}}=0.$$

Since $\alpha<1$,

$$\left\|C_q^{\alpha}\right\| \quad =\sup_n\sum_{k=0}^{\infty}|c_{nk}| \quad =\sup_n\sum_{k=0}^{\infty}\frac{(\alpha q)^{n-k}}{1+q+\cdots+q^n}\le 1 \tag{2.1}$$

So, it is obtained that $C_q^{\alpha}\in B(c_0)$. Also, we have

$$\begin{aligned}\left\|C_q^{\alpha}\right\| &=\sup_{x\neq\theta}\frac{\left\|C_q^{\alpha}(x)\right\|_{c_0}}{\|x\|_{c_0}} \quad =\sup_{x\neq\theta}\frac{\left\|\left(x_0,\frac{\alpha q}{1+q}x_0+\frac{1}{1+q}x_1,\cdots\right)\right\|_{c_0}}{\|x\|_{c_0}}\\ &\overset{x=e_0}{\geq}\left\|\left(1,\frac{\alpha q}{1+q},\frac{\alpha^2q^2}{1+q+q^2},\cdots\right)\right\|_{c_0}\\ &=\sup\left(1,\frac{\alpha q}{1+q},\frac{\alpha^2q^2}{1+q+q^2},\cdots\right)=1\end{aligned} \tag{2.2}$$

From 2.1 and 2.2, we get $\left\|C_q^{\alpha}\right\|=1$. Thus, the conditions of Lemma 2.1 are met, that is, we obtain $C_q^{\alpha}\in B(c_0)$. □

If $0<q<1$ and $0<\alpha<1$, there is always $m\in\mathbb{N}_0$ such that $\alpha<q^m$. It should be noted that $\alpha<q^0=1$.
Now let's determine the point spectrum. For brevity, we will take $A_m:=\sum_{k=0}^{m}q^k$, $C_m:=\frac{1}{A_m}$ for $m=0,1,2,\dots$ and

$$E_m \quad :=\{C_1,C_2,\dots,C_m:\ \alpha<q^m\}.$$

**Theorem 2.3.** *Let $0<q<1$ and $0<\alpha\le 1$. Then, if $\alpha<q^m$ then*

$$\sigma_p\left(C_q^{\alpha},c_0\right)=E_m$$

*and if $\alpha=1$ then $\sigma_p\left(C_q^{\alpha},c_0\right)=\emptyset$.*

*Proof.* Let $C_q^{\alpha}x=\lambda x$. Thus, the following equations can be written;

$$\begin{aligned} x_0 &= \lambda x_0\\ \frac{1}{\sum_{k=0}^{1}q^k}\left(\alpha q x_0+x_1\right) &= \lambda x_1\\ \frac{1}{\sum_{k=0}^{2}q^k}\left((\alpha q)^2x_0+\alpha q x_1+x_2\right) &= \lambda x_2\\ &\ \vdots\\ \frac{1}{\sum_{k=0}^{n}q^k}\left((\alpha q)^n x_0+(\alpha q)^{n-1}x_1+\cdots+\alpha q x_{n-1}+x_n\right) &= \lambda x_n\\ &\ \vdots \end{aligned} \tag{2.3}$$

If $x_0\neq 0$, then since $(1-\lambda)x_0=0$ in the 1st row of equation (2.3), we get $\lambda=1$. Again, we find $x_1=\alpha x_0$, $x_2=\alpha^2x_0,\dots,x_n=\alpha^n x_0$ from the 2nd row, the 3rd row and the (n+1)th row of the equation (2.3), respectively. Hence, $x_n=\alpha^n x_0$ is obtained for every $n$. Thus, since $0<\alpha<1$, the eigenvector corresponding to $\lambda=1$ is $x=(\alpha^n)\in c_0$.
Similarly, let $x_m$ be the first nonzero term of the sequence $(x_n)$. Thus, from the mth row of the equation (2.3)

$$\left(\lambda-\frac{1}{\sum_{k=0}^{m}q^k}\right)x_m=0$$

is found. Hence, we have

$$\lambda=\frac{1}{\sum_{k=0}^{m}q^k}, \tag{2.4}$$

since $x_m \neq 0$. Since $x_1 = x_2 = x_3 = \cdots = x_{m-1} = 0$, from the (m+1)th, (m+2)th and (m+n)th rows of the equation (2.4) with this $\lambda$,

$$\begin{aligned} x_{m+1} &= \frac{\alpha q}{q^{m+1}} \left( \sum_{k=0}^{m} q^k \right) x_m = \frac{\alpha q}{q^{m+1}} A_m x_m \\ x_{m+2} &= \frac{\alpha^2 q^2}{(q^{m+1})^2 (1+q)} A_m A_{m+1} x_m \\ x_{m+3} &= \frac{\alpha^3 q^3}{(q^{m+1})^3 (1+q)(1+q+q^2)} A_m A_{m+1} A_{m+2} x_m \\ &\vdots \\ x_{m+n} &= \frac{\alpha^n q^n A_m A_{m+1} \cdots A_{m+n-1}}{(q^{m+1})^n A_1 \cdots A_{n-1}} x_m \\ &\vdots \end{aligned}$$

are obtained. Hence, we have

$$\lim_{n\to\infty} \frac{x_{m+n}}{x_{m+n-1}} = \lim_{n\to\infty} \frac{\alpha}{q^m} \left[ 1 + q^n \frac{1-q^m}{1-q^n} \right] = \frac{\alpha}{q^m}.$$

If $\alpha < q^m$, then $\sum_{n=0}^{\infty} |x_{m+n}|$ series converges from the ratio test. So, for $\alpha < q^m$, we have $\lim_{n\to\infty} x_{m+n} = 0$,. that is, the eigenvector corresponding to $\lambda = C_m$ for $\alpha < q^m$ is $x = (0,0,\ldots,0,x_m,x_{m+1},\ldots,x_{m+n},\ldots) \in c_0$. Continuing in this way, the proof of the theorem is completed. □

**Remark 2.4.** *If $0 < q < 1$ and $0 < \alpha < 1$, then since $\alpha < q^0 = 1$, $C_0 = 1 \in \sigma_p \left( C_q^{\alpha}, c_0 \right)$.*

**Remark 2.5.** *The eigenvector corresponding to $\lambda = 1$ in Theorem 2.3 is $x = (\alpha^n) \in c_0$ provided that $0 < \alpha < 1$. Obviously, if $\alpha = 1$ then this vector is not an eigenvector. Therefore, if $\alpha = 1$ is taken, it does not contradict [51, Theorem 2.3]. If $\alpha = 1$ is taken, [51, Theorem 2.3] is still valid.*

**Remark 2.6.** *[57, p.266] If $T : c_0 \to c_0$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_1 \to \ell_1$ is defined by the transpose of the matrix A. It is well-known that the dual space $c_0^*$ of the space $c_0$ is isomorphic to the space $\ell_1$.*

Let's define $\beta := \frac{1-q}{1-\alpha^2 q^2}$ and $D := \{\lambda \in \mathbb{C} : |\lambda - \beta| < \beta \alpha q\}$ for simplicity.

