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Research Article

Localization of the spectra of dual frames multipliers

ROSARIO CORSO*

ABSTRACT. This paper concerns dual frames multipliers, i.e. operators in Hilbert spaces consisting of analysis, multiplication and synthesis processes, where the analysis and the synthesis are made by two dual frames, respectively. The goal of the paper is to give some results about the localization of the spectra of dual frames multipliers, i.e. to identify regions of the complex plane containing the spectra using some information about the frames and the symbols.

Keywords: Multipliers, dual frames, spectrum.

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1. INTRODUCTION

Frame multipliers have been objects of several studies [6, 9, 17, 18, 19, 20, 21] and applications in physics [13], signal processing (in particular, Gabor multipliers [12, 16] attracted interest as time-variant filters) and acoustics [4, 5]. Details about applications are discussed also in the survey [22].

Frame multipliers are part of the Bessel multipliers which were introduced in [2] and we are going to recall. A *Bessel sequence* of a separable Hilbert space $\mathcal{H}$ is a sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ such that there exists $B_\varphi > 0$ and

$$\sum_{n\in\mathbb{N}} |\langle f, \varphi_n\rangle|^2 \leq B_\varphi \|f\|^2, \qquad \forall f \in \mathcal{H}.$$

The constant B_φ is called a *Bessel bound* of φ. A sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}$ is a *frame* of $\mathcal{H}$ if there exist $A, B > 0$ (*lower bound* and *upper bound* of φ, respectively) such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, \varphi_n\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

Now, let $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}, \psi = \{\psi_n\}_{n\in\mathbb{N}}$ be Bessel sequences of $\mathcal{H}$ and $m = \{m_n\}_{n\in\mathbb{N}} \in \ell^\infty$, i.e. a bounded complex sequence. The operator $M_{m,\varphi,\psi}$ given by

$$M_{m,\varphi,\psi} f = \sum_{n\in\mathbb{N}} m_n \langle f, \psi_n\rangle \varphi_n, \qquad f \in \mathcal{H},$$

is called the *Bessel multiplier* of φ, ψ with *symbol* m. Correspondent versions of Bessel multipliers have been studied also in continuous and distributional contexts (see [3, 10, 23]). A Bessel multiplier $M_{m,\varphi,\psi}$ is called a *frame multiplier* if φ and ψ are frames.

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*Corresponding author: Rosario Corso; rosario.corso02@unipa.it
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This paper deals with the spectra of *dual frames multipliers*, i.e. multipliers $M_{m,\varphi,\psi}$, where φ and ψ are dual frames and $m \in \ell^\infty$. Two frames φ and ψ of $\mathcal{H}$ are called *dual* if $f = \sum_{n\in\mathbb{N}}\langle f, \varphi_n\rangle\psi_n$ for every $f \in \mathcal{H}$ (or, equivalently, $f = \sum_{n\in\mathbb{N}}\langle f, \psi_n\rangle\varphi_n$ for every $f \in \mathcal{H}$). In particular, the study of this paper is inspired by the result for Bessel multipliers shown in Proposition 1 below, which is an immediate consequence of [2, Theorem 6.1], i.e. of the fact that

$$\|M_{m,\varphi,\psi}\| \le \sup_{n\in\mathbb{N}}|m_n|{B_\varphi}^{\frac{1}{2}}{B_\psi}^{\frac{1}{2}}, \tag{1.2}$$

where $M_{m,\varphi,\psi}$ is any Bessel multiplier with B_φ and B_ψ some Bessel bounds of φ and ψ, respectively.

Proposition 1. *The spectrum of any Bessel multiplier $M_{m,\varphi,\psi}$ is contained in the closed disk centered the origin with radius $\sup_{n\in\mathbb{N}}|m_n|{B_\varphi}^{\frac{1}{2}}{B_\psi}^{\frac{1}{2}}$, where B_φ and B_ψ are Bessel bounds of φ and ψ, respectively.*

Proposition 1 provides information about the location of the spectra of Bessel multipliers in the complex plane. However, the given estimate may be too large for the spectra of dual frames multipliers[1]. The main results of the paper, Theorems 4.1 and 5.2, provide more accurate localization results for the spectra of dual frames multipliers $M_{m,\varphi,\psi}$ under some conditions on φ and ψ. We also stress that these conditions are satisfied by many frames used in applications (see Remark 4.2).

