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APPROXIMATE ANALYTICAL SOLUTIONS OF THE SCHRÖDİNGER EQUATION IN CENTRAL POTENTIAL FIELD

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Abstract. We investigate the approximate l-state solutions of the Schrödinger equation for Hulthn plus a class of Yukawa potential. In this context, we construct the bound-state energy equation and the wave function expressed by the Gauss hypergeometric function by means of asymptotic iteration approach in detail.

1. Introduction

Bound state solutions of the Schrödinger equation for a quantum system interacting with spherical symmetric potential models are among the most important in various fields of physics. The l-state solutions of the non-relativistic wave equation for exponential potentials especially are of great interest in literature[1, 2, 3, 4]. Under consideration of this problem, it cannot be possible to obtain analytical solutions without approximations. For this reason, approximations and their applications are essential in quantum mechanical models. In the present work, we choose the proper approximate expression for investigating of analytical solutions. Asymptotic iteration method proposed by Ciftci et al. [5, 6, 7] is a powerful tool for solving second-order homogeneous linear differential equation. This method gives a precise way to probe the bound state solutions of the Schrödinger wave equation for any l-state. In this context, the purpose of this work is to apply the asymptotic iteration approach to investigate the non-relativistic treatment of Hulthn plus a class of Yukawa potential.
A combination of two potentials has aroused extensive research interest in literature. Many researchers have adopted this type of potential to carry out some works[8, 9, 10, 11, 12, 13, 14, 15]. Motivated by these works, we consider the following form of potential model which is the superposition of Hulthn[16] and a class of Yukawa[17] potentials

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$$V(r) = -\frac{2\alpha Z e^2 e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{Ae^{-\alpha r}}{r} - \frac{Be^{-2\alpha r}}{r^2} \tag{1.1}$$

where the parameter Z denotes the atomic number. α is the screening parameter which determines the range of potential, A and B are the coupling strengths of the potential. Hulthn plus a class of Yukawa potential has been newly proposed by Ahmadov et al.[15]. Bound state solutions of the Dirac equation under the spin and pseudospin symmetries for this potential including Coulomb-like tensor interaction have been presented in [15]. As far as we know, no report has been made so far in literature employing this combined potential within the framework of non-relativistic theory. For this reason, we focus on studying the model of a quantum system with Hulthn plus a class of Yukawa potential by means of asymptotic iteration method.

2. Overview of the asymptotic iteration method

The purpose of this section is to briefly outline the asymptotic iteration approach used to solve the second-order differential equations. The details of this approach have been reported in [5, 6, 7]. We start with the approach by writing a general form of the second-order differential equation

$$y^{''}(r) = \lambda_0(r)\, y^{'}(r) + s_0(r)\, y(r) \tag{2.1}$$

where $\lambda_0(r)$ and $s_0(r)$ functions in $C_\infty(a,b)$ are sufficiently differentiable. The general solution of Equation (2.1) can be obtained in the following form

$$y(r) = \exp(-\textstyle\int^r \alpha(r')dr')\left[C_2 + C_1 \textstyle\int^r \exp\left(\textstyle\int^{r'}\left[\lambda_0(\tau)+2\alpha(\tau)\right]d\tau\right)dr'\right] \tag{2.2}$$

For sufficiently large k,

$$\frac{s_k(r)}{\lambda_k(r)} = \frac{s_{k-1}(r)}{\lambda_{k-1}(r)} = \alpha(r) \tag{2.3}$$

in which

$$\begin{aligned} \lambda_k(r) &= \lambda_{k-1}^{'}(r) + s_{k-1}(r) + \lambda_0(r)\lambda_{k-1}(r) \\ s_k(r) &= s_{k-1}^{'}(r) + s_0(r)\,\lambda_{k-1}(r) \end{aligned} \tag{2.4}$$

If the eigenvalue problem has exact analytical solutions, the termination condition Equation (2.3), or equivalently

$$\delta_k(r) = \lambda_k(r)\, s_{k-1}(r) - \lambda_{k-1}(r)\, s_k(r) = 0 \tag{2.5}$$

produces, at each iteration, an expression that is independent of r. It is noted that k displays the iteration number. Physically meaningful solution of Equation (2.1) is provided by the first term of Equation (2.2) not the second term, so we can use the first term as the wave function generator

$$y(r) = C_2 exp\left(-\int^r \frac{s_k(r')}{\lambda_k(r')}dr'\right) \tag{2.6}$$

in which C_2 denotes the integrant constant which can be determined by normalization.

