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An Application of Trigonometric Quintic B-Spline Collocation Method for Sawada- Kotera Equation

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Abstract

In this paper, we deal with the numerical solution of Sawada-Kotera (SK) equation classified as the type of fifth order Korteweg-de Vries (gfKdV) equation. In the first step of our study consisting of several steps, nonlinear model problem is split into the system with the coupled new equations by using the transformation $w_{xxx} = v$. In the second step, to get rid of the nonlinearity of the problem, Rubin-Graves type linearization is used. After these applications, the approximate solutions are obtained by using the trigonometric quintic B-Spline collocation method. The efficiency and accuracy of the present method is demonstrated with the tables and graphs. As it is seen in the tables given with the error norms L_2 and L_∞ for different time and space steps, the present method is more accurate for the larger element numbers and smaller time steps.

Keywords: Sawada-Kotera Equation; Collocation Finite Element Method; Trigonometric Quintic B-Spline; Rubin-Graves Type Linearization.

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Sawada-Kotera Denklemi için Trigonometrik Beşli Baz Fonksiyonları Kollokasyon Yönteminin Bir Uygulaması

Öz

Bu çalışmada, beşinci dereceden Korteweg-de Vries (gfKdV) denklemlerinin türü olarak sınıflandırılan Sawada-Kotera (SK) denkleminin nümerik çözümü ele alınmaktadır. Birkaç adımdan oluşan çalışmamızın ilk adımında, lineer olmayan model problem $w_{xxx} = v$ dönüşümü kullanılarak iki yeni denklem sistemine ayrıştırılmıştır. İkinci adımda, problemin lineer olmama durumundan kurtulmak için Rubin-Graves tipi lineerleştirme kullanılmıştır. Bu uygulamalardan sonra trigonometrik beşli B-Spline kollokasyon yöntemi kullanılarak yaklaşık çözümler elde edilmiştir. Mevcut yöntemin etkinliği ve doğruluğu tablolar ve grafiklerle gösterilmiştir. Farklı zaman ve konum adımı için L_2 ve L_∞ hata normları ile verilen tablolardan görüldüğü üzere, mevcut yöntem daha büyük eleman sayıları ve daha küçük zaman adımları için yüksek doğruluktadır.

***Anahtar Kelimeler*:** Sawada- Kotera Denklemi; Kollokasyon Sonlu Eleman Yöntemi; Trigonometrik Beşli B-Spline; Rubin- Graves Tipi Lineerleştirme.

1. Introduction

Many problems in various areas of scientific and engineering fields can be expressed as partial differential equations. These equations frequently appear in the fields such as fluid dynamics, plasma physics, mathematical biology, nonlinear optics, quantum mechanics etc. One of the most important problems studied in these fields is the generalized fifth order Korteweg-de Vries (gfKdV) equation. Although there isn't a general solution corresponding to the solution of the problem, the exact solutions are available for the special cases of solitary waves [1]. The generalized fifth order Korteweg-de Vries (gfKdV) equation is modelled with the relation

$$\frac{\partial w}{\partial t} + \alpha w^2 \frac{\partial w}{\partial x} + \beta \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \gamma \mathrm{w} \frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0, \tag{1}$$

where α, β and γ are random real parameters and w is a differentiable function related to t time and x space variables. It should be noted here that equation (1) not only expresses the motion of the long waves in shallow water under gravity and in a one dimensional nonlinear lattice, but it is also a significant mathematical model for a chain of coupled non-linear oscillators and magneto-sound propagation in plasmas [2]. In the last few years, many researchers have studied nonlinear gfKdV equations to obtain their exact solutions and numerical solutions. When the literature is investigated, there are many studies in which analytical and numerical solutions are obtained by using the Adomian Decomposition Method [3-6], Variational Iteration Method [7, 8], Homotopy Perturbation Method [9, 10], Laplace Decomposition Approach [11, 12] and finite difference schemes based on a predictor-corrector algorithm [13]. Since the gfKdV equation doesn't have general solution expect for special cases of solitary waves, their numerical solutions are commonly studied [3]. For the numerical solutions, Bakodah [4] generalized on appropriate polynomials for the gfKdV and implemented the new modified Adomian Decomposition Method, Kaya [5] calculated explicit and numerical solutions for a various fifth-order KdV equations, Odibat and Momani [8] used the variational iteration method by obtaining a correction functional for the differential equation and Djidjeli and the others [13] proposed two methods derived using central differences with a predictor-corrector time stepping and linearizing implicit corrector scheme. In this study, different from the existing studies for numerical solutions, we are going to use the finite element collocation method. In order to obtain better and more effective numerical results, trigonometric quintic basis functions will be used.

