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A NEW MEASURE OF PREFERRED DIRECTION FOR CIRCULAR DATA USING ANGULAR WRAPPING

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Abstract. The statistical techniques which are developed for the analysis of data in the linear number system cannot be applied to directional data directly. Circular data may be discontinuous in some principal interval. These discontinuities cause failure results in the circular statistics. Because of that the proposed wrapping operator must be used for data, which are defined in the discontinuous range. However, in both continuity and discontinuity, the wrapping operator works correctly. The most common preferred directions for circular data are circular mean and variance summarizing and comparing them. Although circular data has a very important role in statistics, the literature is weak in terms of statistical analysis of circular data. It creates a gap in this field. This study examines the preferred direction of circular data to fill this gap and presents a new measure of preferred direction for circular data using angular wrapping. Four different artificial and three real datasets are employed to evaluate the performance of the proposed methods. The results demonstrate the superiority of the proposed methods in terms of the absolute error and absolute percentage error. Consequently, it has been seen that the proposed methods give more consistent and more accurate results than the vectorial methods.

1. Introduction

The obtained data from observation can be existed in various measurement spaces. One of the measurement spaces is an angular space in which data are

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expressed as the angular. For instance, a biologist may be measuring the orientation of an animal depending on a factor in the nature or a geologist may be interested in the direction of the earth's magnetic pole. Such directions may be univariate as in the first example or bivariate like the second one ([1]). In general, data identifying in angular space are referred to as directional data. Data showing univariate angular change are called as circular data; data showing bivariate angular change are called as spherical data. If data show more than two angular changes are called as hyper-spherical data.

The statistical techniques which are developed for the analysis of data in the linear number system cannot be applied to directional data directly. The most illustrative example of this situation is to consider a sample of size two on the circle consisting of the angles 350° and 10°, mean direction of these angles is 0° when linear mean formula is applied to them, their linear mean is 180°. Many problems arise when the other statistics such as dispersion measurement and correlation are applied to circular data ([2]). Therefore, circular statistics have been developed for circular data as a branch of statistical science. Circular statistics include statistical techniques to summarize the obtained data in angular space and to interpret that. In the literature several studies were performed for the analysis of directional data and circular statistics were applied to different field of study. Mardia ([3]) is the first reference book for the analysis of circular data. Fisher ([4]), Mardia and Jupp ([5]), and Jammalamadaka and SenGupta ([1]) are good alternative reference books in this field. Statistics of circular data are used in different scientific disciplines such as earth sciences ([6]), meteorology, biology, physics, psychology ([7,8]), mathematics and statistics ([9–18]), image analysis ([19]), medicine ([20–22]), astronomy and agriculture ([23]), geography and marine sciences ([24–27]), computer sciences ([28, 29]).

In many research, the usage of appropriate descriptive statistics is useful to summarize the data. Representation of two-dimensional data in the form of angle and vector on the unit circle is not only one. Because the value of circular observation may be changed according to zero direction and the selection of clockwise or anticlockwise. The obtained results are a function of the given observation, and the function does not depend on the arbitrary value. Owing to these properties, circular data analysis is quite different from statistical analysis. The need of arbitrary zero direction and orientation often make many statistical techniques and measures incorrect and meaningless. Therefore, various methods for descriptive statistics of circular data have been developed in the literature. Firstly, these methods were developed by Fisher ([4]). Batschelet ([30]), Fisher ([4]), Zar ([31]), Jammalamadaka and SenGupta ([1]) are the source books for the descriptive statistics of circular data.

The vectorial methods proposed by Mardia ([3]) define circular data as vectors on the unit circle. If the inspected data are vectorial data, vectorial methods are good approximation. If the inspected data are directional or periodic data, the proposed methods by Mardia ([3]) can give the failure or approximate results. Therefore, several statistics such as the angular mean and the angular variance are proposed for circular data by using wrapping operator to eliminate this approximation in this study. In view of this, the motivation of this paper is to merge the burgeoning field of circular statistics with different disciplines as environmental, biological and ecology science to see how the different areas can be of mutual benefit.

The remainder of the paper is organized as follows. Section 2 presents an overview of circular data. The proposed preferred directions for circular data are discussed in Section 3. Experimental results are highlighted in Section 4. Section 5 focusses on the applications of circular statistics in real environmental, biological and ecological problems. Eventually, the conclusions are drawn in Section 6.