There is also an alternative way to determine the wave function within the framework of AIM. The following second-order homogeneous linear differential equation allows us to find the wave function

$$y^{''} = 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(m+1)}{x}\right)y' - \frac{wx^N}{1-bx^{N+2}}y \tag{2.7}$$

in which N=-1,0,1... and a , w , m are the real numbers which are to be determined. The general solution of Eq.(2.8) is found in the following form

$$y_n(x) = (-1)^n C_2 (N+2)^n (\mu)_n {}_2F_1\left(-n,\ t+n;\ \mu; bx^{N+2}\right) \tag{2.8}$$

where $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)}$, $\mu = \frac{2m+N+3}{N+2}$, $t = \frac{(2m+1)b+2a}{(N+2)b}$ and ${}_2F_1$ denotes to the Gauss hypergeometric function being defined as

$${}_2F_1(-n,\ b,c,x) = \sum_{k=0}^{n} \frac{(-n)_k (b)_k x^k}{(c)_k k!}$$

the Pochhammer symbol $(\alpha)_k$ is defined by $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots L(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ for $k = 1, 2, 3\ldots$ It should be mentioned that the details concerning AIM can be found in [5, 6, 7].

3. Bound state solutions of Hulthn plus a class of Yukawa potential in approximate analytic form

Firstly, we focus on the separation of variables for the Schrödinger equation. The motion of a particle in central potential field is described in non-relativistic theory as follows

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi(\overrightarrow{r}) = E\psi(\overrightarrow{r}) \tag{3.1}$$

where E and μ define non-relativistic energy and reduced mass, $V(r)$ is the central potential, $\hbar$ is the Planck constant.

The following expression is the Laplace operator in three dimensions

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2}{\partial\varphi^2} \tag{3.2}$$

We employ a form of the total spherical wave function as

$$\psi(r,\theta,\varphi) = \frac{R(r)}{r} Y_l^m(\theta,\varphi) \tag{3.3}$$

where $R(r)$ and $Y_l^m(\theta,\phi)$ are the radial wave function and the spherical harmonics. The way of separating variables has been applied to Schrödinger equation. Based on this way, we obtain the non-relativistic wave equation with respect to $R(r)$

$$\frac{d^2R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \tag{3.4}$$

When inserted Equation (1.1) into Equation (3.4), the radial Schrödinger wave equation becomes

$$\frac{d^2R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E - \left(-\frac{2Ze^2\alpha e^{-2\alpha r}}{(1-e^{-2\alpha r})} - \frac{Ae^{-\alpha r}}{r} - \frac{Be^{-2\alpha r}}{r^2}\right)\right) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \tag{3.5}$$

Equation (3.5) cannot be solved analytically for any l-state because of the centrifugal term. Therefore, to solve this equation, we need to use an approximation of the following form

$$\frac{1}{r} \approx \frac{2\alpha e^{-\alpha r}}{(1-e^{-2\alpha r})},\ \frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \tag{3.6}$$

This scheme is called as the Greene-Aldrich approximation[18] which is only suitable for a short range (small α) potential. If we apply this approximation and transformation $x = e^{-2\alpha r}$ to Equation (3.5), then we can rewrite the radial wave equation in non-relativistic theory

$$\frac{d^2R(x)}{dx^2} + \frac{1}{x}\frac{dR(x)}{dx} + \left[-\frac{\gamma^2}{x^2} + \frac{\beta^2}{x\,(1-x)} - \frac{\sigma^2}{(1-x)^2} - \frac{l(l+1)}{x(1-x)^2}\right]R\,(x) = 0 \quad (3.7)$$

In Equation (3.7), we take the abbreviations as

$$\beta^2 = \frac{\mu\,(V_0 + V_0')}{2\alpha^2\hbar^2}, \gamma^2 = -\frac{\mu E}{2\alpha^2\hbar^2}, \sigma^2 = -\frac{\mu B'}{2\alpha^2\hbar^2} \quad (3.8)$$

For simplicity, we take as $V_0 = 2\alpha Ze^2$, $V_0' = 2A\alpha$ and $B' = 4B\alpha^2$ in above expressions. By analyzing the asymptotic behaviour of Equation (3.7) at the origin and infinity, we can propose the wave function in terms of $R(x)$

$$R\,(x) = x^{\gamma}(1-x)^{\delta+1}f(x) \quad (3.9)$$

with

$$\delta = -\frac{1}{2} + \frac{\sqrt{1 + 4\sigma^2 + 4l(l+1)}}{2} \quad (3.10)$$

After taking the proposed wave function given in Equation (3.9) and inserting this into Equation (3.7), we obtain the second-order homogeneous linear differential equation as

$$\frac{d^2f(x)}{dx^2} = \left[\frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x(1-x)}\right]\frac{df(x)}{dx} + \left[\frac{(2\gamma + 1)\,(\delta + 1) - \beta^2 + l(l+1)}{x(1-x)}\right]f(x) \quad (3.11)$$