The model problem discussed in our study is the Sawada-Kotera [14] problem which is the special form of the gfKdV equation for the parameters $\alpha = 45, \beta = 15$ and $\gamma = 15$ is given as

$$\frac{\partial w}{\partial t} + 45w^2\frac{\partial w}{\partial x} + 15\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + 15\mathrm{w}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0\,, \tag{2}$$

subject to initial and boundary conditions

$$w(\pm L, t) = w_x(\pm L, t) = 0, \qquad x \in (-L, L), \quad t > 0,$$

$$w(x, 0) = f(x, k, \lambda),$$

where k and λ are constant numbers.

The organization of this study is built on the main steps given in the following sections. In section 2, the used method and its application schemes are given. In section 3, initial matrix systems are composed. The obtained numerical results are discussed in the last section 4.

2. Implementation of the Collocation Finite Element Method with Trigonometric Quintic B-Spline Basis

In this section, first of all, by using the relation $\frac{\partial^3 w}{\partial x^3} = w_{xxx} = v$, two different equations will be obtained corresponding the equation (2). The new coupled equations can be written as

$$w_{xxx} - v = 0, \tag{3}$$

$$w_t + 45w^2w_x + 15w_xw_{xx} + 15wv + v_{xx} = 0. \tag{4}$$

So, we can create the scheme for equation systems (3) and (4) using the collocation finite element method based on trigonometric quintic B-spline functions. For this purpose, firstly, we consider the domain $[-L, L]$ divided uniformly into sub intervals from the x_m knots as $-L = x_0 < x_1 < x_2 <\ldots< x_m = L$ where spatial step size $\Delta x = h = x_{m+1} - x_m$, for all m, $m = -2, -1, 0, 1, \ldots, M+1, M+2$. The group of trigonometric quintic B-spline basis $\{\phi_{-2}(x), \phi_{-1}(x), \ldots, \phi_{M+1}(x), \phi_{M+2}(x)\}$ forms a basis for the solution region $[-L, L]$. In this way, the approximate solutions of functions $w(x,t)$ and $v(x,t)$ can be represented with $W_m(x,t)$ and $V_m(x,t)$. Also these approximate values can be expressed with the terms $\phi_m(x)$, $\delta_m(t), \sigma_m(t)$ as

$$W_m(x,t) = \sum_{m=-2}^{M+2} \phi_m(x)\delta_m(t), \qquad V_m(x,t) = \sum_{m=-2}^{M+2} \phi_m(x)\sigma_m(t),$$

where $\phi_m(x)$ are values of the x_m knots and $\delta_m(t)$, $\sigma_m(t)$ are time-dependent coefficients which can be found by using the boundary conditions and trigonometric quintic B-spline collocation conditions [15-17]. In our paper, we use the symbolization of W_m, V_m for the approximate solutions at the knots $x = x_m$. Numerical solutions at the knots x_m derived by using the trigonometric quintic B-splines $\phi_m(x)$ are found to be:

$$W_m = W(x_m) = a_1\delta_{m-2} + a_2\delta_{m-1} + a_3\delta_m + a_2\delta_{m+1} + a_1\delta_{m+2},$$

$$W_m' = W'(x_m) = b_1\delta_{m-2} + b_2\delta_{m-1} - b_2\delta_{m+1} - b_1\delta_{m+2},$$

$$W_m'' = W''(x_m) = c_1\delta_{m-2} + c_2\delta_{m-1} + c_3\delta_m + c_2\delta_{m+1} + c_1\delta_{m+2},$$

$$W_m''' = W'''(x_m) = d_1\delta_{m-2} + d_2\delta_{m-1} - d_2\delta_{m+1} - d_1\delta_{m+2},$$

$$V_m = V(x_m) = a_1\sigma_{m-2} + a_2\sigma_{m-1} + a_3\sigma_m + a_2\sigma_{m+1} + a_1\sigma_{m+2},$$

$$V_m' = V'(x_m) = b_1\sigma_{m-2} + b_2\sigma_{m-1} - b_2\sigma_{m+1} - b_1\sigma_{m+2},$$

$$V_m'' = V''(x_m) = c_1\sigma_{m-2} + c_2\sigma_{m-1} + c_3\sigma_m + c_2\sigma_{m+1} + c_1\sigma_{m+2},$$

