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Holomorphically Planar Conformal Vector Field On Almost $\alpha-$Cosymplectic $(\kappa,\mu)-$ Spaces

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Abstract

The aim of the present paper is to study holomorphically planar conformal vector (HPCV) fields on almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces. This is done assuming various conditions such as i) U is pointwise collinear with ξ (in this case, the integral manifold of the distribution D is totally geodesic, or totally umbilical), ii) M has a constant $\xi-$sectional curvature (under this condition the integral manifold of the distribution D is totally geodesic (or totally umbilical) or the manifold is isometric to sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$), iii) M an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces (in this case the manifold has constant curvature, or the integral manifold of the distribution D is totally geodesic(or totally umbilical) or U is an eigenvector of h).

1. Introduction

Killing vector fields are of great importance in terms of having an impact on the the geometry in addition to the topology of Riemannian manifolds and being incompressible fields has a significant role in physics. The importance of Killing vector fields in Riemannian geometry is associated with the fact that the flows preserve the given metric and determine the symmetry degree of the manifold. Also, in terms of physics, Killing vectors allow the energy and momentum of a freely moving particle to be conserved in flat space- times. In a general manner, special vector fields such as Killing vector fields are conformal vector fields of which flow maintains a conformal class of metrics. A vector field V satisfying $\pounds_V g = 2fg$ on a Riemannian manifold (M,g) is said to be a conformal vector field or conformal transformation on M, where $\pounds$ denotes the Lie derivative on M and f is a smooth function. If f is constant, then V is called homothetic. Also, it has been stated that if the metrically associated $1-$form of V is closed, it is described as closed. V is named as the gradient conformal vector field in case of the fact that the conformal vector field V is the gradient of any differentiable function. The conformal vector fields have been carried out in numerous studies ([1]- [3]).
As a result of these studies, Sharma [4] introduced a holomorphically planar conformal vector (HPCV) field U as a generalization of a closed conformal vector field on an almost Hermitian manifolds. Then, Ghosh–Sharma [5] extensively studied this concept in various conditions. Later, in [6], HPCV fields studied on contact metric manifolds and Einstein contact metric manifolds under some curvature conditions. An HPCV field on a contact metric manifold refers to a vector field U on (M,φ,ζ,η) which satisfies

$$\nabla_X U = aX + b\varphi X \tag{1.1}$$

for arbitrary $X\in\chi(M)$, where a and b are smooth functions on M.

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It must be considered that another class of almost contact manifold, named almost cosymplectic manifold, has gained much attention in various studies [7]. The notion was introduced in [8] as an almost contact metric manifold of which the fundamental $2-$form Φ and $1-$form η are closed. It has been reported that if the almost contact structure is normal then the manifold is called cosymplectic (in the present case the term "cosymplectic" has been adopted to refer to the term "coKähler" in [7]). Also, Endo [9] defined almost cosymplectic $(\kappa,\mu)-$spaces that the curvature tensor of the manifold satisfies

$$R(X,Y)\zeta = \kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY) \tag{1.2}$$

for any $X,Y\in\chi(M)$, where κ,μ are constant and $h=\frac{1}{2}\pounds_{\zeta}\varphi$. On the other hand, Kenmotsu [10] defined the almost Kenmotsu manifold, an almost contact manifold satisfying $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$. According to this definition, Kim [11] introduced the notion of almost $\alpha-$cosymplectic manifold, referring to an almost contact manifold satisfying $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$ (α is real constant). Aktan et al. [12] carried out an extensive study on almost $\alpha-$cosymplectic $(\kappa,\mu,\upsilon)-$spaces and revealed some outcomes of substantial importance. In recent studies numerous studies were carried out on this subject (cf. [13]-[17]). In the light of these studies, the aim of the paper is to study the HPCV fields on almost $\alpha-$cosymplectic manifolds and almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces. Firstly, we give some basic definitions and properties of such structures. In main section, we consider an almost $\alpha-$cosymplectic and almost cosymplectic $(\kappa,\mu)-$ spaces admits a non zero HPCV field U. This is done assuming various conditions such as i) U is pointwise collinear with ζ (in this case the integral manifold of the distribution D is totally geodesic or totally umbilical), ii) M has a constant $\zeta-$sectional curvature (under this condition the integral manifold of the distribution D is totally geodesic (or totally umbilical) or the manifold is isometric to sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$), iii) M an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces (i this case the manifold is constant negative curvature or the integral manifold of the distribution D is totally geodesic(or totally umbilical) or U is an eigenvector of h).