This equation is convenient to apply the asymptotic iteration approach. Comparison of Equation (3.11) and Equation (2.1) gives the values of λ_0 and s_0. With Equation (2.4), it is then easy to obtain the values of $\lambda_n(x)$ and $s_n(x)$ in the following forms

$$\lambda_0 = \frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x(1-x)}$$

$$s_0 = \frac{(2\gamma + 1)\,(\delta + 1) - \beta^2 + l(l+1)}{x(1-x)}$$

$$\lambda_1 = \frac{(2\gamma + 2\delta + 3)\,x}{x(1-x)} - \frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x^2\,(1-x)} + \frac{(2\gamma + 2\delta + 3)\,x - (2\gamma + 1)}{x(1-x)^2}$$
$$+ \frac{(1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2}{x\,(1-x)} + \frac{((2\gamma + 2\delta + 3)\,x - (2\gamma + 1))^2}{x^2(1-x)^2}$$

$$s_1 = -\frac{(1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2}{x^2\,(1-x)} + \frac{(1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2}{x(1-x)^2}$$
$$+ \frac{\left((1+\delta)\,(2\gamma + 1) + l\,(l+1) - \beta^2\right)\left((2\gamma + 2\delta + 3)\,x - (2\gamma + 1)\right)}{x^2(1-x)^2}$$
$$\vdots \quad (3.12)$$

To calculate the radial energy eigenvalues, we employ the termination condition given by Equation (2.3). Thus, these energy eigenvalues are obtained as

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Rightarrow \gamma_0 = -\frac{l(l+1) - \beta^2 + \delta + 1}{2\,(\delta + 1)}$$

$$\begin{aligned}
&\frac{s_1}{\lambda_1}=\frac{s_2}{\lambda_2} \Rightarrow \gamma_1=-\frac{l(l+1)-\beta^2+3\delta+4}{2\left(\delta+2\right)}\\
&\frac{s_2}{\lambda_2}=\frac{s_3}{\lambda_3} \Rightarrow \gamma_2=-\frac{l(l+1)-\beta^2+5\delta+9}{2\left(\delta+3\right)}\\
&\vdots
\end{aligned} \tag{3.13}$$

Based on the preceding expressions, we can generalize in the following form

$$\gamma_n=-\frac{l(l+1)-\beta^2+(2n+1)\delta+(n+1)^2}{2\left(\delta+n+1\right)}, n=0,1,2,\dots \tag{3.14}$$

Substituting the values of γ, β and σ given in Equation (3.8) and the value of δ given in Equation (3.10) into Equation (3.14), it can be built as

$$E=-\frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu}{\hbar^2\alpha}\left(Ze^2+A\right)-l\left(l+1\right)-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-\frac{2\mu B}{\hbar^2}+l\left(l+1\right)}}{2n+1+2\sqrt{\frac{1}{4}-\frac{2\mu B}{\hbar^2}+l\left(l+1\right)}}\right)^2 \tag{3.15}$$

Thus, we find the energy spectrum for Hulthn plus a class of Yukawa potential in spherical coordinates. By comparing with Equation (3.11) and Equation (2.7) and following expressions below Equation (2.8), we can easily find as

$$\begin{aligned}
&b=1, N=-1, a=\delta+1, m=2\gamma-1\\
&\mu=2\gamma+1, \mathrm{t}=2\gamma+2\delta+2
\end{aligned} \tag{3.16}$$

Directly the function of $f(x)$ can be obtained from Equation (2.8) with the substitution of Equation (3.16) in the following form

$$f\left(x\right)=(-1)^n C_2(2\gamma+1)_n {}_2F_1\left(-n,\quad 2\gamma+2\delta+2+n;2\gamma+1\ ;x\ \right) \tag{3.17}$$

If we put Equation (3.17) into Equation (3.9), we can obtain the unnormalized radial wave function for Hulthn plus a class of Yukawa potential

$$\begin{aligned}
R\left(x\right)=&(-1)^n C_2(2\gamma+1)_n(1-x)^{\delta+1}x^{\gamma}\\
&\times\ {}_2F_1\quad\left(-n,\quad 2\gamma+2\delta+2+n;2\gamma+1\ ;x\ \right)
\end{aligned} \tag{3.18}$$

Then, substituting $x=e^{-2\alpha r}$ into Equation (3.18), we write the unnormalized radial wave function for considered potential with respect to r

$$\begin{aligned}
&R\left(r\right)=(-1)^n C_2(2\gamma+1)_n\left(1-e^{-2\alpha r}\right)^{\delta+1}e^{-2\alpha\gamma r}\\
&\times {}_2F_1\left(-n,\quad 2\gamma+2\delta+2+n;2\gamma+1\ ;e^{-2\alpha r}\ \right)
\end{aligned} \tag{3.19}$$

in which C_2 is the integration constant.

4. Conclusion

We consider Hulthn plus a class of Yukawa potential because of the importance of the combined potentials. In this connection, bound state solutions of the Schrödinger equation have been established for any l-state within the framework of asymptotic iteration method. To achieve this, we apply a proper approximation scheme which is called as Greene-Aldrich approximation. Therefore, we construct the energy eigenvalues and unnormalized wave function in approximate analytic form. We note that the theoretical results obtained for the considered potential may shed light on the applications in different fields.

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