2. Overview of Circular Data

Circular variables are defined on a circle curve unlike number line. For this reason, they show periodic changes. In this way, periodic variables are defined as

$$\theta = mod\left(\theta + 2k\pi, 2\pi\right), \left(k = 0, \pm 1, \pm 2, \ldots\right).$$

These variables are periodic with 2π radian period. In the same way, the stability of periodicity in different phase values can alter circle number line's starting point location or definition interval boundaries. The most common mathematical representation of starting point accepts the positive x-axis as starting point and counterclockwise as orientation. Two different approaches are used as the definition interval of radian unit. These are one-sided principal interval $[0, 2\pi)$ and symmetric principal interval $(-\pi, \pi]$ ([32]). The symmetric principal interval is preferred in this study.

Although directional data are continuous at each point on the circle, when directional data addressed linearly, it creates the illusion of discontinuities at the 0 radian point according to the one-sided principal interval and at the π point according to the symmetric principal interval. Therefore, classical statistical techniques are insufficient and occasionally give failure results in the analysis of directional data.

Generally, observed circular data are measured in degrees; however, this study is assumed that circular data are measured in radians. Circular data are converted from degrees (α) to radians (θ) by using following equation

$$\theta = \frac{\alpha}{180}\pi.$$

Circular data can be applied to periodic data as well as data which show the angular change. Periodic data, such as the days of the week and time of the day can be exemplified for this situation. Periodic data (x) are converted into angular space by

$$\theta = \frac{2\pi x}{T},$$

where T gives the period of observed data.

Circular data can be cut across interval boundaries because of some arithmetic operations. In this situation, principal interval can be reduced to symmetric principal interval by using wrapping process. Wrapping process is given in the following equation

$$\theta = mod\,(\phi + \pi, 2\pi) - \pi.$$

In this study, wrapping process is represented by $Wrap_\pi(\cdot)$ operation and it is defined as

$$\theta = Wrap_\pi\,(\phi)\,.$$

2.1. **Addition of Two Circular Values.** The sum of two variables which are in the same unit and show the angular change is the same as in the linear number system. However, in this operation, principal interval boundaries can be cut across. In this situation, the obtained values from the result of addition can be reduced to the symmetric principal interval by

$$\psi = Wrap_\pi\,(\phi + \theta)\,.$$

2.2. **Subtraction of Two Circular Values.** The subtraction involves some complexity. Some equations such as equation (1) may lead to failure results by the reason of the characteristics of the circular data.

$$\psi = \phi - \theta \tag{1}$$

Therefore, if counterclockwise is assumed as positive and clockwise is assumed as negative, the subtraction will be easier. The angle θ is accepted as starting point, so that in the range $(\theta - \pi, \theta)$ is assumed as negative region and in the range $(\theta, \theta + \pi]$ is assumed positive region (Figure 1).

Therefore, we assumed that a value (ϕ) in the range of $(\theta - \pi,\ \theta)$ is smaller than θ and a value (ϕ) in the range of $(\theta,\ \theta + \pi]$ is bigger than θ. Under these circumstances, the smallest difference between two angles is calculated by

$$\psi = Wrap_\pi\,(\phi - \theta)\,.$$

2.3. **Distance of Two Circular Values.** Every pair of distinct points on a circle determines two arcs. If two points are not directly opposite each other, one of these arcs, the minor arc, will subtend an angle at the center of the circle that is less than π radians. The other arc, the major arc, will subtend an angle greater than π radians ([33]).

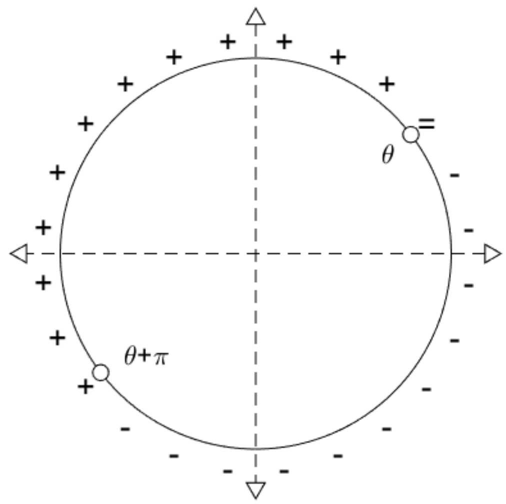


FIGURE 1. Positive and negative region according to θ radian

When the distance between two points on the circle number line is calculated, the minor arc length is preferred. The minor arc length is calculated by using the following equation ([34])