2. Preliminaries

Let M be a $(2n+1)-$dimensional smooth manifold. An almost contact structure on M is a triple (φ,ζ,η) which carries a field φ of endomorphisms of the tangent spaces, a vector field ζ, called characteristic vector field, and a $1-$form η satisfying

$$\varphi^2=-I+\eta\otimes\zeta,\ \eta(\varphi)=0,\ \varphi(\zeta)=0. \tag{2.1}$$

A smooth manifold with such a structure is called an almost contact manifold. It is known that any almost contact manifold M admits a Riemannian metric g satisfying

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),\ g(X,\zeta)=\eta(X),$$

then g is called compatible metric with the structure. Then the manifold (M,φ,ζ,η,g) is called an almost contact metric manifold. An almost contact structure (φ,ζ,η) is said to be normal if the Nijenheus tensor of φ vanishes identically. The fundamental $2-$form Φ on M is defined by $\Phi(X,Y)=g(\varphi X,Y)$ for any vector fields $X,\ Y\in\chi(M)$.
An almost $\alpha-$cosymplectic manifold is an almost contact metric manifold defined by $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$, for any real number α . If $\alpha=0$ then the manifold reduce to almost cosymplectic manifold. Furthermore, normal almost $\alpha-$cosymplectic manifold is called $\alpha-$cosymplectic manifold (For more detail [11], [12]).
Let M be an almost $\alpha-$cosymplectic manifold and $D=\{X:\eta(X)=0\}$ which denote the distribution orthogonal to ζ. Since the $1-$form is closed, then we have $\pounds_{\zeta}\eta=0$ and $[X,\zeta]\in D$ for any $X\in D$. The Levi-Civita connection satisfies $\nabla_{\zeta}\zeta=0$ and $\nabla_{\zeta}\varphi\in D$, which implies that $\nabla_{\zeta}X\in D$ for any $X\in D$.
In addition, an almost $\alpha-$cosymplectic manifold satisfies the following equations [11]:

$$h\zeta=0,\ g(hX,Y)=g(X,hY),\ trace(h)=0,\ \varphi h+h\varphi=0,$$

$$\nabla_X\zeta=-\alpha\varphi^2X-\varphi hX=-A,$$

$$(\nabla_X\varphi)Y+(\nabla_{\varphi X}\varphi)\varphi Y=-\alpha[\eta(Y)\varphi X+2g(X,\varphi Y)\zeta]-\eta(Y)hX, \tag{2.2}$$

$$(\nabla_X\eta)Y=\alpha[g(X,Y)-\eta(X)\eta(Y)]+g(\varphi Y,hX), \tag{2.3}$$

$$tr(A\varphi)=tr(\varphi A)=0,\ tr(h\varphi)=tr(\varphi h)=0,$$

$$tr(A)=-2\alpha n,\ tr(h)=0,$$

for any vector fields $X,Y\in\chi(M)$.

3. Holomorphically planar conformal vector fields on almost $\alpha-$cosymplectic manifolds

In this part, we study almost $\alpha-$cosymplectic manifolds with respect to a HPCV field U. We first state and prove the following lemma for the our main theorem.

Lemma 3.1. *Let M be an almost $\alpha-$cosymplectic manifold with respect to a HPCV field U.Then*

$$\varphi U(a) = -U(b) + (\zeta(b) + 2\alpha nb)\eta(U) \tag{3.1}$$

holds on M.