A localization of the spectrum of $M_{m,\varphi,\psi}$ may show that $M_{m,\varphi,\psi}$ is invertible. The invertibility of multipliers was a subject faced in [6, 9, 17, 18, 19, 20] and Theorems 4.1 and 5.2 bring new results in this direction (see Remark 5.4).

Moreover, the knowledge of a region containing the spectrum of $M_{m,\varphi,\psi}$ gives, in particular, information about the distribution of the possible eigenvalues of $M_{m,\varphi,\psi}$. In connection with this subject, recently in [9] some types of dual frames multipliers with at most countable spectra have been studied.

The paper is organized as follows. In Section 2, we recall some basic notions of frame theory, while we give some preliminary localization results about the spectra of dual frames multipliers in Section 3. Finally, Sections 4 and 5 contain the main results mentioned above together with examples.

2. Preliminaries

Throughout the paper, $\mathcal{H}$ indicates a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Given an operator T acting between two Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $R(T)$ and $N(T)$ the *range* and *kernel* of T, respectively, and by T^* its *adjoint* when T is bounded.

If $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator, then we write $\rho(T)$ and $\sigma(T)$ for the *resolvent set* and *spectrum* of T, respectively. We recall that $\rho(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has bounded inverse $(T - \lambda I)^{-1}$ everywhere defined on $\mathcal{H}$ and $\sigma(T)$ is the complement set of $\rho(T)$. We recall that two bounded operators $T, T' : \mathcal{H} \to \mathcal{H}$ are said to be *similar* if there exists a bounded and bijective operator $S : \mathcal{H} \to \mathcal{H}$ such that $T = S^{-1}TS$. Throughout the paper, we will apply the following standard perturbation result (for a reference see, for instance, Theorem IV.1.16 of [15]).

[1] Indeed, if $m \in \ell^\infty$, φ and ψ are dual Riesz bases (for the definition see the end part of Section 2), then the spectrum of $M_{m,\varphi,\psi}$ is the closure of the set $\{m_n\}_{n\in\mathbb{N}}$ (see [1, Proposition 2.1] or [8, Section 5.1]), which is in general smaller than the closed disk centered the origin with radius $\sup_{n\in\mathbb{N}}|m_n|{B_\varphi}^{\frac{1}{2}}{B_\psi}^{\frac{1}{2}}$.

Lemma 2.1. *Let $T, B : \mathcal{H} \to \mathcal{H}$ be bounded operators. If T is bijective and $\|B\| < \|T^{-1}\|^{-1}$, then $T + B$ is bijective.*

We denote by ℓ^2 (respectively, ℓ^∞) the usual spaces of square summable (respectively, bounded) complex sequences indexed by $\mathbb{N}$. A *limit point* for $m \in \ell^\infty$ is the limit of a converging subsequence of m.

In the introduction, we gave the definitions of Bessel sequences and frames. Here, we recall some other notions and elementary results about frame theory [7]. A sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{N}}$ is *complete* in $\mathcal{H}$ if its linear span is dense in $\mathcal{H}$ if and only if $\langle \varphi_n, f\rangle = 0$ for every $n \in \mathbb{N}$ implies $f = 0$. A frame for $\mathcal{H}$ is, in particular, complete in $\mathcal{H}$.

Let φ be a frame for $\mathcal{H}$. We say that φ is a *Parseval frame* if (1.1) holds with $A = B = 1$. The operator $S : \mathcal{H} \to \mathcal{H}$, defined by

$$Sf = \sum_{n\in\mathbb{N}} \langle f, \varphi_n\rangle \varphi_n, \qquad f \in \mathcal{H},$$

is well-defined, bounded, bijective and it called the *frame operator* of φ. The sequence $\{S^{-1}\varphi_n\}_{n\in\mathbb{N}}$ is a dual frame of φ, called the *canonical dual*, and $\{S^{-\frac{1}{2}}\varphi_n\}_{n\in\mathbb{N}}$ is a Parseval frame for $\mathcal{H}$.

A *Riesz basis* φ for $\mathcal{H}$ is a complete sequence in $\mathcal{H}$ satisfying for some $A, B > 0$

$$A \sum_{n\in\mathbb{N}} |c_n|^2 \le \left\| \sum_{n\in\mathbb{N}} c_n \varphi_n \right\|^2 \le B \sum_{n\in\mathbb{N}} |c_n|^2, \qquad \forall \{c_n\} \in \ell^2. \tag{2.3}$$

A Riesz basis φ for $\mathcal{H}$ is a frame for $\mathcal{H}$, the constants in (1.1) can be chosen as in (2.3) and the canonical dual of φ is a Riesz basis too (called *dual Riesz basis* of φ).