**Theorem 2.7.** *Let $0 < q < 1$ and $0 < \alpha < 1$. Then*

$$\sigma_p \left( \left[ C_q^{\alpha} \right]^*, (c_0)^* \simeq \ell_1 \right) = D \cup E_m.$$

*Proof.* Let $x \neq 0$ and $C_1^*(q)x = \lambda x$. Then, we conclude that;

$$\begin{aligned} x_0 + \frac{(\alpha q)}{\sum_{k=0}^{1} q^k} x_1 + \frac{(\alpha q)^2}{\sum_{k=0}^{2} q^k} x_2 + \frac{(\alpha q)^3}{\sum_{k=0}^{3} q^k} x_3 + \cdots &= \lambda x_0 \\ \frac{1}{\sum_{k=0}^{1} q^k} x_1 + \frac{(\alpha q)}{\sum_{k=0}^{2} q^k} x_2 + \frac{(\alpha q)^2}{\sum_{k=0}^{3} q^k} x_3 + \cdots &= \lambda x_1 \\ \frac{1}{\sum_{k=0}^{2} q^k} x_2 + \frac{(\alpha q)}{\sum_{k=0}^{3} q^k} x_3 + \cdots &= \lambda x_2 \\ &\vdots \end{aligned}$$

Thus,

$$\begin{aligned} x_1 &= \frac{1}{\alpha q \lambda} (\lambda - 1) x_0 \\ x_2 &= \frac{1}{(\alpha q \lambda)^2} (\lambda - 1) \left( \lambda - \frac{1}{\sum_{k=0}^{1} q^k} \right) x_0 \\ x_3 &= \frac{1}{q^2} (\lambda - 1) \left( \lambda - \frac{1}{1+q} \right) \left( \lambda - \frac{1}{\sum_{k=0}^{2} q^k} \right) x_0 \\ &\vdots \end{aligned}$$

where $x_0 \neq 0$ and so

$$x_n = \frac{x_0}{(\alpha q \lambda)^n} \prod_{k=1}^{n} \left( \lambda - \frac{1}{\sum_{v=0}^{k-1} q^v} \right), x_0 \neq 0, n = 1,2,\ldots.$$

Using the assumption that one obtains $\lambda \in \left\{1, \frac{1}{\sum_{k=0}^{1} q^k}, \frac{1}{\sum_{k=0}^{2} q^k}, \ldots, \frac{1}{\sum_{k=0}^{n} q^k}, \ldots\right\}$.

If $\lambda = 1 = C_0$, then $C_1^*(q)x = x$ for $x = (x_0, 0, 0, \ldots) \neq \theta$. That is, we have obtained that $C_0 = 1$ is at $\sigma_p\left(C_1^*(q), \ell_1\right)$.
In this case, if $\lambda = \frac{1}{1+q} = C_1$, then $C_1^*(q)x = \frac{1}{1+q}x$ for $x = \left(x_0, -\frac{x_0}{\alpha}, 0, 0, \ldots\right) \neq \theta$. So we have obtained that $C_1 \in \sigma_p\left(C_1^*(q), \ell_1\right)$.
If we choose $\lambda = \frac{1}{1+q+q^2} = C_2$, then $C_1^*(q)x = \frac{1}{1+q+q^2}x$ for $x = \left(x_0, -\frac{(1+q)}{\alpha}x_0, \frac{q}{\alpha^2}x_0, 0, 0, \ldots\right) \neq \theta$. So we get $C_2 \in \sigma_p\left(C_1^*(q), \ell_1\right)$.

Using a similar technique, it can be easily seen for other $\lambda$ values;

$\{C_n\}_{n=1}^{\infty} \subset \sigma_p\left(C_1^*(q), (c_0)^* \simeq \ell_1\right)$.

Let $\lambda \notin \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$. If we apply the ratio test to the series $\sum x_n$;

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| &= \lim_{n\to\infty}\left|\frac{1}{\alpha q\lambda}\left(\lambda - \frac{1}{\sum_{v=0}^{n} q^v}\right)\right| \\ &= \lim_{n\to\infty}\left|\frac{1}{\alpha q\lambda}\left(\lambda - \frac{1-q}{1-q^{n+1}}\right)\right| \\ &= \left|\frac{1}{\alpha q\lambda}\left(\lambda - \frac{1-q}{\lim_{n\to\infty}(1-q^{n+1})}\right)\right| \end{aligned} \tag{2.5}$$

procedured which implies

$$\begin{aligned} &\left|\frac{1}{\alpha q\lambda}(\lambda - (1-q))\right| < 1 \Leftrightarrow \left|1 - \frac{1-q}{\lambda}\right| < \alpha q \\ &\overset{\lambda = u+iv}{\Leftrightarrow} \left|1 - \frac{1-q}{u^2+v^2}u + \frac{1-q}{u^2+v^2}vi\right| < \alpha q \\ &\Leftrightarrow \left(1 - \frac{(1-q)}{u^2+v^2}u\right)^2 + \frac{(1-q)^2}{(u^2+v^2)^2}v^2 < \alpha^2 q^2 \\ &\Leftrightarrow \left|\lambda - \frac{(1-q)}{(1-\alpha^2 q^2)}\right| < \frac{(1-q)\alpha q}{1-\alpha q}. \end{aligned} \tag{2.6}$$

(2.6) shows us, if $\left|\lambda - \frac{1}{1+q}\right| < \frac{q}{1+q}$, then $(x_n) \in \ell_1$. Thus, $\sigma_p\left(\left[C_q^{\alpha}\right]^*, \ell_1\right)$ is as follows;

$$\sigma_p\left(\left[C_q^{\alpha}\right]^*, \ell_1\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| < \frac{(1-q)\alpha q}{1-\alpha^2 q^2}\right\} \cup \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}.$$

Since $C_{m+1}, C_{m+2}, \ldots \in D$ for $\alpha > q^m$, $m = 0, 1, 2, \ldots$,

$$\sigma_p\left(\left[C_q^{\alpha}\right]^*, \ell_1\right) = D \cup E_m$$

is obtained. □

Since $c_0$ is a Banach space, $\sigma\left(\left[C_q^{\alpha}\right]^*, \ell_1\right) = \sigma\left(C_q^{\alpha}, c_0\right)$ and $\sigma_p\left(C_q^{\alpha}, c_0\right) \subset \sigma\left(C_q^{\alpha}, c_0\right)$. Let's determine the spectrum of $C_q^{\alpha}$ using these. It was also shown in [29] that the lower triangular double band matrix $\triangle_{a,b}$ can be the inverse of the $C_1(q)$ $q$-Cesaro matrix. Using the spectrum of the inverse matrix, he determined the spectrum of $C_1(q)$ on the sequence space $\ell_p$ with the help of the spectral transformation theorem. We will use this idea in the following Theorem. First, let's define the lower triangular double band matrix. An infinite matrix defined as

$$\triangle_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & 0 & \ddots \\ 0 & 0 & b_2 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is called a lower triangular double band matrix with a variable sequence, where $(a_k)$ and $(b_k)$ are two nonzero sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b \neq 0. \tag{2.7}$$

This matrix defines a $\triangle_{a,b} : c_0 \to c_0$ operator with

$$\triangle_{a,b}x = \triangle_{a,b}(x_k) = (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty} \text{ with } x_{-1} = b_{-1} = 0 \tag{2.8}$$

It has been shown by El-Shabrawy in [27] that this operator is a bounded linear operator on $c_0$ and its spectrum is determined.
The generalized $\alpha q$-Cesàro matrix $C_q^{\alpha}$ for $0 < \alpha, q < 1$ is also given by

$$C_q^{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{\alpha q}{1+q} & \frac{1}{1+q} & 0 & 0 & \cdots \\ \frac{\alpha^2 q^2}{1+q+q^2} & \frac{\alpha q}{1+q+q^2} & \frac{1}{1+q+q^2} & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{2.9}$$