Proof. Using equation (1.1) in the formula Riemannian curvature tensor $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, we have

$$R(X,Y)U = X(a)Y - Y(a)X + X(b)\varphi Y - Y(b)\varphi X + b\left[(\nabla_X\varphi)Y - (\nabla_Y\varphi)X\right]. \tag{3.2}$$

Replacement of X with φX and Y with φY in the previous equation yields

$$\begin{aligned} R(\varphi X, \varphi Y)U &= \varphi X(a)\varphi Y - \varphi Y(a)\varphi X - \varphi X(b)Y + \varphi X(b)\eta(Y)\zeta + \varphi Y(b)X \\ &\quad - \varphi Y(b)\eta(X)\zeta + b\left[(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla\varphi_Y\varphi)\varphi X\right]. \end{aligned} \tag{3.3}$$

By adding (3.2) and (3.3) and using (2.2), we obtain

$$\begin{aligned} R(X,Y)U + R(\varphi X, \varphi Y)U &= \varphi X(a)\varphi Y - \varphi Y(a)\varphi X - \varphi X(b)Y + \varphi X(b)\eta(Y)\zeta \\ &\quad + \varphi Y(b)X - \varphi Y(b)\eta(X)\zeta + X(a)Y - Y(a)X \\ &\quad + X(b)\varphi Y - Y(b)\varphi X + b\alpha[\eta(X)\varphi Y + 2g(\varphi X, Y)\zeta \\ &\quad + \eta(X)hY - \eta(Y)\varphi X - 2g(X, \varphi Y)\zeta + \eta(Y)hX]. \end{aligned} \tag{3.4}$$

Using (2.1) and considering the inner product of (3.4) with U and then replacing X with φX and Y with φY, we have

$$\begin{aligned} & [X - \eta(X)\zeta](a)g(\varphi Y, \varphi U) - [Y - \eta(Y)\zeta](a)g(\varphi X, \varphi U) \\ & - [-X + \eta(X)\zeta](a)g(\varphi Y, U) + [-Y + \eta(Y)\zeta](a)g(\varphi X, U) \\ & + \varphi X(a)g(\varphi Y, U) - \varphi Y(a)g(\varphi X, U) - \varphi X(b)g(\varphi Y, \varphi U) \\ & + \varphi Y(b)g(\varphi X, \varphi U) - 4\alpha b g(X, \varphi Y)\eta(U) \\ = \; & 0. \end{aligned} \tag{3.5}$$

Putting φY for Y in (3.5), we have

$$\begin{aligned} & g(Da, \varphi X)[-g(Y,U) + \eta(Y)\eta(U)] - [-g(Da,Y) + \eta(Y)\zeta(a)]g(\varphi X, U) \\ & - g(Db, \varphi X)g(\varphi Y, U) - [-g(Db,Y) + \eta(Y)\zeta(b)][-g(X,U) + \eta(X)\eta(U)] \\ & + [g(Da,X) - \eta(X)\zeta(a)]g(\varphi Y, U) + g(Da, \varphi Y)[-g(X,U) + \eta(X)\eta(U)] \\ & + [g(Db,X) - \eta(X)\zeta(b)][-g(Y,U) + \eta(Y)\eta(U)] - g(Db, \varphi Y)g(\varphi X, U) \\ & - 4\alpha b[g(X,Y) - \eta(X)\eta(Y)] \\ = \; & 0. \end{aligned} \tag{3.6}$$

Contracting X and Y in (3.6), we obtain

$$\varphi U(a) = -U(b) + \zeta(b)\eta(U) + 2\alpha bn\eta(U).$$

□

Theorem 3.2. *([11]) Let M be an almost $\alpha-$cosymplectic $f-$manifold and $\widetilde{M}$ be integral manifold of D. Then*

i) when $\alpha = 0$, $\widetilde{M}$ is totally geodesic if and only if all the operators h_i vanishes.
ii) when $\alpha \neq 0$ $\widetilde{M}$ is totally umbilical if and only if all the operators h_i vanishes.