where

$$a_1 = \frac{\sin^5(h/2)}{\theta},$$

$$a_2 = \frac{2\sin^5(h/2)\cos(h/2)(16\cos^2(h)/2 - 3)}{\theta},$$

$$a_3 = \frac{2\sin^5(h/2)(1+48\cos^4(h/2)-16\cos^2(h)/2)}{\theta},$$

$$b_1 = -\frac{5\sin^4(h/2)\cos(h/2)}{2\theta},$$

$$b_2 = -\frac{5\sin^4\left(\frac{h}{2}\right)\cos^2\left(\frac{h}{2}\right)\left(8\cos^2\left(\frac{h}{2}\right)-3\right)}{\theta},$$

$$c_1 = \frac{5\sin^3\left(\frac{h}{2}\right)\left(5\cos^2\left(\frac{h}{2}\right)-1\right)}{4\theta},$$

$$c_2 = \frac{5\sin^3(h/2)\cos(h/2)(-15\cos^2(h/2)+16\cos^4(h/2)+3)}{2\theta},$$

$$c_3 = -\frac{5\sin^3\left(\frac{h}{2}\right)\left(16\cos^6\left(\frac{h}{2}\right)-5\cos^2\left(\frac{h}{2}\right)+1\right)}{2\theta},$$

$$d_1 = -\frac{5\sin^2\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)\left(25\cos^2\left(\frac{h}{2}\right)-13\right)}{8\theta},$$

$$d_2 = -\frac{5\sin^2\left(\frac{h}{2}\right)\cos^2\left(\frac{h}{2}\right)\left(8\cos^4\left(\frac{h}{2}\right)-35\cos^2\left(\frac{h}{2}\right)+15\right)}{4\theta},$$

for $\theta = \sin(h/2)\sin(h)\sin(3h/2)\sin(2h)\sin(5h/2)$ [18].

In the collocation finite element method, a differential equation is satisfied at the collocation points. We use the Crank-Nicolson approach and forward difference approximation for equations (3) and (4) at two time levels n and $n+1$

$$w_m = \frac{w_m^{n+1}+w_m^n}{2} \text{ and } v_m = \frac{v_m^{n+1}+v_m^n}{2}, \tag{5}$$

$$\dot{w}_m = \frac{w_m^{n+1}-w_m^n}{\Delta t} \text{ and } \dot{v}_m = \frac{v_m^{n+1}-v_m^n}{\Delta t},$$

where w_m^n, v_m^n are the parameters at the time $n\Delta t$, $\Delta t = k = t_{k+1}-t_k$ is time step and $n = 0,1,\ldots,N$ and " • " demonstrates the derivative with respect to t. Before placing these equations in equations (3) and (4), we must apply the Rubin-Graves type linearization method [19] to the non-linear terms of the equation given in equation (4). By applying Rubin-Graves to the non-linear terms return the form

$$(w^2 w_x)^{n+1} = 2w^n w^{n+1} w_x^n + (w^n)^2 w_x^{n+1} - 2(w^n)^2 w_x^n,$$

$$(w_x w_{xx})^{n+1} = w_x^{n+1} w_{xx}^n + w_x^n w_{xx}^{n+1} - w_x^n w_{xx}^n, \tag{6}$$

$$(wv)^{n+1} = w^{n+1} v^n + w^n v^{n+1} - w^n v^n.$$

By substituting equations (5) and (6) into equations (3) and (4), we obtain the following equations

$$w_{xxx}^{n+1} - v^{n+1} = -w_{xxx}^n + v^n \tag{7}$$

$$\left(1 + 45\Delta t z_m (z_m)_x + \frac{15\Delta t}{2} g_m\right) w^{n+1} + \left(\frac{45\Delta t}{2}(z_m)^2 + \frac{15\Delta t}{2}(z_m)_{xx}\right)(w_x)^{n+1}$$

$$+\frac{15\Delta t}{2}(z_m)_x (w_{xx})^{n+1} + \frac{15\Delta t}{2} z_m v^{n+1} + \frac{\Delta t}{2}(v_{xx})^{n+1} \tag{8}$$

$$= w^n + \frac{45\Delta t}{2}(z_m)^2 (w_x)^n - \frac{\Delta t}{2}(v_{xx})^n,$$

where Δt is the time step, the values $z_m = w$, $g_m = v$ and their derivatives $(z_m)_x = w_x$, $(z_m)_{xx} = w_{xx}$ approximation values are used for the purpose of the linearization of nonlinear terms in the finite element schemes at time step.