3. Basic localization results

In this section, we give two preliminary localization results of the spectra of dual frames multipliers (Propositions 2 and 3) without requiring specific properties of the two frames. For the first one, we need the notion of numerical range. Given a bounded operator $T : \mathcal{H} \to \mathcal{H}$ the *numerical range* of T is the set $\mathfrak{n}_T = \{\langle Tf, f\rangle : f \in \mathcal{H}, \|f\| = 1\}$. We recall also that the spectrum of T is contained in the closure of $\mathfrak{n}_T$ (see [15, Corollary V.3.3]).

In addition, we are going to use the following lemma, which states that to examine the spectrum of a dual frame multiplier $M_{m,\varphi,\psi}$ where ψ is, in particular, the canonical dual frame of φ, we can just consider a multiplier determined by a Parseval frame.

Lemma 3.2. *Let φ be a frame for $\mathcal{H}$, ψ its canonical dual frame and $m \in \ell^\infty$. Then $M_{m,\varphi,\psi}$ is similar to $M_{m,\rho,\rho}$, where ρ is the Parseval frame associated to φ, and so $\sigma(M_{m,\varphi,\psi}) = \sigma(M_{m,\rho,\rho})$. In particular, if m is a real (resp., non-negative) sequence, then $M_{m,\varphi,\psi}$ is similar to a self-adjoint (resp., non-negative) operator and $\sigma(M_{m,\varphi,\psi})$ is real (resp., non-negative).*

Proof. Let S be the frame operator of φ, which is a bijective operator. It is immediate to see that $S^{-\frac{1}{2}} M_{m,\varphi,\psi} S^{\frac{1}{2}} = M_{m,\rho,\rho}$, where $\rho = S^{-\frac{1}{2}}\varphi$ is the Parseval frame associated to φ. The rest of the statement follows easily. □

Proposition 2. *Let φ be a frame for $\mathcal{H}$, ψ its canonical dual and $m \in \ell^\infty$. Then $\sigma(M_{m,\varphi,\psi})$ is contained in the closed convex hull of m, i.e. the closure of the set $\{\sum_{n\in\mathbb{N}} a_n m_n : \sum_{n\in\mathbb{N}} |a_n|^2 = 1\}$.*

Proof. By Lemma 3.2, we can confine to the case where $\varphi = \psi$ is a Parseval frame. We note that

$$\langle M_{m,\varphi,\varphi} f, f\rangle = \sum_{n\in\mathbb{N}} m_n |\langle f, \varphi_n\rangle|^2,$$

therefore, because $\|f\|^2 = \sum_{n\in\mathbb{N}}|\langle f, \varphi_n\rangle|^2$, the numerical range (and then also the spectrum) of $M_{m,\varphi,\varphi}$ is contained in the closed convex hull of m. □

Remark 3.1.

(i) Under the hypothesis of Proposition 2, if in addition m is a real sequence, we have that $\sigma(M_{m,\varphi,\psi}) \subseteq [\inf_{n\in\mathbb{N}} m_n, \sup_{n\in\mathbb{N}} m_n]$.

(ii) If φ is not a Parseval frame and ψ is its canonical dual, then the numerical range of $M_{m,\varphi,\psi}$ is not necessarily contained in the closed convex hull of m (even though by Proposition 2 $\sigma(M_{m,\varphi,\psi})$ is). For example, let

$$\varphi = \{e_1, e_1 + e_2, e_3, \dots, e_n, \dots\} \text{ and } \psi = \{e_1 - e_2, e_2, e_3, \dots, e_n, \dots\},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $m = \{2, 1, 1, \dots\}$. The sequences are frames and ψ is the canonical dual of φ. Moreover, $\frac{3+i}{2}$ belongs to the numerical range of $M_{m,\varphi,\psi}$, because $\langle M_{m,\varphi,\psi} f, f\rangle = \frac{3+i}{2}$, where $f = \frac{e_1+ie_2}{\sqrt{2}}$. Nevertheless, $\frac{3+i}{2}$ is not in the convex hull of m.