Theorem 3.3. *Let M be an almost $\alpha-$cosymplectic manifold. If M admits a non zero HPCV field U such that U is pointwise collinear with ζ, then the integral manifold of the distribution D is totally geodesic or totally umbilical.*

Proof. Let U be a nonzero HPCV field on M and U is pointwise collinear with ζ such that

$$U = \rho\zeta \tag{3.7}$$

where $\rho \neq 0$ a smooth function on M. Substituting (3.7) in (3.1), we obtain $2n\alpha b\rho = 0$, which implies $b = 0$ since $\rho \neq 0$.Using these obtained result together with equations (3.7) and (1.1), we get $\rho = \eta(U)$. From

$$X(\rho) = \nabla_X \rho = \nabla_X(\eta(U)) = (\nabla_X \eta)U + \eta(\nabla_X U)$$

and using (1.1), (2.3), (3.7) , we have

$$X(\rho) = a\eta(X).$$

Taking the covariant derivative of equation (3.7) along X, we get

$$aX = -\alpha\rho\varphi^2 X - \rho\varphi hX + \alpha a\eta(X)\zeta. \tag{3.8}$$

Considering by the inner product of (3.8) with Y, then contracting X and Y last equation $\rho\alpha = a$. Then equation (3.8) reduces to

$$a\varphi hX = 0, \tag{3.9}$$

since $\rho \neq 0$ a smooth function, so $a \neq 0$. Hence from (3.9) we obtain the result $h = 0$. The proof is completed. □

Theorem 3.4. *Let M be a complete almost $\alpha-$cosymplectic manifold that admits an HPCV field U. If M has a constant $\zeta-$sectional curvature, then*

i) the integral manifold of the distribution D is totally geodesic or totally umbilical,
ii) the manifold M is isometric to sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$.

Proof. Let $K(\zeta, X) = c$ is the positive constant sectional curvature of an almost $\alpha-$cosymplectic manifold. Then by a simple calculation, we obtain:

$$R(\zeta, X)\zeta = -c[X - \eta(X)\zeta]. \tag{3.10}$$

for any vector field $X \in \chi(M)$.
Using $X = \zeta$ in (3.2) yield

$$R(\zeta, Y)U = \zeta(a)Y - Y(a)\zeta + \zeta(b)\varphi Y + b\alpha\varphi Y + bhY. \tag{3.11}$$

By considering the inner product of (3.11) with ζ, we obtain

$$g(R(\zeta, Y)U, \zeta) = \zeta(a)\eta(Y) - Y(a). \tag{3.12}$$

Then, using (3.10), we obtain

$$g(R(\zeta, Y)U, \zeta) = -g(R(\zeta, Y)\zeta, U) = c[g(Y, U) - \eta(Y)\eta(U)]. \tag{3.13}$$

From the equations (3.12) and (3.13), we obtain

$$Da - \zeta(a)\zeta + cU - c\eta(U)\zeta = 0. \tag{3.14}$$

By considering the inner product of (3.11) with U implies

$$\zeta(a)U - (Da)\eta(U) - \zeta(b)\varphi U - b\alpha\varphi U + bhU = 0. \tag{3.15}$$

If we eleminate Da from the last two equations, we obtain

$$-\zeta(a)\varphi^2 U - c\eta(U)\varphi^2 U - \zeta(b)\varphi(U) - b\alpha\varphi U + bhU = 0. \tag{3.16}$$

Then differentiating (3.14) covariantly along any vector field X and further the inner product with Y, we obtain

$$\begin{aligned} g(\nabla_X Da, Y) = {} & \zeta(a)[\alpha g(X,Y)\zeta - \alpha\eta(X)\eta(Y) - g(\varphi hX, Y)] \\ & + X(\zeta(a))\eta(Y) - c[ag(X,Y) + bg(\varphi X, Y)] \\ & + c\eta(Y)[a\eta(X) + \alpha g(X,U) - \alpha\eta(X)\eta(U)] \\ & + c\eta(U)[\alpha g(X,Y) - \alpha\eta(X)\eta(Y) - g(\varphi hX, Y)] \\ & - cg(U, \varphi hX)\eta(Y). \end{aligned} \tag{3.17}$$