The matrix $C_q^\alpha : c_0 \to c_0$ has an inverse $\left[C_q^\alpha\right]^{-1}$ and this inverse matrix is given by

$$\left[C_q^\alpha\right]^{-1} = \begin{pmatrix} A_0 & 0 & 0 & 0 & \dots \\ B_0 & A_1 & 0 & 0 & \dots \\ 0 & B_1 & A_2 & 0 & \ddots \\ 0 & 0 & A_2 & A_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where

$$A_n = 1+q+q^2+\cdots+q^n \text{ and } B_n = -\alpha q a_n \text{ for all } n \in \mathbb{N}_0. \tag{2.10}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} A_n &= \lim_{n\to\infty}\left(1+q+q^2+\cdots+q^n\right) = \lim_{n\to\infty}\frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} = A, \\ \lim_{n\to\infty} B_n &= \lim_{n\to\infty} -\alpha q\left(1+q+q^2+\cdots+q^n\right) = -\frac{\alpha q}{1-q} = B \end{aligned} \tag{2.11}$$

is obtained from (2.10). It is clearly seen that the operators $\left[C_q^\alpha\right]^{-1}$ and $C_q^\alpha$ are bijective. It has also been shown in [27] that $\triangle_{a,b} = \left[C_q^\alpha\right]^{-1}$ is bounded on the sequence space $c_0$ and $0 \notin \sigma(\triangle_{a,b}, c_0)$.

**Theorem 2.8.** *Let $0<q<1$ and $0<\alpha<1$. Then*

$$\sigma\left(C_q^\alpha, c_0\right) = \overline{D} \cup E_m.$$

*Proof.* From Theorem 2.7, we get

$$D \cup \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \dots\right\} \subset \sigma\left(\left[C_q^\alpha\right]^*, \ell^1\right) = \sigma\left(C_q^\alpha, c_0\right).$$

Since

$$\begin{aligned} \overline{D} \cup \overline{\left\{1, \tfrac{1}{1+q}, \tfrac{1}{1+q+q^2}, \dots\right\}} &= \{\lambda \in \mathbb{C} : |\lambda-\beta| \le \beta\alpha q\} \\ &\cup \left\{1, \tfrac{1}{1+q}, \tfrac{1}{1+q+q^2}, \dots\right\} \cup \{1-q\} \\ &= \{\lambda \in \mathbb{C} : |\lambda-\beta| \le \beta\alpha q\} \cup \{C_0, C_1, C_2, \dots\} \cup \{1-q\}, \end{aligned}$$

$\{C_{m+1}, C_{m+2}, \dots\} \in \overline{D}$ and $1-q \in \overline{D}$, we have

$$\{\lambda \in \mathbb{C} : |\lambda-\beta| \le \beta\alpha q\} \cup E_m \subset \sigma\left(C_q^\alpha, c_0\right).$$

Also, since

$$\begin{aligned} |C_m - \beta| \le \beta\alpha q;\ \alpha = q^m \quad &\Leftrightarrow \left|\tfrac{1-q}{1-q^{m+1}} - \tfrac{1-q}{1-\alpha^2 q^2}\right| \le \tfrac{(1-q)\alpha q}{1-\alpha^2 q^2} \\ &\overset{\alpha=q^m}{\Leftrightarrow} (1-q)\left|\tfrac{1}{1-\alpha q} - \tfrac{1}{1-\alpha^2 q^2}\right| \le \tfrac{(1-q)\alpha q}{1-\alpha^2 q^2} \\ &\Leftrightarrow \left|\tfrac{1+\alpha q}{1-\alpha^2 q^2} - \tfrac{1}{1-\alpha^2 q^2}\right| \le \tfrac{\alpha q}{1-\alpha^2 q^2} \\ &\Leftrightarrow \tfrac{\alpha q}{1-\alpha^2 q^2} \le \tfrac{\alpha q}{1-\alpha^2 q^2}, \end{aligned}$$

if $\alpha = q^m$ for a $m \in \mathbb{N}$, it should be noted that $\lambda = C_m$ is the point at the right end of the circle and on the $x$-axis.
From the explanation the above theorem $\left[C_q^\alpha\right]^{-1}$ is invertible and bounded on $c_0$. In [27], It is known that $\sigma\left(\triangle_{a,b}, c_0\right) = \sigma\left(\left[C_q^\alpha\right]^{-1}, c_0\right) = F \cup G$, where $F = \{\lambda \in \mathbb{C} : |\lambda - a| \le |b|\}$ and $G = \{a_m : k \in \mathbb{N}, |a_m - a| > |b|\}$. Since $a = \frac{1}{1-q}$ and $b = -\frac{\alpha q}{1-q}$ from (2.11), we have

$$|\lambda - a| \le |b| \Leftrightarrow \left|\lambda - \frac{1}{1-q}\right| \le \frac{\alpha q}{1-q}$$

and

$$\begin{aligned} |a_m - a| > |b| \quad &\Leftrightarrow \left|\tfrac{1-q^{m+1}}{1-q} - \tfrac{1}{1-q}\right| > \tfrac{\alpha q}{1-q} \\ &\Leftrightarrow \tfrac{q^{m+1}}{1-q} > \tfrac{\alpha q}{1-q} \Leftrightarrow q^{m+1} > \alpha q \\ &\Leftrightarrow \alpha < q^m. \end{aligned}$$

From the spectral maping Theorem, it must be as follows;

$$\begin{aligned} \sigma\left(C_q^\alpha, c_0\right) &= \left\{\tfrac{1}{\lambda} \in \mathbb{C} : \lambda \in \sigma\left(\left[C_q^\alpha\right]^{-1}, c_0\right)\right\} \\ &= \left\{\tfrac{1}{\lambda} \in \mathbb{C} : |\lambda - a| \le |b|\right\} \cup \left\{\tfrac{1}{a_m} : m \in \mathbb{N}, |a_m - a| > |b|\right\}. \\ &= \left\{\tfrac{1}{\lambda} \in \mathbb{C} : \left|\lambda - \tfrac{1}{1-q}\right| \le \tfrac{\alpha q}{1-q}\right\} \cup \left\{\tfrac{1}{a_m} : m \in \mathbb{N}, \alpha < q^m\right\}. \\ &= \left\{\tfrac{1}{\lambda} \in \mathbb{C} : \left|\lambda - \tfrac{1}{1-q}\right| \le \tfrac{\alpha q}{1-q}\right\} \cup E_m. \end{aligned}$$

Let $\mu = \frac{1}{\lambda}$. Since

$$\mu = \frac{1}{\lambda} = x+iy \Leftrightarrow \lambda = \frac{1}{\mu} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u+iv,$$

we have

$$\begin{aligned} \left|\lambda - \tfrac{1}{1-q}\right| \le \tfrac{\alpha q}{1-q} \quad & \Leftrightarrow \left|\tfrac{1}{\mu} - \tfrac{1}{1-q}\right| \le \tfrac{\alpha q}{1-q} \\ & \Leftrightarrow \left|\tfrac{1-q-\mu}{\mu(1-q)}\right| \le \tfrac{\alpha q}{1-q} \Leftrightarrow |\mu-(1-q)| \le \alpha q|\mu| \\ & \Leftrightarrow (u-(1-q))^2 + v^2 \le (\alpha q)^2\left(u^2+v^2\right) \\ & \Leftrightarrow u^2+v^2-2(1-q)u \le (\alpha q)^2\left(u^2+v^2\right) \\ & \Leftrightarrow \left|\mu - \tfrac{1-q}{1-(\alpha q)^2}\right| \le \alpha q \tfrac{(1-q)}{1-(\alpha q)^2} \\ & \Leftrightarrow |\mu-\beta| \le \alpha q \beta. \end{aligned}$$