The statement of Proposition 2 may not hold if ψ is just a dual frame of φ. For example, take

$$\varphi = \{e_1, e_1, e_2, \dots, e_n, \dots\} \text{ and } \psi = \{ie_1, (1-i)e_1, e_2, \dots, e_n, \dots\},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $m = \{2, 1, 1, \dots\}$. Then, a straightforward calculation shows that $M_{m,\varphi,\psi} f = f + (1-i)\langle f, e_1\rangle e_1$ for every $f \in \mathcal{H}$, so $M_{m,\varphi,\psi}$ is not self-adjoint and, in particular, its spectrum is not contained in the closed convex hull of m, which is a subset of the real line. For generic dual frames, we can actually state the following.

Proposition 3. *Let φ, ψ be dual frames for $\mathcal{H}$ with upper bounds B_φ, B_ψ, respectively. Let $m \in \ell^\infty$. If $\lambda, \mu \in \mathbb{C}$ and*

$$\sup_{n\in\mathbb{N}}|m_n - \mu| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}} < |\mu - \lambda|, \tag{3.4}$$

then $\lambda \in \rho(M_{m,\varphi,\psi})$. In particular,

(1) if m is contained in the disk of center μ with radius r, then $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center μ with radius $rB_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$;

(2) if m is real, then $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center $\frac{1}{2}(\sup_{n\in\mathbb{N}} m_n + \inf_{n\in\mathbb{N}} m_m)$ with radius $\frac{1}{2}(\sup_{n\in\mathbb{N}} m_n - \inf_{n\in\mathbb{N}} m_m) B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$.

Proof. For simplicity, we write $m - \lambda$ and $m - \mu$ for the complex sequences $\{m_n - \lambda\}$ and $\{m_n - \mu\}$, respectively. If (3.4) holds, then by (1.2) we have $\|M_{m-\mu,\varphi,\psi}\| < |\mu - \lambda|$ and by Lemma 2.1 we have $\lambda - \mu \in \rho(M_{m-\mu,\varphi,\psi})$, i.e. $\lambda \in \rho(M_{m,\varphi,\psi})$. Now, the rest of the statement is immediate. □

When ψ is the canonical dual of φ, Proposition 2 gives a more accurate result than Proposition 3.

4. Main result 1

For the localization result in this section, we make the assumption that a frame contains a Riesz basis. Note that this is not a very strong requirement. Indeed, it is satisfied by many frames used in applications (see Remark 4.2 below for a consideration about Gabor and wavelet frames). For simplicity, we write the statement of the result in terms of the resolvent set.

Theorem 4.1. *Let φ, ψ be frames for $\mathcal{H}$ such that for some $I \subseteq \mathbb{N}$, $\{\varphi_n : n \in I\}$, $\{\psi_n : n \in I\}$ be Riesz bases for $\mathcal{H}$ with lower frame bounds $A_{\varphi,1}$ and $A_{\psi,1}$, respectively. Moreover, let $B_{\varphi,2}$ and $B_{\psi,2}$ be Bessel bounds of $\{\varphi_n : n \in \mathbb{N}\backslash I\}$ and $\{\psi_n : n \in \mathbb{N}\backslash I\}$, respectively. Let $m \in \ell^\infty$. If*

$$\sup_{n\in\mathbb{N}\backslash I} |m_n| {B_{\varphi,2}}^{\frac{1}{2}} {B_{\psi,2}}^{\frac{1}{2}} < \inf_{n\in I} |m_n| {A_{\varphi,1}}^{\frac{1}{2}} {A_{\psi,1}}^{\frac{1}{2}}, \tag{4.5}$$

then $M_{m,\varphi,\psi}$ is bijective.
If, in addition, φ and ψ are dual frames and

$$\sup_{n\in\mathbb{N}\backslash I} |m_n - \lambda| {B_{\varphi,2}}^{\frac{1}{2}} {B_{\psi,2}}^{\frac{1}{2}} < \inf_{n\in I} |m_n - \lambda| {A_{\varphi,1}}^{\frac{1}{2}} {A_{\psi,1}}^{\frac{1}{2}}, \tag{4.6}$$

then $\lambda \in \rho(M_{m,\varphi,\psi})$.