If we recall the Hessian operator, that is, $Hess_a(X,Y) = g(\nabla_X Da, Y) = g(\nabla_Y Da, X)$ and using the antisymmetrizing of the preceding equation

$$\begin{aligned} & X(\zeta(a))\eta(Y) + \alpha c\eta(Y)g(X,U) - cg(\varphi hX,U)\eta(Y) - Y(\zeta(a))\eta(X) \\ & -\alpha c\eta(X)g(Y,U) + cg(\varphi hY,U)\eta(X) + 2bcg(\varphi X,Y) \\ = & \ 0. \end{aligned} \tag{3.18}$$

Replacing X by φX and Y by φY in (3.18), we obtain $2bcg(\varphi X,Y) = 0$, then from $b = 0$ ($c \neq 0$). Thus using (3.15), we obtain

$$\zeta(a)U = Da\eta(U). \tag{3.19}$$

On the other hand, since $b = 0$ equation (3.16) reduce to

$$[\zeta(a) + c\eta(U)]\varphi^2 U = 0,$$

which implies either $\varphi^2 U = 0$ or $\zeta(a) = -c\eta(U)$.

Case1. If $\varphi^2 U = 0$, then $U = \eta(U)\zeta$ which implies U is pointwise collinear with ζ. Thus, from Theorem 3.3. the integral submanifold of the distribution D is totally geodesic or totally umbilical.

Case2. If $\zeta(a) = -c\eta(U)$, then from (3.19) we have $(Da + cU)\eta(U) = 0$.Thus, in both cases, $Da = -cU$ obtained. By considering (1.1) and using covariant differentiation, we obtain $\nabla_X Da = -caX$, any $X \in \chi(M)$. By view of ([18]-Theorem 3), condition ii) is proved.

□

In this part, we suppose that (M,φ,ζ,η) is an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces, namely the Riemannian curvature tensor satisfies (1.2). Furthermore, the following relations are provided.

4. Holomorphically planar conformal vector fields on almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces

Proposition 4.1. *[12] Let M be an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces. Then the following relations are hold.*

$$h^2 = \left(\kappa + \alpha^2\right)\varphi^2, \text{ for } \kappa \leq -\alpha^2, \tag{4.1}$$

$$\nabla_\zeta h = -\mu\varphi h,$$

$$R(\zeta,X)Y = \kappa(g(Y,X)\zeta - \eta(Y)X) + \mu(g(hY,X)\zeta - \eta(Y)hX), \tag{4.2}$$

$$(\nabla_X\varphi)Y = g(\alpha\varphi X + hX, Y)\zeta - \eta(Y)(\alpha\varphi X + hX),$$

for any $X,Y \in \chi(M)$,where h is symmetric operator $h = \frac{1}{2}\pounds_\zeta\varphi$.
From (4.1), we find easily that $\kappa \leq 0$ and $\kappa = 0$ if and only if M is a cosymplectic manifold, thus in the following we always suppose $\kappa < 0$.

Theorem 4.2. *Let (M,φ,ζ,η) be an almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces that admits an HPCV field U, then*

i) the manifold has constant curvature,
ii) the integral manifold of the distribution D is totally geodesic or totally umbilical,
iii) U is an eigenvector of h.

Proof. Using $X = \zeta$ in (3.2), we obtain

$$R(\zeta,Y)U = \zeta(a)Y - Y(a)\zeta + \zeta(b)\varphi Y + b\alpha\varphi Y + bhY. \tag{4.3}$$

By considering the inner product of (4.3) with ζ, we have

$$g(R(\zeta,Y)U,\zeta) = \zeta(a)\eta(Y) - Y(a). \tag{4.4}$$

Using the (4.2) in the preceding equation, we obtain

$$g(R(\zeta,Y)U,\zeta) = \kappa g(Y,U) - \kappa\eta(Y)\eta(U) + \mu g(hY,U). \tag{4.5}$$