As a result, we get

$$\sigma\left(C_q^\alpha, c_0\right) \quad = \left\{\lambda \in \mathbb{C} : \left|\lambda - \tfrac{1-q}{1-(\alpha q)^2}\right| \le \tfrac{1-q}{1-(\alpha q)^2}\alpha q\right\} \cup E_m.$$

□

**Remark 2.9.** *In Theorem 2.8, since* $E_m \subset \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| \le \frac{q}{1+q}\right\}$ *as* $\alpha \to 1^-$, *we get*

$$\begin{aligned} \sigma\left(C_1(q), c_0\right) \quad & = \left\{\lambda \in \mathbb{C} : \left|\lambda - \tfrac{1-q}{1-q^2}\right| \le \tfrac{1-q}{1-\alpha q^2}q\right\} \cup E_m \\ & = \left\{\lambda \in \mathbb{C} : \left|\lambda - \tfrac{1}{1+q}\right| \le \tfrac{q}{1+q}\right\}. \end{aligned}$$

This shows that [51, Theorem 2.6] is still valid when as $\alpha \to 1^-$.

## 3. Various Spectral Decompositions

First of all, let's decompose the spectrum into the continuous spectrum, point spectrum and residual spectrum.

### 3.1. Classical decomposition of the spectrum

From Theorem 2.3, if $\alpha < q^m$ then we know that

$$\sigma_p\left(C_q^\alpha, c_0\right) = \{1, C_1, \ldots, C_m\}$$

for $0 < q < 1$ and $0 < \alpha < 1$ where $C_m = \frac{1}{A_m} = \frac{1}{1+q+\cdots+q^m}$.

**Theorem 3.1.** *Let* $0 < q < 1$ *and* $0 < \alpha < 1$. *Then*

$$\sigma_r\left(C_q^\alpha, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - \beta| < \beta\alpha q\}$$

*where* $\beta = \frac{1-q}{1-\alpha^2 q^2}$.

*Proof.* Since $\sigma_r\left(C_q^\alpha, c_0\right) = \sigma_p\left(C_1^*(q), \ell_1\right) \setminus \sigma_p\left(C_q^\alpha, c_0\right)$, it is clear that

$$\sigma_r\left(C_q^\alpha, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - \beta| < \beta\alpha q\}$$

from Theorem 2.3 and Theorem 2.7. □

**Theorem 3.2.** *Let* $0 < q < 1$ *and* $0 < \alpha < 1$. *If* $\alpha = q^m$, *then*

$$\sigma_c\left(C_q^\alpha, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - \beta| = \beta\alpha q\} \setminus \{C_m\}$$

*and if* $\alpha \ne q^m$, *then*

$$\sigma_c\left(C_q^\alpha, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - \beta| = \beta\alpha q\}$$

*where* $\beta = \frac{1-q}{1-\alpha^2 q^2}$.

*Proof.* Since

$$\sigma_c\left(C_q^\alpha, c_0\right) = \sigma\left(C_1(q), c_0\right) \setminus \left\{\sigma_p\left(C_1(q), c_0\right) \cup \sigma_r\left(C_q^\alpha, c_0\right)\right\},$$

the result is clear from Theorem 2.3, Theorem 2.8, Theorem 3.1 and Table 1. □

### 3.2. Goldberg's Classification of Spectrum

Now let's give the Goldberg classification of the spectrum for the $C_q^\alpha$ operator.
Let us give the following Lemma to be used in calculation of $\mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)$.

**Lemma 3.3.** *[32, p.60]A linear operator $T$ has a bounded inverse if and only if $T^*$ is onto.*

**Theorem 3.4.** *Let $0<q<1$ and $0<\alpha<1$. Then*

$$\mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)=\{C_{m+1}, C_{m+2}, \dots\}$$

*where $C_n=\frac{1}{A_n}=\frac{1}{1+q+\cdots+q^n}$ and $\alpha<q^m$.*

*Proof.* Since $C_{m+1}, C_{m+2}, \dots \in \{\lambda \in \mathbb{C} : |\lambda-\beta|<\beta\alpha q\}$ for $\alpha<q^m$, $\{C_{m+1}, C_{m+2}, \dots\} \subseteq \sigma_r\left(C_q^\alpha, c_0\right)$ is obtained from Theorem 3.1. Hence, $\{C_{m+1}, C_{m+2}, \dots\} \in \mathscr{I}\mathscr{I}\mathscr{I}_1\sigma\left(C_q^\alpha, c_0\right) \cup \mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)$ is valid. Let us now show that $\{C_{m+1}, C_{m+2}, \dots\} \in \mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)$. For this we will use Lemma 3.3.
For every $y \in \ell_1$, is there $x \in \ell_1$ such that $\left(C_q^\alpha-\lambda I\right)^* x=y$?
Let $\left(C_q^\alpha-\lambda I\right)^* x=y$. In this case, the equations

$$\begin{aligned}
(1-\lambda)x_0 &= y_0\\
(1-\lambda)x_1+\tfrac{\alpha q}{1+q}x_2+\tfrac{(\alpha q)^2}{1+q+q^2}x_3+\tfrac{(\alpha q)^3}{1+q+q^2+q^3}x_4+\cdots &= y_1\\
\left(\tfrac{1}{1+q}-\lambda\right)x_2+\tfrac{(\alpha q)}{1+q+q^2}x_3+\tfrac{(\alpha q)^2}{1+q+q^2+q^3}x_4+\cdots &= y_2\\
&\vdots
\end{aligned}$$

are valid. Let's consider these equations sequentially, two by two. If we subtract multiple $\alpha q$ of the lower equation from the upper equation and subtract $x_n$, then we get

$$\begin{aligned}
x_n \;=\; & \frac{x_0}{(\alpha q\lambda)^{n-1}}\prod_{i=0}^{n-1}(\lambda-C_i)+\frac{y_0}{(\lambda q\alpha)^{n-1}}\prod_{i=1}^{n-1}(\lambda-C_i)\\
&+\sum_{i=1}^{n-1}\frac{1}{\lambda A_i}\frac{y_i}{(q\alpha)^{n-i}}\prod_{v=i+1}^{n-1}(\lambda-C_v)-\frac{1}{\lambda}y_{n+1}.
\end{aligned} \tag{3.1}$$

where $A_i=\sum_{k=0}^{i}q^k$ and $C_v=\frac{1}{A_v}$. If $\lambda \in \{C_{m+1}, C_{m+2}, \dots, C_n\}$, then $x_n=-\frac{1}{\lambda}y_n$ with $x \in \ell_1$. Hence, we get $(y_n) \in \ell^1$ because $(y_n) \in \ell^1$. As a result, the operator $\left(C_q^\alpha-\lambda I\right)^*$ is surjective. So, from Lemma 3.3, the operator $C_q^\alpha-\lambda I$ for $\lambda \in \{C_{m+1}, C_{m+2}, \dots\}$ has bounded inverse and , so we have

$$\{C_{m+1}, C_{m+2}, \dots\} \subset \mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)$$