Then, by substituting the nodal values W_m, V_m and their required derivatives at the collocation points into equations (7) and (8), the following finite element schemes are obtained as

$$d_1\delta_{m-2}^{n+1} + d_2\delta_{m-1}^{n+1} - d_2\delta_{m+1}^{n+1} - d_1\delta_{m+2}^{n+1}$$

$$-a_1\sigma_{m-2}^{n+1} - a_2\sigma_{m-1}^{n+1} - a_3\sigma_m^{n+1} - a_2\sigma_{m+1}^{n+1} - a_1\sigma_{m+2}^{n+1}$$

$$= -d_1\delta_{m-2}^n - d_2\delta_{m-1}^n + d_2\delta_{m+1}^n + d_1\delta_{m+2}^n$$

$$+a_1\sigma_{m-2}^n + a_2\sigma_{m-1}^n + a_3\sigma_m^n + a_2\sigma_{m+1}^n + a_1\sigma_{m+2}^n, \tag{9}$$

$$\lambda_1\delta_{m-2}^{n+1} + \lambda_2\delta_{m-1}^{n+1} + \lambda_3\delta_m^{n+1} + \lambda_4\delta_{m+1}^{n+1} + \lambda_5\delta_{m+2}^{n+1}$$

$$+\mu_1\sigma_{m-2}^{n+1} + \mu_2\sigma_{m-1}^{n+1} + \mu_3\sigma_m^{n+1} + \mu_4\sigma_{m+1}^{n+1} + \mu_5\sigma_{m+2}^{n+1}$$

$$= \lambda_6\delta_{m-2}^n + \lambda_7\delta_{m-1}^n + \lambda_8\delta_m^n + \lambda_9\delta_{m+1}^n + \lambda_{10}\delta_{m+2}^{n+1}$$

$$+\mu_6\sigma_{m-2}^n + \mu_7\sigma_{m-1}^n + \mu_8\sigma_m^n + \mu_9\sigma_{m+1}^n + \mu_{10}\sigma_{m+2}^n, \tag{10}$$

for $m = 0,1,\dots,M$ where the coefficients of the δ values are

$$\lambda_1 = \alpha_1 + 45a_1\Delta t z_m (z_m)_x + \frac{15a_1\Delta t}{2} g_m + \frac{45b_1\Delta t}{2}(z_m)^2 + \frac{15b_1\Delta t}{2}(z_m)_{xx} + \frac{15c_1\Delta t}{2}(z_m)_x,$$

$\lambda_2 = \alpha_2 + \ 45a_2\Delta t z_m (z_m)_x + \frac{15a_2\Delta t}{2} g_m \ + \frac{45b_2\Delta t}{2}(z_m)^2 + \frac{15b_2\Delta t}{2}(z_m)_{xx} \ + \frac{15c_2\Delta t}{2}(z_m)_x,$

$\lambda_3 = \alpha_3 + \ 45a_3\Delta t z_m (z_m)_x + \frac{15a_3\Delta t}{2} g_m \ + \frac{15c_3\Delta t}{2}(z_m)_x,$

$\lambda_4 = \alpha_2 + \ 45a_2\Delta t z_m (z_m)_x + \frac{15a_2\Delta t}{2} g_m \ - \frac{45b_2\Delta t}{2}(z_m)^2 - \frac{15b_2\Delta t}{2}(z_m)_{xx} \ + \frac{15c_2\Delta t}{2}(z_m)_x,$

$\lambda_5 = \alpha_1 + \ 45a_1\Delta t z_m (z_m)_x + \frac{15a_1\Delta t}{2} g_m \ - \frac{45b_1\Delta t}{2}(z_m)^2 - \frac{15b_1\Delta t}{2}(z_m)_{xx} \ + \frac{15c_1\Delta t}{2}(z_m)_x,$

$\lambda_6 = \alpha_1 + \frac{45b_1\Delta t}{2}(z_m)^2,\ \lambda_7 = \alpha_2 + \frac{45b_2\Delta t}{2}(z_m)^2,\ \lambda_8 = \alpha_3,$

$\lambda_9 = \alpha_2 - \frac{45b_2\Delta t}{2}(z_m)^2,\ \lambda_{10} = \alpha_1 - \frac{45b_1\Delta t}{2}(z_m)^2,$

and the coefficients of the σ values are

$\mu_1 = \frac{15a_1\Delta t}{2} z_m \ + \frac{c_1\Delta t}{2}, \mu_2 = \frac{15a_2\Delta t}{2} z_m \ + \frac{c_2\Delta t}{2},\ \mu_3 = \frac{15a_3\Delta t}{2} z_m \ + \frac{c_3\Delta t}{2},$