Eqs. (4.4)-(4.5) yield to

$$\zeta(a)\eta(Y) - Y(a) = \kappa g(Y,U) - \kappa\eta(Y)\eta(U) + \mu g(hY,U) \tag{4.6}$$

which implies

$$-\kappa\eta(U)\zeta + \mu hY + \kappa U = \zeta(a)\zeta - Da. \tag{4.7}$$

On the other hand,taking the inner product both sides of equation (4.3) with U,

$$\zeta(a)g(Y,U) - Y(a)\eta(U) + \zeta(b)g(\varphi Y,U) + b\alpha g(\varphi Y,U) + bg(hY,U) = 0.$$

Remove Y in preceding equation

$$\zeta(a)U - Da\eta(U) - \zeta(b)\varphi(U) - b\alpha\varphi U + bhU = 0. \tag{4.8}$$

Eliminating Da from (4.7) and (4.8), we have

$$\mu\eta(U)hY + bhU - \zeta(b)\varphi(U) - b\alpha\varphi U - \zeta(a)\varphi^2 U - \kappa\eta(U)\varphi^2 U = 0. \tag{4.9}$$

On the other hand, substituting $Y = \zeta$ and then taking the covariant derivative of equation (4.7) along X, we have

$$-\kappa[\eta(\nabla_X U)\zeta + g(U,\nabla_X\zeta)\zeta + \eta(U)\nabla_X\zeta - \nabla_X U] = X(\zeta(a))\zeta - \zeta(a)\nabla_X\zeta - \nabla_X Da.$$

Then using (1.1) and $AX = -\nabla_X\zeta$ in preceding equation, also taking the inner product with Y, we obtain

$$\begin{aligned} & -\kappa[a\eta(X)\eta(Y) - g(AX,U)\eta(Y) - g(AX,Y)\eta(U) - ag(X,X_2) - bg(\varphi X,Y)] \\ = \ & X(\zeta(a))\eta(Y) + \zeta(a)g(AX,Y) - g(\nabla_X Da,Y) \end{aligned} \tag{4.10}$$

Using the symmetry of the Hessian operator, we have

$$\kappa[2bg(\varphi X,Y) - g(U,AX)\eta(Y) - g(U,AY)\eta(X)] - X(\zeta(a))\eta(Y) - Y(\zeta(a))\eta(X) = 0$$

Replacing X with φX and Y with φY in the previous equation, we obtain that

$$2\kappa bg(\varphi X,Y) = 0$$

which implies $b = 0$ as $\kappa < 0$. Therefore, from equation (4.8), we get

$$\zeta(a)U = (Da)\eta(U). \tag{4.11}$$

Considering that $Y \in [\lambda]'$ in (4.6), we obtain that

$$(\kappa + \mu\lambda)g(Y,U) = -Y(a). \tag{4.12}$$

Substituting $Y = \zeta$ in the last equality, then (4.11) and (4.12) implies that

$$\zeta(a) = -(\kappa + \mu\lambda)\eta(U) \text{ and } Da = -(\kappa + \mu\lambda)U. \tag{4.13}$$

By using equality of $\zeta(a)$ and $b = 0$ in (4.9), we obtain

$$-\zeta(a)\varphi^2 U - \kappa\eta(U)\varphi^2 U + \mu\eta(U)hY = 0$$

which implies

$$\mu\left[\lambda\eta(U)\varphi^2 U + hU\right]\eta(U) = 0.$$

Case1. If $\mu = 0$, then from (4.2) the manifold has constant curvature.

Case2. If $\lambda\eta(U)\varphi^2 U + hU = 0$, then from (2.1), we obtain $hU = \lambda U - \lambda\eta(U)\zeta$. If we apply h to both sides of the equation, we infer that $h^2 U = \lambda hU$. From that, $tr(h^2) = 0$, so we obtain $h = 0$. Under the same conditions of Theorem 3.3 the integral manifold of the distribution D is totally geodesic or totally umbilical.

Case3. If $\eta(U) = 0$, then from (4.13), $\zeta(a) = 0$. Using that value in (4.7), we obtain

$$Da = \kappa\eta(U)\zeta - \mu hY - \kappa U.$$

When this result is considered together with the value of Da in (4.13), we infer that

$$hU = \lambda U.$$

which implies that V is an eingenvector of h.