$\{C_{m+1}, C_{m+2}, \dots\}$ inclusion.
Now let $\lambda \notin \{C_{m+1}, C_{m+2}, \dots\}$. In Table 1, since $C_0$, $C_1$, … , $C_m \notin \mathscr{I}\mathscr{I}\mathscr{I}$, we can assume that $\lambda \neq C_n$ for every $n \in \mathbb{N}$. In this case, considering that $y \in \ell_1$ from (3.1), sequence of $(x_n)$ is convergent if and only if the product

$$\prod_{v=0}^{\infty}\left(\lambda-\frac{1}{\sum\limits_{k=0}^{v}q^k}\right)$$

is convergent. If the infinity product is convergent, the limit of the general term is 1, so it must be

$$\lim_{v\to\infty}\left(\lambda-\frac{1}{\sum\limits_{k=0}^{v}q^k}\right)=\lim_{v\to\infty}\lambda-\frac{1-q}{1-q^{v+1}}=\lambda-(1-q)=1,$$

that is, $\lambda=2-q$. If $\lambda\neq 2-q$, then the infinite product becomes divergent. Hence, if $\lambda \neq 2-q$, then $x \notin \ell_1$. That is, if

$$\lambda \notin \{C_{m+1}, C_{m+2}, \dots\} \cup \{2-q\},$$

the operator $\left(C_q^\alpha-\alpha I\right)^*$ is not onto. So from Lemma 3.3, $C_q^\alpha-\alpha I$ has not bounded inverse for $\lambda \notin \{C_{m+1}, C_{m+2}, \dots\} \cup \{2-q\}$. Therefore, we have

$$\{C_{m+1}, C_{m+2}, \dots\} \subseteq \mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right) \subseteq \{C_{m+1}, C_{m+2}, \dots\} \cup \{2-q\}.$$

Let's assume that $\lambda=2-q$. From (3.1), the first component of sequence $(x_n)$ which is the first component

$$\frac{x_0}{(\alpha q(2-q))^n}\prod_{v=0}^{n-1}(\lambda-C_v)$$

go to zero even if the infinite product bounded, $\frac{1}{(\alpha q(2-q))^n} \to \infty$ is necessary. However, we know that $0<\alpha,q<1$, so $0<\alpha q(2-q)<1$ is valid. From here, since $(\alpha q(2-q))^n \to 0$ for $\lambda = 2-q$, $x \notin c_0$ and thus $x \notin \ell^1$ are obtained. This means that the operator $\left(C_q^\alpha - \lambda I\right)^*$ is not surjective for $\lambda = 2$. Hence, with Lemma 3.3, the operator $C_q^\alpha - \lambda I$ has no bounded inverse for $\lambda = 2-q$. Thus

$$\mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right) = \{C_{m+1}, C_{m+2}, \dots\}$$

is obtained. □

**Corollary 3.5.** *Let $0<q<1$ and $0<\alpha<1$. Then*

$$\mathscr{I}\mathscr{I}\mathscr{I}_1\sigma\left(C_q^\alpha, c_0\right) = D \setminus \{C_{m+1}, C_{m+2}, \dots\}$$

*where $C_n = \frac{1}{A_n} = \frac{1}{1+q+\cdots+q^n}$ and $\alpha < q^m$*

*Proof.* We know from Table 1 that $\sigma_r(C_q^\alpha, c_0) = \mathscr{I}\mathscr{I}\mathscr{I}_1\sigma\left(C_q^\alpha, c_0\right) \cup \mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)$. Now, if we consider Theorem 3.1 and Theorem 3.4, it is seen that $\mathscr{I}\mathscr{I}\mathscr{I}_1\sigma\left(C_q^\alpha, c_0\right) = D \setminus \{C_{m+1}, C_{m+2}, \dots\}$. □

**Corollary 3.6.** *Let $0<q<1$ and $0<\alpha<1$. If $\alpha = q^m$, then*

$$\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - \beta| = \beta\alpha q\} \setminus \{C_m\}$$

*and if $\alpha \neq q^m$, then*

$$\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - \beta| = \beta\alpha q\}$$

*where $\beta = \frac{1-q}{1-\alpha^2 q^2}$.*

*Proof.* The proof is clear from Theorem 3.2. □

Let us give the following Lemma to be used in calculation of $\mathscr{I}\mathscr{I}\mathscr{I}_2\sigma\left(C_q^\alpha, c_0\right)$.

**Lemma 3.7.** *[32, Theorem II 3.7]A linear operator $T$ has a dense range if and only if the adjoint operator $T^*$ is one-to-one.*

**Theorem 3.8.** *Let $0<q<1$ and $0<\alpha<1$. Then*

$$\mathscr{I}\mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) = \{C_1, C_2, \dots, C_m\}$$

*where $C_n = \frac{1}{A_n} = \frac{1}{1+q+\cdots+q^n}$ and $\alpha < q^m$.*

*Proof.* We know from Table 1 and Theorem 2.3 that

$$\begin{aligned}\sigma_p\left(C_q^\alpha, c_0\right) &= \mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) \cup \mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) \cup \mathscr{I}\mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right)\\ &= \{C_1, C_2, \dots, C_m\}\end{aligned}$$

for $\alpha < q^m$. Let $\left(C_q^\alpha - I\right)^* x = \theta$ and $x_0 = 1$. Thus, we get

$$\begin{aligned} x_1 &= \frac{1}{\lambda\alpha q}(\lambda - 1)\\ x_2 &= \frac{1}{(\lambda\alpha q)^2}(\lambda - 1)\left(\lambda - \frac{1}{1+q}\right)\\ &\vdots\\ x_m &= \frac{1}{(\lambda\alpha q)^m}\prod_{k=0}^{m}(\lambda - C_k).\end{aligned}$$

From the expressions

$$\begin{aligned} \lambda = C_1 = \frac{1}{1+q} \quad &\Rightarrow \quad \left(C_q^\alpha - I\right)^* x^1 = 0\\ &\qquad \text{where } x^1 = \left(1, \frac{1}{\alpha q C_1}(C_1 - 1), 0, \dots\right) \neq \theta\\ \lambda = C_2 = \frac{1}{1+q+q^2} \quad &\Rightarrow \quad \left(C_q^\alpha - I\right)^* x^2 = 0 \end{aligned}$$

where

$$x^2 = \left(1, \frac{1}{\alpha q C_2}(C_2 - 1), \frac{1}{(C_2\alpha q)^2}(C_2 - 1)\left(C_2 - \frac{1}{1+q}\right), 0, \dots\right) \neq \theta$$

$$\vdots$$

$$\lambda = C_m = \frac{1}{\sum_{k=0}^{m} q^k} \quad \Rightarrow \quad \left(C_q^\alpha - I\right)^* x^m = 0$$

where

$$x^m = \left(1, \frac{1}{C_m\alpha q}(C_m - 1), \frac{1}{(C_m\alpha q)^2}(C_m - 1)(C_m - \frac{1}{1+q}), \dots, \frac{1}{(C_m\alpha q)^2}\prod_{k=0}^{m}(C_m - C_k), 0, \dots\right) \neq \theta,$$

there is $\theta \neq x \in \ell^1$ such that $\left(C_q^\alpha - \lambda I\right)^* x = 0$ for $\lambda \in \{C_1, C_2, \dots, C_m\}$. Thus $\left(C_q^\alpha - \lambda I\right)^*$ is not 1:1 for $\lambda \in \{C_1, C_2, \dots, C_m\}$. From Lemma 1, the operator $C_q^\alpha - \lambda I$ does not have a dense image for $\lambda \in \{C_1, C_2, \dots, C_m\}$. Consequently, we have $\mathscr{I}\mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) = \{C_1, C_2, \dots, C_m\}$. □

**Corollary 3.9.** *Let $0<q<1$ and $0<\alpha<1$. $\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) = \mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) = \emptyset$.*

*Proof.* Since $\sigma_p\left(C_q^\alpha, c_0\right) = \mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) \cup \mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right) \cup \mathscr{I}\mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha, c_0\right)$ from Table 1, the required result is obtained from Theorems 2.3 and 3.8. □

### 3.3. Non-discrete Spectrum of the spectrum (defect spectrum, approximate point spectrum, compression spectrum)

Now, let's determine the defect spectrum, the approximate point spectrum, the compression spectrum of the operator $C_q^\alpha$ using Table 2.