$$\psi_0 = \pi - |\pi - |\phi - \theta|| . \tag{2}$$

This equation gives accurate results in radians. However, ϕ and θ must be in the principal interval. Another alternative equation is given in the following equation ([35])

$$\psi_v = 1 - \cos(\phi - \theta). \tag{3}$$

This equation takes values in $[0, 2]$. It is not equal to the length of the $[0, \pi]$ radian. If this equation multiplies by $\pi/2$, it will be in the desired principal interval. In spite of this improvement, it may not always produce the desired results due to the curvature of the cosine function. Proposed method which is given in the equation (4) gives the best results by taking the absolute value of the subtraction.

$$\psi_a = |Wrap_\pi (\phi - \theta)| \tag{4}$$

2.4. **Circular Mean and Variance.** The mean of the circular data cannot be inherently calculated like mean of linear data. The most illustrative example of

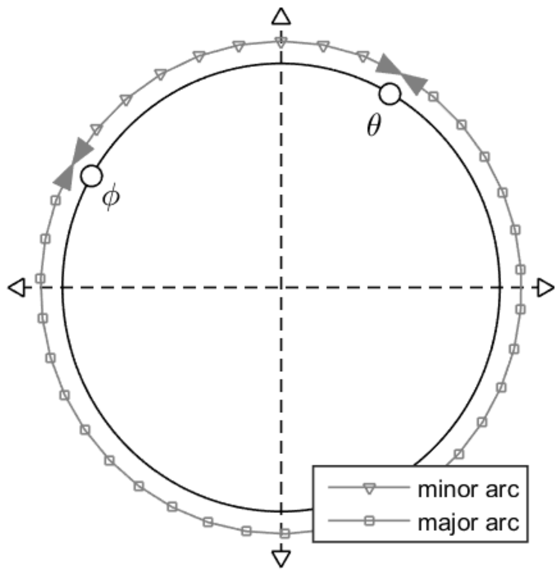


FIGURE 2. The minor arc and the major arc of two points

this situation is to consider a sample of size two on the circle consisting of the angles 355° and 5°, mean direction of these angles is 0° when linear mean formula is applied to these angles, their linear mean is 180°. Angular continuity is exposed to numerical discontinuity; therefore, mean of the circular data cannot be calculated properly. When Mardia and Jupp ([5]) proposed a method to calculate the circular mean; in this method, each observation regarded as unit vectors and the resultant length of these vectors is calculated. The average horizontal component of circular data is calculated by

$$\bar{C} = \frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i),$$

and the average vertical component of circular data is calculated by using equation (5)

$$\bar{S} = \frac{1}{n}\sum_{i=1}^{n}\sin(\theta_i). \tag{5}$$

The resultant length of these components is calculated by the following equation

$$\bar{R} = \sqrt{\bar{C}^2 + \bar{S}^2}.$$

The angle of this resultant length is defined as

$$\bar{\theta} = atan2\left(\bar{S}, \bar{C}\right).$$

In this equation, $\bar{\theta}$ also gives the circular mean where atan2 is an arc tangent function that ranges between $(-\pi, \pi]$. In statistical analysis, variance is the most widely used measure of variability. The sample variance is the sum of the squared differences around the arithmetic mean divided by the sample size minus one. Circular variance is calculated by using vectorial approach as

$$V_v = 1 - \bar{R}.$$

In which the circular variance (V_v) takes value in $[0, 1]$. This method was proposed by Mardia and Jupp ([5]) but this approach is unsuitable for the definition of variance in the linear data.

3. Proposed Preferred Directionsfor Circular Data

The definitions of the mean and the variance are given clearly in the literature. These definitions were altered for circular data because of the special nature of circular data, but these are not suit to the standard definition. Thus, in the literature, several statistics were obtained by using vectorial mean or resultant length. The most common preferred directions for circular data are the circular mean and the variance ([4]). Circular mean and circular variance are the most commonly used parameters to summarize and compare the circular data. Circular variance shows the spread of a dataset. If all circular data are concentrated in one direction, the average resultant vector length will be close to one. If the circular data show a widespread over the unit circle, that is, if they show a uniform distribution, the average resultant vector length will be close to or equal to zero.