□

5. Conclusion and discussion

The notion of conformality is an important object that appears in various fields of mathematics (e.g., real and complex analysis, Riemannian geometry, classical geometry) as well as in physics (e.g., general relativity, conformal field theory) and also,has many applications in military aircraft, electronics, cartography, and so on. The notion, which started with conformal functions between Euclidean spaces, conformal maps between Riemannian or semi-Riemannian manifolds, was later extended to conformal vector fields. Recently, it is an important tool used in many mathematical and physical subjects with many special types.

Considering the importance and wide application of this notion, we characterize and classify almost $\alpha-$cosymplectic $(\kappa,\mu)-$spaces admitting holomorphically conformal vector fields which a generalization of the conformal vector field. In this direction, many results have been given in the third section and an important characterization of the given structure has been obtained. This study will shed light on our future investigations. Our further studies will be denote applications of some types of conformal vector fields like $\varphi-$holomorphic planar conformal vector fields and Ricci biconformal vector fields.

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References

[1] S. Desmukh, F. Alsolamy, *Conformal vector fields on a Riemannian manifold*, Balkan J. Geom. Appl. **19** (2014), 86-93.
[2] M. Obata, *Conformal transformations of Riemannian manifolds*, J. Diff. Geom. **4** (1970), 311-333.
[3] S. Tanno, W. Weber, *Closed conformal vector fields*, J. Diff. Geom. **3** (1969), 361-366.
[4] R. Sharma, *Holomorphically planar conformal vector fields on almost Hermitian manifolds*, Contemp. Math. **337** (2003), 145-154.
[5] A. Ghosh, R. Sharma, *Almost Hermitian manifolds admitting holomorphically planar conformal vector fields*, J. Geom., **84** (2005), 45-54.
[6] A. Ghosh, *Holomorphically planar conformal vector fields on contact metric manifolds*, Acta Math. Hungar., **129** (2010), 357-367.
[7] B. Cappelletti-Montano, De Nicola, A.I. Yudin, *A survey on cosymplectic geometry*. Rev. Math. Phys., **25** (2013), 1343002.
[8] S.I. Goldberg, K. Yano, *Integrability of almost cosymplectic structure*. Pac. J. Math., **31** (1969), 373-382.
[9] H. Endo, *Non-existence of almost cosymplectic manifolds satisfying a certain condition,* Tensor (N.S.), **284**(3) (2002), 272-289.
[10] K. Kenmotsu, *A class of almost contact Riemannian manifolds*. Tôhoku Math. J., **24** (1972), 93-103.
[11] T.W. Kim, H.K. Pak, *Canonical foliations of certain classes of almost contact metric structures,* Acta Math. Sin. Eng. Ser., **21** (2005), 841–846.
[12] N. Aktan, H. Öztürk, C. Murathan, *Almost $\alpha-$cosymplectic $(\kappa,\mu,\nu)-$spaces*, arXiv:1007.0527v1.
[13] S. Beyendi, G. Ayar, N. Aktan, *On a type of $\alpha-$cosymplectic manifolds*, Commun. Fac. Sci. Univ. Ankara, Ser. A1, Math. Stat., **68** (2019), 852-861.
[14] S. Beyendi, M. Yıldırım, *On generalized weakly symmetric $\alpha-$cosymplectic manifolds*, Hacet. J. Math. Stat., **50**(6) (2021), 1745-1755.
[15] M. Yıldırım, S. Beyendi, *On almost generalized weakly symmetric $\alpha-$cosymplectic manifolds*, Univers. J. Math. Appl., **3**(4) (2020), 156-159, doi:10.32323/ujma.730960.
[16] H. İ.Yoldaş, *Some results on cosymplectic manifolds admitting certain vector fields*, JGSP, **60** (2021), 83-94.
[17] H.İ. Yoldaş, Ş. Eken Meriç, E. Yaşar, *Some characterizations of $\alpha-$cosymplectic manifolds admitting Yamabe solitons*, Palestine Journal of Math., **10** (2021), 234-241.
[18] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan., **14** (1962), 333-340.