**Theorem 3.10.** *Let* $0<q<1$ *and* $0<\alpha<1$. *The following expressions yield;*
*(a)* $\sigma_{ap}\left(C_q^\alpha,c_0\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|=\frac{(1-q)\alpha q}{1-\alpha^2q^2}\right\}\cup\{C_{m+1},C_{m+2},\dots\}$, *if* $\alpha<q^m$,
*(b)* $\sigma_\delta\left(C_q^\alpha,c_0\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|\le\frac{(1-q)\alpha q}{1-\alpha^2q^2}\right\}\cup\{C_0,C_1,\dots,C_m\}$ *if* $\alpha<q^m$,
*(c)* $\sigma_{co}\left(C_q^\alpha,c_0\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|<\frac{(1-q)\alpha q}{1-\alpha^2q^2}\right\}\cup\{C_0,C_1,\dots,C_m\}$ *if* $\alpha<q^m$.

*Proof.* (a) We know that the following expression exists from Table 2;

$$\sigma_{ap}\left(C_q^\alpha,c_0\right)=\sigma\left(C_q^\alpha,c_0\right)\setminus\mathscr{I}\mathscr{I}\mathscr{I}_1\sigma\left(C_q^\alpha,c_0\right). \tag{3.2}$$

The desired result is obtained by using the above expression (3.2), Theorem 2.8 and Corollary 3.5.
(b) We know that the following expression exists, from Theorem 3.4 and Table 2;

$$\sigma_\delta\left(C_q^\alpha,c_0\right)=\sigma\left(C_q^\alpha,c_0\right)\setminus\mathscr{I}_1\sigma\left(C_q^\alpha,c_0\right). \tag{3.3}$$

The desired result is obtained by using the above expression (3.3), Theorem 2.8 and Corollary 3.9.
(c) We know that the following expression exists from Table 2;

$$\sigma_{co}\left(C_q^\alpha,c_0\right)=\sigma_r\left(C_q^\alpha,c_0\right)\cup\mathscr{I}\mathscr{I}\mathscr{I}_3\sigma\left(C_q^\alpha,c_0\right) \tag{3.4}$$

The desired result is obtained by using the above expression (3.4), Theorem 3.1 and Theorem 3.8. □

**Corollary 3.11.** *Let* $0<q<1$ *and* $0<\alpha<1$.
*(a)* $\sigma_{ap}\left(\left[C_q^\alpha\right]^*,\ell_1\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|\le\frac{(1-q)\alpha q}{1-\alpha^2q^2}\right\}\cup\{C_0,C_1,\dots,C_m\}$ *if* $\alpha<q^m$,
*(b)*

$$\begin{aligned}\sigma_\delta\left(C_1^{\alpha*}(q),\ell_1\right)&=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|=\frac{(1-q)\alpha q}{1-\alpha^2q^2}\right\}\\&\cup\{C_{m+1},C_{m+2},\dots\},\ \text{if } \alpha<q^m.\end{aligned}$$

*Proof.* We know from [8] that

$$\sigma_{ap}(\left[C_q^\alpha\right]^*,\ell_1)=\sigma_\delta(C_q^\alpha,c_0)$$

and

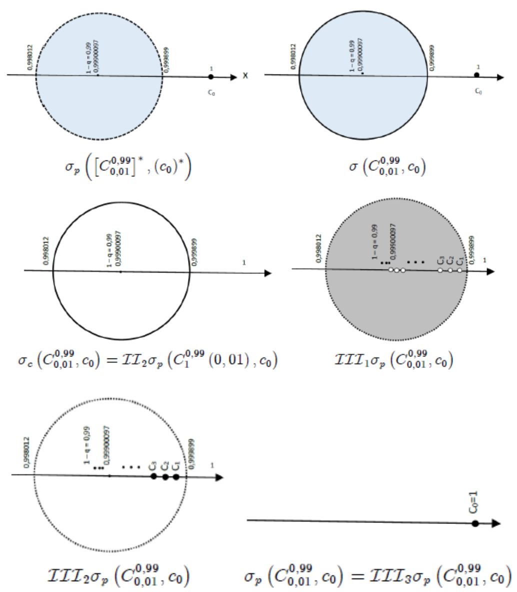
$$\sigma_\delta(\left[C_q^\alpha\right]^*,\ell_1)=\sigma_{ap}(C_q^\alpha,c_0).$$

By using Theorem 3.10, we conclude the proof. □

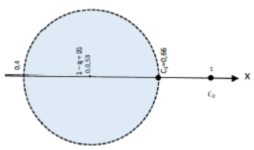
## 4. Applications

In the examples below, we will assign values for $0<q<1$ and $0<\alpha<1$. We will try to show the spectrum and spectral decompositions of $C_q^\alpha$ corresponding to these values with the figure.

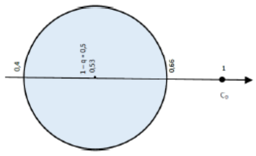
**Example 4.1.** $\sigma_p\left(C_{0,01}^{0,99},c_0\right)=\{1\}$, $\sigma\left(C_{0,01}^{0,99},c_0\right)=\{\lambda\in\mathbb{C}:|\lambda-0,99900097|\le 0,000989\}\cup\{1\}$,



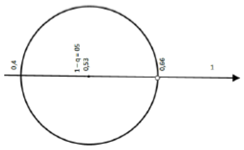
**Example 4.2.** $\sigma_p\left(C_{0,5}^{0,5}, c_0\right) = \{1, 0.66\}$, $\sigma\left(C_{0,5}^{0,5}, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - 0,99900097| \leq 0,000989\} \cup \{1\}$,



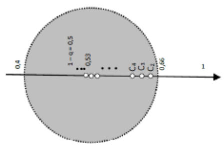
$\sigma_p\left(\left[C_{0,5}^{0,5}\right]^*, (c_0)^*\right)$



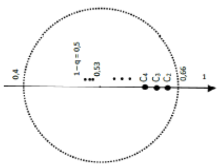
$\sigma\left(C_{0,5}^{0,5}, c_0\right)$



$\sigma_c\left(C_{0,5}^{0,5}, c_0\right) = II_2\sigma_p\left(C_1^{0,5}(0,5), c_0\right)$



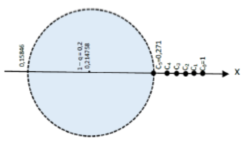
$III_1\sigma_p\left(C_{0,5}^{0,5}, c_0\right)$



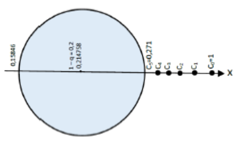
$III_2\sigma_p\left(C_{0,5}^{0,5}, c_0\right)$

$\sigma_p\left(C_{0,5}^{0,5}, c_0\right) = III_3\sigma_p\left(C_{0,5}^{0,5}, c_0\right)$