$\mu_4 = \frac{15a_2\Delta t}{2} z_m \ + \frac{c_2\Delta t}{2},\ \mu_5 = \frac{15a_1\Delta t}{2} z_m \ + \frac{c_1\Delta t}{2}, \mu_6 = -\frac{c_1\Delta t}{2},$

$\mu_7 = -\frac{c_2\Delta t}{2},\ \mu_8 = -\frac{c_3\Delta t}{2}, \mu_9 = -\frac{c_2\Delta t}{2}, \mu_{10} = -\frac{c_1\Delta t}{2}.$

The pentadiagonal matrix system (9) and (10) consists of $2M + 2$ linear equations and $2M + 10$ unknown parameters $\boldsymbol{\delta}_m = (\delta_{-2}, \delta_{-1}, \ldots, \delta_{M+1}, \delta_{M+2})^T$ and $\boldsymbol{\sigma}_m = (\sigma_{-2}, \sigma_{-1}, \ldots, \sigma_{M+1}, \sigma_{M+2})^T$. To solve this system uniquely, we must eliminate eight unknowns from this system. We can obtain these equations from the left and right boundary conditions and use them in the system to eliminate the fictitious values δ_{-2}, δ_{-1}, δ_{M+1}, δ_{M+2}, σ_{-2}, σ_{-1}, σ_{M+1}, σ_{M+2}. So, the system becomes matrix in the following form

$$A\delta_m^{n+1} = B\delta_m^n + E, \tag{11}$$

$$C\sigma_m^{n+1} = D\sigma_m^n + F, \tag{12}$$

where A, B, C and D are $(M + 1) \times (M + 1)$ pentadiagonal matrixes and E, F are $(M + 1)$ column vectors. This system can be solved by a suitable numerical method. To obtain the solution at the $(n + 1)th$ time level, we need to know the solution at the nth time level. Thinking iteratively, we should firstly know initial vectors $\boldsymbol{\delta}_m^0 = (\delta_0^0, \delta_1^0 \ldots, \delta_M^0)^T$ and $\boldsymbol{\sigma}_m^0 = (\sigma_0^0, \sigma_1^0 \ldots, \sigma_M^0)^T$ for initializing the iterative solution.

3. Initial Vector

To start time evaluation of the approximate solution firstly, we must determine the initial vectors $\boldsymbol{\delta}_m^0$ and $\boldsymbol{\sigma}_m^0$. For this aim, the initial conditions

$$\alpha_1\delta_{m-2}^0+\alpha_2\delta_{m-1}^0+\alpha_3\delta_m^0+\alpha_2\delta_{m+1}^0+\alpha_1\delta_{m+2}^0=W(x_m,0),$$

$$\alpha_1\sigma_{m-2}^0+\alpha_2\sigma_{m-1}^0+\alpha_3\sigma_m^0+\alpha_2\sigma_{m+1}^0+\alpha_1\sigma_{m+2}^0=V(x_m,0),$$

are used at the initial time t_0 for $m=0,1,\ldots,M$. To attain a solvable system, we need to remove the fictitious eight parameters δ_{-2}, δ_{-1}, δ_{M+1}, δ_{M+2}, σ_{-2}, σ_{-1}, σ_{M+1}, σ_{M+2}. After eliminating the fictitious parameters in the system, we can uniquely solve the system which is the type of $(2M+2)\times(2M+2)$. Here, the essential equations are obtained by using the first and the second derivatives of approximate initial conditions based on trigonometric quintic B-Splines. So, the initial matrix systems are written with the following form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_2 & \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_2 & \alpha_1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & \cdots & 0 & 0 & \alpha_1 & a_{2M+1,2M} & a_{2M+1,2M+1} & a_{2M,2M+2} \\ 0 & \cdots & 0 & 0 & 0 & a_{2M+2,2M} & a_{2M+2,2M+1} & a_{2M+2,2M+2} \end{bmatrix} \begin{bmatrix} \delta_{m-2}^0 \\ \delta_{m-1}^0 \\ \vdots \\ \delta_M^0 \\ \sigma_{m-2}^0 \\ \sigma_{m-1}^0 \\ \vdots \\ \sigma_M^0 \end{bmatrix} = \begin{bmatrix} W(x_0,0) \\ W(x_1,0) \\ \vdots \\ W(x_M,0) \\ V(x_0,0) \\ V(x_1,0) \\ \vdots \\ V(x_M,0) \end{bmatrix},$$