Proof. We can write $M_{m,\varphi,\psi} = M_1 + M_2$, where $M_1 = M_{m^{(1)},\varphi^{(1)},\psi^{(1)}}$ and $M_2 = M_{m^{(2)},\varphi^{(2)},\psi^{(2)}}$, $m^{(1)} = \{m_n : n \in I\}$, $m^{(2)} = \{m_n : n \in \mathbb{N}\backslash I\}$, $\varphi^{(1)} = \{\varphi_n : n \in I\}$, $\varphi^{(2)} = \{\varphi_n : n \in \mathbb{N}\backslash I\}$, $\psi^{(1)} = \{\psi_n : n \in I\}$, $\psi^{(2)} = \{\psi_n : n \in \mathbb{N}\backslash I\}$. First of all, $\inf_{n\in I} |m_n| > 0$ holds by (4.5), so M_1 is bijective by [17, Theorem 5.1]. Moreover, (4.5) allows to apply Lemma 2.1 because

$$\|M_2\| \leq \sup_{n\in\mathbb{N}\backslash I} |m_n| {B_{\varphi,2}}^{\frac{1}{2}} {B_{\psi,2}}^{\frac{1}{2}}$$

by (1.2), and

$$\inf_{n\in I} |m_n| {A_{\varphi,1}}^{\frac{1}{2}} {A_{\psi,1}}^{\frac{1}{2}} \leq \|M_1^{-1}\|^{-1}$$

by Propositions 7.7 and 7.2 of [2] and the fact that a Bessel bound of the canonical dual of φ (resp., ψ) is ${A_{\varphi,1}}^{-1}$ (resp., ${A_{\psi,1}}^{-1}$). The second part of the statement now follows from the fact that $M_{m,\varphi,\psi} - \lambda I = M_{m-\lambda,\varphi,\psi}$ when φ and ψ are dual frames (here, we write $m - \lambda$ for the sequence $\{m_n - \lambda\}$). □

We show an application of Theorem 4.1 with an example of multiplier with $0-1$ symbol (i.e. a sequence made only of 0 and 1)[2].

Example 4.1. *Let φ be a Parseval frame for $\mathcal{H}$ such that $\{\varphi_{2n}\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ with lower bound A. Clearly, we have $0 < A < 1$. Consequently, $\{\varphi_{2n-1}\}$ is a Bessel sequence with bound $1 - A$. Let, moreover, m be a sequence of 0 and 1. With these choices, we apply Theorem 4.1 to $M_{m,\varphi,\varphi}$. Condition (4.6) is*

$$\sup_{n\in\mathbb{N}} |m_{2n-1} - \lambda|(1 - A) < \inf_{n\in\mathbb{N}} |m_{2n} - \lambda| A. \tag{4.7}$$

We have

$$\inf_{n\in\mathbb{N}} \{|-\lambda|, |1-\lambda|\} = \begin{cases} -\lambda, & \lambda < 0, \\ \lambda, & 0 \leq \lambda \leq \frac{1}{2}, \\ 1-\lambda, & \frac{1}{2} < \lambda \leq 1, \\ \lambda - 1, & 1 < \lambda, \end{cases}$$

and

$$\sup_{n\in\mathbb{N}} \{|-\lambda|, |1-\lambda|\} = \begin{cases} 1-\lambda, & \lambda < 0, \\ 1-\lambda, & 0 \leq \lambda \leq \frac{1}{2}, \\ \lambda, & \frac{1}{2} < \lambda \leq 1, \\ \lambda, & 1 < \lambda. \end{cases}$$

[2] Such a multiplier often occurs in applications, see e.g. [22].

Since, by Proposition 2, we already know that $\sigma(M_{m,\varphi,\varphi}) \subseteq [0,1]$, we need to check the validity of (4.7) only for $0 \le \lambda \le 1$. We note that if $0 \le \lambda \le \frac{1}{2}$, then (4.7) is true if and only if $\lambda > 1 - A$, which makes sense only if $A > \frac{1}{2}$. On the other hand, if $\frac{1}{2} < \lambda \le 1$, then (4.7) is true if and only if $\lambda < A$, which makes sense again only if $A > \frac{1}{2}$. Thus, by Theorem 4.1, we can write that if $A > \frac{1}{2}$

$$\sigma(M_{m,\varphi,\varphi}) \subseteq [0, 1 - A] \cup [A, 1].$$

As particular case of Theorem 4.1, we get the following.