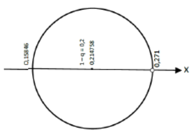
**Example 4.3.** $\alpha = q^5$



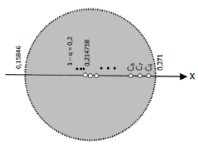
$\sigma_p\left(\left[C_{0,8}^{0,32768}\right]^*, (c_0)^*\right)$



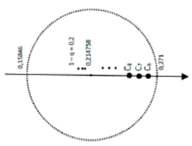
$\sigma\left(C_{0,8}^{0,32768}, c_0\right)$



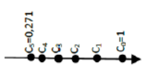
$\sigma_c\left(C_{0,8}^{0,32768}, c_0\right) = II_2\sigma_p\left(C_1^{0,32768}(0,8), c_0\right)$



$III_1\sigma_p\left(C_{0,8}^{0,32768}, c_0\right)$



$III_2\sigma_p\left(C_{0,8}^{0,32768}, c_0\right)$



$\sigma_p\left(C_{0,8}^{0,32768}, c_0\right) = III_3\sigma_p\left(C_{0,8}^{0,32768}, c_0\right)$

## 5. Discussion and Conclusion

The spectra of Cesàro operators and generalizations of this operator with different techniques have been discussed by various authors. Their properties, characterization and examination of several sequence spaces has an extensive literature (See [2]-[6], [13]-[16],[40]-[44], [49], [50]).
For example, in the following theorems, spectral decompositions of the generalized $q-$Cesàro operator in various sequence spaces are investigated.

**Theorem 5.1.** *[24]Let $0 < q < 1$ and $0 < \alpha < 1$. Then $C_q : c \to c$ is a bounded operator with $\|C_q\|_c = 1$ and*
*1. $\sigma\left(C_q, c\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| \leq \frac{q}{1+q}\right\}$.*
*2. $\sigma_p\left(C_q, c\right) = \{1\}$.*
*3. $\sigma_r\left(C_q, c\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| < \frac{q}{1+q}\right\} \cup \{1\}$.*
*4. $\sigma_c\left(C_q, c\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| = \frac{q}{1+q}\right\} \setminus \{1\}$.*

**Theorem 5.2.** *[51]Let $0 < q < 1$ and $0 < \alpha < 1$. Then $C_q : c \to c$ is a bounded operator with $\|C_q\|_{c_0} = 1$ and*
*1. $\sigma\left(C_q, c_0\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| \leq \frac{q}{1+q}\right\}$.*
*2. $\sigma_p\left(C_q, c_0\right) = \emptyset$.*

*3.* $\sigma_r\left(C_q, c_0\right)=\left\{\lambda \in \mathbb{C}:\left|\lambda-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\}$.

*4.* $\sigma_c\left(C_q, c_0\right)=\left\{\lambda \in \mathbb{C}:\left|\lambda-\frac{1}{1+q}\right|=\frac{q}{1+q}\right\} \backslash\{1\}$.

**Theorem 5.3.** *[29]Let $0<q<1$ and $0<\alpha<1$. Then $C_q: \ell_p \rightarrow \ell_p$ is a bounded operator and*
*1.* $\sigma\left(C_q, \ell_p\right)=\left\{\lambda \in \mathbb{C}:\left|\lambda-\frac{1}{1+q}\right| \leq \frac{q}{1+q}\right\}$.
*2.* $\sigma_p\left(C_q, \ell_p\right)=\emptyset$.

*3.* $\sigma_r\left(C_q, \ell_p\right)=\left\{\lambda \in \mathbb{C}:\left|\lambda-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\} \cup\{1\}$.

*4.* $\sigma_c\left(C_q, \ell_p\right)=\left\{\lambda \in \mathbb{C}:\left|\lambda-\frac{1}{1+q}\right|=\frac{q}{1+q}\right\} \backslash\{1\}$.

The theorems mentioned above have been a source of motivation for this study. The purpose of this article is to examine various spectral decompositions of $C_q^\alpha=\left(c_{nk}^\alpha(q)\right)$ such as the spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum, and the compression spectrum on the sequence space $c_0$. The results are original and may inspire various studies on this subject.

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## Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

## References

[1] A.M. Akhmedov and S.R. El-Shabrawy, Spectra and fine spectra of lower triangular double-band matrices as operators on $\ell_p$ $(1 \leq p<\infty)$, Math. Slovaca, 65 (5) (2015), 1137–1152.
[2] A.M. Akhmedov and F. Başar, On the fine spectrum of the Cesàro operator in $c_0$, Math. J. Ibaraki Univ., 36 (2004), 25-32.
[3] A.M. Akhmedov and F. Başar, On the fine spectra of the difference operator $\Delta$ over the sequence space $l_p$, $(1 \leq p<\infty)$, Demonstratio Math., 39 (3) (2006), 585-595.
[4] A.M. Akhmedov and F. Başar, The fine spectra of the Cesàro operator $C_1$ over the sequence space $bv_p$, $(1 \leq p<\infty))$, Math. J. Okayama Univ., 50 (2008), 135-147.
[5] A.M. Akhmedov and F. Başar, The fine spectra of the difference operator $\Delta$ over the sequence space $bv_p$, $(1 \leq p<\infty))$, Acta Math. Sin. Eng. Ser., 23 (10) (2007), 1757-1768.
[6] B. Altay and F. Başar, The fine spectrum and the matrix domain of the difference operator on the sequence space $l_p$, $(0<p<1)$, Commun. Math. Anal., 2 (2) (2007), 1-11.
[7] RKh.Amirov, N. Durna and M. Yildirim, Subdivisions of the spectra for Cesàro, Rhaly and Weighted mean operator on $\ell_p$, $c$ and $\ell_p$, IJST, A3 (2011), 175-183.
[8] J. Appell, E. De Pascale and A. Vignoli, Nonlinear Spectral Theory. Berlin, New York, Walter de Gruyter, 2004.
[9] F. Başar, N. Durna and M. Yildirim, Subdivisions of the Spectra for Generalized Difference Operator $\Delta_v$ on the Sequence Space $\ell_1$, ICMS., 1309 (2010), 254-260.
[10] F. Başar, N. Durna and M. Yildirim, Subdivisions of the spectra for the triple band matrix over certain sequence spaces, Gen. Math. Notes, 4 (1) (2011), 35-48.
[11] F. Başar, N. Durna and M. Yildirim, Subdivisions of the spectra for genarilized difference operator over certain sequence spaces, Thai J. Math., 9 (1) (2011), 285-295.
[12] F. Başar, N. Durna and M. Yildirim, Subdivision of the spectra for difference operator over certain sequence space, Malays. J. Math. Sci., 6 (2012), 151-165.
[13] F. Başar, Summability Theory and its Applications, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton London New York, (in press) 2022.
[14] A. Brown, P.R. Halmos and A.L. Shields, Cesàro operators. Acta Sci. Math., (Szeged). 26 (1965), 125–137.
[15] D.W. Boyd, The spectrum of the Cesàro operator, Acta Sci. Math., (Szeged). 29 (1968), 31–34.
[16] G.P. Curbera and W.J. Ricker, Spectrum of the Cesàro operator in $\ell_p$, Arch. Math., 100 (2013), 267–271.
[17] N. Durna and M.cYildirim, Subdivision of the spectra for factorable matrices on $c$ and $\ell_p$, Math. Commun., 16 (2) (2011), 519-530.
[18] N. Durna and M. Yildirim, Subdivision of the Spectra for Factorable Matrices on $c_0$, Gazi Univ. J. Sci., 24 (1) (2011), 45-49.
[19] N. Durna, M. Yildirim and Ç. Ünal, On The Fine Spectrum of Generalized Lower Triangular Double Band Matrices $\Delta_{uv}$ Over The Sequence Space $c_0$, Cumhuriyet Sci. J., 37 (3) (2016), 281-291.
[20] N. Durna, Subdivision of the spectra for the generalized upper triangular double-band matrices $\Delta^{uv}$ over the sequence spaces $c_0$ and $c$, ADYUSCI, 6 (1) (2016), 31-43.
[21] N. Durna, Subdivision of the spectra for the generalized difference operator $\Delta_{a,b}$ on the sequence space $\ell_p$ $(1<p<\infty)$, CBU J. Sci., 13 (2) (2017), 359–364.
[22] N. Durna, M. Yildirim and R. Kılıç, Partition of the spectra for the generalized difference operator $B(r,s)$ on the sequence space $cs$, Cumhuriyet Sci. J., 39 (1) (2018), 7–15.
[23] N. Durna, Subdivision of spectra for some lower triangular doule-band matrices as operators on $c_0$, Ukr. Mat. Zh., 70 (7) (2018), 913–922.