where the entries of the matrix are given the following equations

$$a_{11}=a_{2M+2,2M+2}=\alpha_3+\frac{\alpha_1\beta_2\gamma_3-\alpha_2\beta_1\gamma_3}{\beta_1\gamma_2-\beta_2\gamma_1},$$

$$a_{21}=a_{2M,2M+2}=\alpha_2-\frac{\alpha_1\beta_1\gamma_3}{\beta_1\gamma_2-\beta_2\gamma_1},$$

$$a_{12}=a_{2M+2,2M+1}=\frac{2\alpha_1\beta_2\gamma_2-2\alpha_2\beta_2\gamma_1}{\beta_1\gamma_2-\beta_2\gamma_1},$$

$$a_{22}=a_{2M+1,2M+1}=\alpha_3-\frac{\alpha_1\beta_1\gamma_2+\alpha_1\beta_2\gamma_1}{\beta_1\gamma_2-\beta_2\gamma_1},$$

$$a_{13}=a_{2M+2,2M}=\frac{2\alpha_1\beta_1\gamma_2-2\alpha_2\beta_1\gamma_1}{\beta_1\gamma_2-\beta_2\gamma_1},$$

$$a_{23}=a_{2M+1,2M}=\alpha_2-\frac{2\alpha_1\beta_1\gamma_1}{\beta_1\gamma_2-\beta_2\gamma_1}.$$

4. Numerical Results and Discussion

In this section, the accuracy, efficiency and computational complexity of the present scheme is demonstrated on the model problem (2). To measure the accuracy of the present method and see how the approximation of the numerical solution approaches to exact ones, we used the error norms given as following equations

$$L_2 = \|W - W_M\|_2 \simeq \sqrt{h \sum_{j=0}^{M} \left|W_j - (W_M)_j\right|^2},$$

and

$$L_\infty = \|W - W_M\|_\infty \simeq \max_{j=0} \left|W_j - (W_M)_j\right|.$$

The exact soliton solution of the model problem Sawada-Kotera is

$$w(x,t) = 2k^2 \mathrm{sech}^2[k(x - 16k^4 t - \lambda)],$$

with the initial condition

$$w(x,0) = 2k^2 \mathrm{sech}^2[k(x - \lambda)],$$

where $\lambda = 0$ [4].

The numerical values obtained by applying the collocation finite element method are shown in Table 1. For the numerical calculations, the values are used as $\Delta t = 0.01,\ t = 10,\ k = 0.2$ and the solution region is taken as $[-20,20]$. In order to view the convergence of the method, L_2 and L_∞ error norms are given with the different values of t and smaller space steps $h = \Delta x = 0.25,\ 0.1,\ 0.05$ and 0.025. It is seen from Table 1 that the method provides a reasonable approximation for the increasing element numbers. The process can be associated with the efficiency of collocation method based on space knots. It should also be noted here, the method can be used with the accuracy desired for the smaller step sizes.

Table 1: Comparison of the error norms L_2 and L_∞ for the values $\Delta t = 0.01, k = 0.2$ and different values of t and Δx

	$\Delta x = 0.25$		$\Delta x = 0.1$	
t	L_2	L_∞	L_2	L_∞
2	3.896238×10^{-7}	1.632945×10^{-7}	2.377189×10^{-8}	1.285998×10^{-8}

4	7.100182×10^{-7}	2.888393×10^{-7}	3.421736×10^{-8}	1.695229×10^{-8}
6	1.063679×10^{-6}	4.651501×10^{-7}	4.672788×10^{-8}	2.163092×10^{-8}
8	1.506294×10^{-6}	6.774396×10^{-7}	5.614304×10^{-8}	2.370217×10^{-8}
10	2.036621×10^{-6}	8.864584×10^{-7}	7.370401×10^{-8}	3.021153×10^{-8}

	$\Delta x = 0.05$		$\Delta x = 0.025$	
t	L_2	L_∞	L_2	L_∞
2	5.279064×10^{-9}	3.082057×10^{-9}	1.306097×10^{-9}	7.658709×10^{-10}
4	7.279155×10^{-9}	4.075803×10^{-9}	1.787015×10^{-9}	1.010783×10^{-9}
6	9.740206×10^{-9}	5.405006×10^{-9}	2.364103×10^{-9}	1.338265×10^{-9}
8	1.163328×10^{-8}	6.400158×10^{-9}	2.928870×10^{-9}	1.637632×10^{-9}
10	1.284928×10^{-8}	6.645599×10^{-9}	3.050049×10^{-9}	1.597710×10^{-9}

In Table 1 which is composed of decreasing Δx and increasing final times t, it is aimed to see the variation of the error norms L_2 and L_∞ in a broad perspective. According to the obtained results, it is seen that the error norms L_2 and L_∞ are minimum at final time $t = 2$ and $\Delta x = 0.025$ spatial step size. In order to investigate the effect of temporal and spatial step sizes on the error norms L_2 and L_∞, the newly obtained error norms L_2 and L_∞ at different temporal and spatial step sizes based on a constant final time t are presented in Tables 2 and 3.