Corollary 1. *Let φ be a frame for $\mathcal{H}$ with bounds A and B such that for some $I \subseteq \mathbb{N}$ $\{\varphi_n : n \in I\}$ is a Riesz basis for $\mathcal{H}$ with lower frame bound A'. Let ψ be the canonical dual of φ and $m \in \ell^\infty$. If*

$$\sup_{n \in \mathbb{N}\setminus I} |m_n - \lambda| \frac{B - A'}{A} < \inf_{n \in I} |m_n - \lambda| \frac{A'}{B},$$

then $\lambda \in \rho(M_{m,\varphi,\psi})$.

Proof. The statement follows by Theorem 4.1 once noticed that $\{\psi_n\}_{n \in I} = \{S^{-1}\varphi_n\}_{n \in I}$ is a Riesz basis with lower bound $\frac{A'}{B^2}$, $\{\psi_n\}_{n \in \mathbb{N}\setminus I}$ has Bessel bound $B - A'$ and $\{\psi_n\}_{n \in \mathbb{N}\setminus I} = \{S^{-1}\varphi_n\}_{n \in \mathbb{N}\setminus I}$ has Bessel bound $\frac{B-A'}{A^2}$. □

Remark 4.2. *Gabor and wavelet frames are classical frames which occur in applications (see [7, 11, 14]). A (regular) Gabor frame for $L^2(\mathbb{R})$ is a frame of the form*

$$\mathcal{G}(g, a, b) = \{E_b^m T_a^n g\}_{m,n \in \mathbb{Z}},$$

where $g \in L^2(\mathbb{R})$, $a, b > 0$, $(T_a f)(x) = f(x - a)$ and $(E_b f)(x) = e^{2\pi i b x} f(x)$ for $x \in \mathbb{R}$. A Gabor frame which is a finite union of Riesz bases can be easily constructed in this way. Let $N \in \mathbb{N}$ and $\mathcal{G}(g, a, b)$ a Riesz basis for $L^2(\mathbb{R})$. A simple calculation shows that $\mathcal{G}(g, \frac{a}{N}, b)$ (as well as $\mathcal{G}(g, a, \frac{b}{N})$) is a frame for $L^2(\mathbb{R})$ which is a union of N Riesz bases.

Frames which are unions of Riesz bases can be found also in the context of wavelet frames. In particular, the frame multiresolution analysis technique (see [7, Ch. 17]) gives a way to construct wavelet frames which are unions of Riesz bases.

5. Main result 2

In this section, we consider Parseval frames φ for $\mathcal{H}$ which are unions of multiples of orthonormal bases. In other words, we can think that there exists $k \in \mathbb{N}$ such that

$$\{\varphi_{(i-1)k+j} : i \in \mathbb{N}\} = \{\alpha_j e_i^j : i \in \mathbb{N}\}, \tag{5.8}$$

where $\alpha_j \in \mathbb{C}\setminus\{0\}$ and $\{e_i^j : i \in \mathbb{N}\}$ is orthonormal basis for $\mathcal{H}$ for $j = 1, \dots, k$. Also here, we remark that this condition occurs for frames used in application. For instance, following Remark 4.2, if $\mathcal{G}(g, a, b)$ is an orthonormal basis for $L^2(\mathbb{R})$, then $\frac{1}{N}\mathcal{G}(g, \frac{a}{N}, b)$ and $\frac{1}{N}\mathcal{G}(g, a, \frac{b}{N})$ are Parseval frames and unions of N multiples of orthonormal bases.

Theorem 5.2. *Let φ be as in (5.8), $m \in \ell^\infty$ and $l_1, \dots, l_k \in \mathbb{C}$. If $\lambda \in \mathbb{C}$ and*

$$\sum_{j=1}^{k} |\alpha_j|^2 \sup_{i \in \mathbb{N}} |m_{(i-1)k+j} - l_j| < \left| \sum_{j=1}^{k} |\alpha_j|^2 l_j - \lambda \right|, \tag{5.9}$$

then $\lambda \in \rho(M_{m,\varphi,\varphi})$. As a consequence, if m is a real sequence, then

$$\sigma(M_{m,\varphi,\varphi}) \subseteq \left[\sum_{j=1}^{k} |\alpha_j|^2 \inf_{i \in \mathbb{N}} m_{(i-1)k+j}, \sum_{j=1}^{k} |\alpha_j|^2 \sup_{i \in \mathbb{N}} m_{(i-1)k+j} \right]. \tag{5.10}$$