[24] N. Durna and M.E. Türkay, The spectrum of $q$-Cesàro matrices on $c$ and Its various spectral decomposition for $0 < q < 1$, Oper. Matrices, 15 (3) (2020), 795-813.
[25] E. Dündar and F. Başar, On the fine spectrum of the upper triangle double band matrix $\Delta^+$ on the sequence space $c_0$, Math. Commun., 18 (2) (2013), 337–348.
[26] S.R. El-Shabrawy, On the fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence space $\ell_p$, $(1 < p < \infty)$, Appl. Math. Inf. Sci., 6 (1S) (2012), 111–118.
[27] S.R. El-Shabrawy, Spectra and fine spectra of certain lower triangular double-band matrices as operators on $c_0$, J. Inequal. Appl., 2014 (1) (2014), 1-9.
[28] S.R. El-Shabrawy and S.H. Abu-Janah, Spectra of the generalized difference operator on the sequence spaces $bv_0$ and $h$, Linear and Multilinear Algebra 66 (8) (2018), 1691–1708.
[29] S.R. El-Shabrawy, On $q$-Cesàro Operators:Boundness, Compactness and Spectra, Numer.Funct.Anal.Optim., 41 (2) (2021), 1019-1037.
[30] J. Fathi and L. Rahmatollah, On the fine spectra of the generalized difference operator $\Delta_{uv}$ over the sequence space $\ell_p$, JMMRC, 1 (1) (2012), 1-12.
[31] H. Furkan, H. Bilgiç and F. Başar, On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces $\ell_p$ and $bv_p$, $(1 < p < \infty)$, Comput. Math. Appl., 60 (7) (2010), 2141–2152.
[32] S. Goldberg, Unbounded Linear Operators, McGraw Hill, New York, 1966.
[33] M. Gonzalez, The fine spectrum of Cesàro operator in $\ell_p$ $(1 < p < \infty)$, Arch. Math., 44 (1985), 355-358.
[34] V. Karakaya and M. Altun, Fine spectra of upper triangular double-band matrices, J. Comput. Appl. Math., 234 (2010), 1387–1394.
[35] V. Karakaya, M.D. Manafov and N. Şimşek, On the fine spectrum of the second order difference operator over the sequence spaces $\ell_p$ and bvp, $(1 < p < \infty)$. Math. Comput. Modelling, 55 (3-4) (2012), 426–431.
[36] V. Karakaya, M.D. Manafov and N. Şimşek, On fine spectra and subspectrum (approximate point, defect and compression) of operator with periodic coefficients, J. Nonlinear Convex Anal., 18 (4) (2017), 709–717.
[37] E. Kreyszing, Introductory Functional Analysis with Applications, John Wiley & Sons Inc., New York Chichester Brisbane Toronto, 1978.
[38] I. J. Maddox, Elements of Functional Analysis, Cambridge University Press, 1970.
[39] M. Mursaleen, M. Yildirim and N. Durna, On the spectrum and Hilbert Schimidt properties of generalized Rhaly matrices. Commun. Fac. Sci. Univ. Ank. Series A1, 68 (1) (2019),712–723.
[40] M. Mursaleen and F. Başar, Sequence Spaces: Topics in Modern Summability Theory, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton London New York, 2020.
[41] J.I. Okutoyi, On the spectrum of Cesàro operator, P.h.D. Thesis Birmingham University, 1986.
[42] J.I. Okutoyi, On the spectrum of $C_1$ as an operator on $bv_0$, J. Aust. Math. Soc., A 48 (1) (1990), 79–86.
[43] J.T. Okutoyi, On the spectrum of $C_1$ as an operator on $bv$, Commun. Fac. Sci. Univ. Ank. Series A1, 41 (1992), 197–207.
[44] J.B. Reade, On the spectrum of the Cesàro operator, Bull. Lond. Math. Soc., 17 (3) (1985), 263–267.
[45] B.E. Rhoades and M. Yildirim, Spectra and fine spectra for factorable matrices, Integral Equations Operator Theory, 53 (1) (2005), 127–144.
[46] B.E. Rhoades, The fine spectra for weighted mean operators in $B(\ell_p)$, Integral Equations Operator Theory, 12 (1) (1989), 82–98.
[47] B.C. Tripathy and R. Das, Fine spectrum of the upper triangular matrix $U(r,0,0,s)$ over the sequence spaces $c_\mathbf{0}$ and $c$, Proyecciones, 37 (1) (2018), 85–101.
[48] T. Yaying, B. Hazarika and M. Mursaleen, On sequence space derived by the domain of q Cesàro matrix in $\ell_p$ space and the associated operator ideal, J. Math. Anal. Appl., 493 (1) (2021), 124453.
[49] M. Yeşilkayagil and F. Başar, On the ne spectrum of the operator defined by a lambda matrix over the sequence spaces of null and convergent sequences, Abstr. Appl. Anal., 2013, Article ID 687393 (2013), 13 pages.
[50] M. Yeşilkayagil and F. Başar, A survey for the spectrum of triangles over sequence spaces, Numer. Funct. Anal. Optim., 40 (16) (2019), 1898-1917.
[51] M.E. Yildirim, The spectrum and fine spectrum of $q-$Cesàro matrices with $0 < q < 1$ on $c_0$, Numer. Func.Anal. Optim., 41 (3) (2020), 361–377.
[52] M. Yildirim, The spectrum and fine spectrum of the compact Rhaly operator, Indian J. Pure Appl. Math., 27 (8) (1996), 779-784.
[53] M. Yildirim, The spectrum of Rhaly operator on $\ell_p$, Indian J. Pure Appl. Math., 32 (2) (2001), 191-198.
[54] M. Yildirim and N. Durna, The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on $\ell_p$, $(1 < p < \infty)$, J. Inequal. Appl. 193 (2017), 1–13.
[55] M. Yildirim, M. Mursaleen and Ç. Doğan, The Spectrum and fine spectrum of generalized Rhaly-Cesàro matrices on $c_0$ and $c$, Oper. Matrices, 12 (4) (2018), 955–975.
[56] R.B. Wenger, The fine spectra of the Hölder summability operator, Indian J. Pure Appl. Math., 6 (6) (1975), 695–712.
[57] A. Wilansky, Summability Through Functional Analysis, North Holland, 1984.