Table 2 shows the approximate solutions for the fixed values $t = 10$, $k = 0.2$, $\Delta x = 0.1$ and the decreasing values of time step Δt. It can be clearly seen from the table, the error norms L_2 and L_∞ have decreased with the temporal mesh Δt.

Table 2: Comparison of the error norms for the values $\mathrm{t} = 10$, $\Delta\mathrm{x} = 0.1$ and different values of Δt

Δt	L_2	L_∞
0.04	1.864118×10^{-7}	5.589674×10^{-8}
0.02	1.450729×10^{-7}	4.430777×10^{-8}
0.01	7.370401×10^{-8}	3.021153×10^{-8}
0.005	1.912650×10^{-8}	6.346669×10^{-8}

For the smaller space step, the error norms L_2 and L_∞ are given in Table 3 for different space steps Δx and the fixed values $t = 10$, $k = 0.2$, $\Delta t = 0.01$. The numerical results have smallest errors with the larger element numbers.

Table 3: Comparison of the error norms for the values $t = 10$, $\Delta t = 0.01$ and different values of Δx.

Δx	L_2	L_∞
0.25	2.036621×10^{-6}	8.864584×10^{-7}
0.1	7.370401×10^{-8}	3.021153×10^{-8}
0.05	1.284928×10^{-8}	6.645599×10^{-9}
0.025	9.852056×10^{-10}	5.924715×10^{-10}

In Table 2 and Table 3, the error norms L_2 and L_∞ obtained by decreasing spatial and temporal time steps, respectively, are compared. As it can be seen from the tables, the decrease of the spatial step sizes seriously affects the shrinking of the the error norms L_2 and L_∞. Since the collocation finite element method based on location points has been used, it is expected that the error norms in Table 3 will also decrease significantly for much more smaller values of spatial step sizes.

The model problem Sawada-Kotera equation often accompanies the non-linear wave phenomena occur in shallow water, acoustic waves in plasma, fluid dynamics etc. In order to understand the behavior of the physical phenomena for Sawada-Kotera problem, the graphics and simulations of the problem are pictured for the different values of $k = 0.2$ and $k = 0.4$. Figure 1 and Figure 2 illustrate the wave's behavior for the $\Delta t = 0.01$, $\Delta x = 0.1$ and different values of t. Both of the parameters $k = 0.2$ and $k = 0.4$, the shape of the solutions do not change.