Proof. First of all, we note that $\lambda \neq \sum_{j=1}^{k} |\alpha_j|^2 l_j$ by (5.9). We have $M_{m,\varphi,\varphi} f = \sum_{j=1}^{k} |\alpha_j|^2 l_j f + M_{m',\varphi,\varphi}$, where $m' = \{m'_n\}$ and $m'_{(i-1)k+j} = m_{(i-1)k+j} - l_j$ for $i \in \mathbb{N}$ and $j = 1, \ldots, k$. Thus, the first statement follows by Lemma 2.1 noting that

$$\|M_{m',\varphi,\varphi}\| \leq \sum_{j=1}^{k} |\alpha_j|^2 \sup_{i\in\mathbb{N}} |m_{(i-1)k+j} - l_j|.$$

Now assume that m is real, i.e. $M_{m,\varphi,\varphi}$ is self-adjoint. Therefore, (5.9) implies that $\sigma(M_{m,\varphi,\varphi})$ is contained in the interval

$$\left[\sum_{j=1}^{k} |\alpha_j|^2 (l_j - \sup_{i\in\mathbb{N}} |m_{(i-1)k+j} - l_j|), \sum_{j=1}^{k} |\alpha_j|^2 (l_j + \sup_{i\in\mathbb{N}} |m_{(i-1)k+j} - l_j|) \right]. \tag{5.11}$$

Choosing in (5.11), first $l_j < \inf_{i\in\mathbb{N}} m_{(i-1)k+j}$ and then $l_j > \sup_{i\in\mathbb{N}} m_{(i-1)k+j}$ for every $j = 1, \ldots, k$, we find (5.10). ☐

Example 5.2. *Let $\varphi = \{\frac{1}{\sqrt{2}} e_1, \frac{1}{\sqrt{2}} f_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} f_2, \ldots\}$, where $\{e_n\}$ and $\{f_n\}$ are orthonormal bases for $\mathcal{H}$. Furthermore, let $m = \{m_n\}$ be such that $m_{4n-3} = 0$, $m_{4n-2} = \frac{1}{3}$, $m_{4n-1} = \frac{2}{3}$ and $m_{4n} = 1$, $n \in \mathbb{N}$. Taking into account Proposition 2, the spectrum of $M_{m,\varphi,\varphi}$ is contained in $[0, 1]$. This estimate can be improved by Theorem 5.2: in particular, we obtain that $\sigma(M_{m,\varphi,\varphi}) \subseteq [\frac{1}{6}, \frac{5}{6}]$.*

Remark 5.3. *Theorem 5.2 is not a special case of Theorem 4.1 (and vice-versa). In particular, Theorem 4.1 gives no improvement on the localization of the spectrum in Example 5.2. On the other hand, Theorem 5.2 does not add any further information about the spectrum of the multiplier in Example 4.1, even in the case where $\{\varphi_{2n}\}_{n\in\mathbb{N}}$ and $\{\varphi_{2n+1}\}_{n\in\mathbb{N}}$ are multiples of orthonormal bases.*

Remark 5.4. *Theorems 4.1 and 5.2 give, in particular, new criteria of invertibility in comparison to the results in [17]. For instance, let*

$$\varphi = \left\{ \frac{1}{\sqrt{2}} e_1, \frac{1}{\sqrt{2}} f_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} f_2, \ldots \right\},$$

where $\{e_n\}$ and $\{f_n\}$ are orthonormal bases for $\mathcal{H}$ and $m = \{m_n\}$ is such that $m_{2n-1} = \frac{1}{n+1}$ and $m_{2n} = 2 - \frac{1}{n+1}$, $n \geq 1$. Both Theorems 4.1 and 5.2 show that $\sigma(M_{m,\varphi,\varphi}) \subseteq [\frac{3}{4}, \frac{5}{4}]$. In particular, $M_{m,\varphi,\varphi}$ is invertible. However, Propositions 4.1, 4.2 and 4.4 of [17] do not apply to this multiplier $M_{m,\varphi,\varphi}$.

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Rosario Corso
Università degli Studi di Palermo
Dipartimento di Matematica e Informatica
Via Archirafi 34, 90123, Palermo, Italy
ORCID: 0000-0001-9123-4977
E-mail address: rosario.corso02@unipa.it