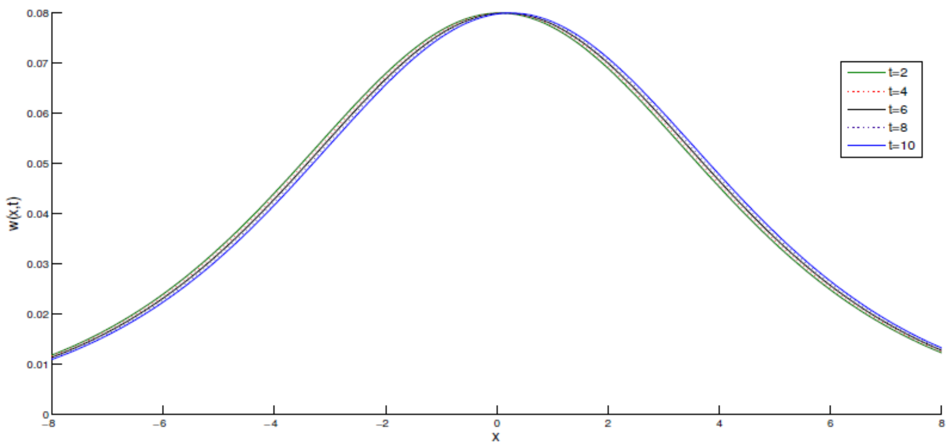

Figure 1: One solitary wave motion for the value k = 0.2

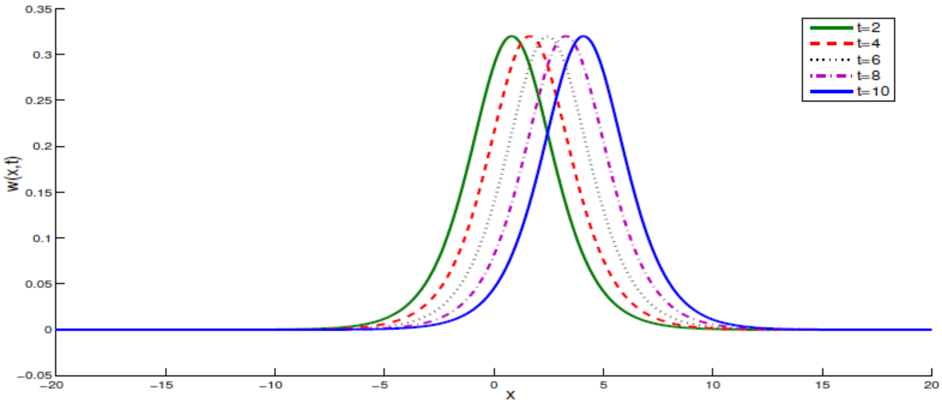

Figure 2: One solitary wave motion for the value $\mathrm{k} = 0.4$

As it is seen in Figure 1, the graphs of the numerical solutions obtained at different times t are close to each other for the amplitude value of $k = 0.2$. In Figure 2, the graphs of the numerical solutions obtained for a larger amplitude value $k = 0.4$, shift significantly at different t times. Both of the figures are drawn at final times $t = 2, 4, 6, 8$ and 10. Also it is clearly seen that in the figures, solitary waves spread of the defination interval according to chosen amplitude.

The exact solutions and numerical solutions are depicted together in Figure 3 for the values $\Delta t = 0.01$, $\Delta x = 0.1, t = 10$ and different values of $k = 0.2$ and $k = 0.4$. In the figures, it can be verified how the approximate solutions are accurate.

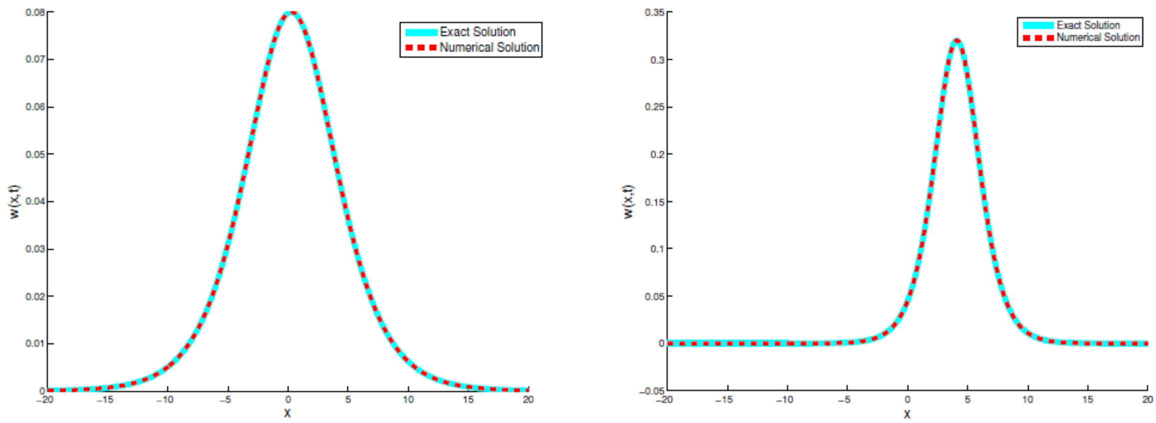

Figure 3: The numerical and exact solutions for $\mathrm{k} = 0.2$ and $\mathrm{k} = 0.4$, respectively

5. Conclusion

In this paper, we have considered the numerical solution of the Sawada-Kotera problem classified with the fifth order KdV equations. The finite element is combined scheme with the collocation finite element method based on the trigonometric quintic B-spline basis is formed and analysed. The detailed analysis of the proposed method is given with the tables and graphs. It is found that the application of the collocation finite element method has high accuracy values for the larger element numbers. Concerning the spatial discretization use of the present method leads to quick convergences in the space. Also given as the graphs and simulations, the numerical solutions are in agreement with the exact ones. In that case, the present method is a useful numerical method for obtaining the numerical solutions to a wide range of non-linear problems in the theory of solitary waves.

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