3.1. **Circular Mean Based on Angular Difference.** In this study, an iterative method which based on the angular distance of circular data is suggested. Let, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ represent the circular data. The first selected value is considered as initial mean and this situation is represented as follows

$$\bar{\theta}_1 = \theta_1.$$

Afterwards, each value is treated with the current mean respectively and equation (6) is obtained.

$$\bar{\theta}_i = \bar{\theta}_{i-1} + \frac{1}{i} Wrap_\pi\left(\theta_i - \bar{\theta}_{i-1}\right), \quad (i = 2, \ldots, n) \tag{6}$$

After all values are treated, general mean is calculated by using equation (7).

$$\bar{\theta} = Wrap_\pi\left(\bar{\theta}_n\right) \tag{7}$$

3.2. **Circular Variance Based on Angular Difference.** The sample variance is the sum of the squared differences around the arithmetic mean divided by the sample size minus one. Vectorial variance proposed by Mardia and Jupp ([5]) is unsuitable for the definition of variance in the linear data. For this reason, vectorial variance method does not perform properly for circular data. Therefore, this study proposes a circular variance by using the angular approach as in equation (8).

$$V_a = \frac{1}{n-1} \sum_{i=1}^{n} Wrap_{\pi}^{2} \left(\theta_i - \bar{\theta}\right) \tag{8}$$

In which $\bar{\theta}$ represents the circular mean which is given in the previous section.

4. Experimental Results

In circular statistics, classical statistical techniques give approximate results because of the special nature of circular data. The angular mean and the angular variance are proposed to eliminate this approximation in this paper. Generally, circular data may be discontinuous in some principal interval. These discontinuities cause failure results in the circular statistics. Because of that the proposed wrapping operator must be used for data, which are defined in the discontinuous range. But discontinuity for variance cannot be mentioned because of the fact that variance is calculated quantitatively. However, in both continuity and discontinuity, the wrapping operator works correctly for circular variance calculation. In this regard, performance criteria are required to show the success of the proposed methods. For this reason, absolute error and dispersion are used as performance criteria for both continuity and discontinuity situations. In addition, a new dispersion measure has been proposed using the wrapping operator as a performance criteria.

4.1. **Performance Criteria for Circular Mean.** The linear sample mean is defined as below

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{9}$$

Equation (10) can be obtained by subtracting the right side of this equation from both sides of equation (9).

$$n\bar{x} - \sum_{i=1}^{n} x_i = 0 \tag{10}$$

If the equation (10) is regulated, the equation (11) is obtained as

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0. \tag{11}$$

According to this equation, the sum of deviations from the mean is zero. If this equation is applied to circular data, equation (12) is obtained as

$$Wrap_\pi\left(\sum_{i=1}^{n} Wrap_\pi\left(\theta_i-\bar{\theta}\right)\right)=0. \tag{12}$$

The fact that this is not zero shows that an error is to be occurred. The absolute error for mean is defined as follows

$$E_{mean}=\left|Wrap_\pi\left(\sum_{i=1}^{n} Wrap_\pi\left(\theta_i-\bar{\theta}\right)\right)\right|.$$

Different approaches have been developed in the literature to calculate the distance of two circular values and given in Section 2.3. Therefore, different dispersion measures have been presented for each of the distance approximations. The dispersion of angles $\theta_1, \theta_2, \ldots, \theta_n$ about a given angle α is defined as in the equation (13) and it was developed based on the minor arc length in the equation (2) ([1,4]).

$$d_0(\alpha)=\pi-\frac{1}{n}\sum_{i=1}^{n}\left|\pi-\left|\theta_i-\alpha\right|\right| \tag{13}$$

The other way of the measuring the dispersion of angles about the angle α is given as the following equation which is based on the equation (3) ([5]).

$$d_v(\alpha)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\cos\left(\theta_i-\alpha\right)\right)$$

This paper presents a new measure of preferred direction for circular data using angular wrapping. Therefore, a new measure of dispersion is proposed as a new performance criteria for the preferred directions which is given equation (14)

$$d_a(\alpha)=\frac{1}{n}\sum_{i=1}^{n}\left|Wrap_\pi\left(\theta_i-\alpha\right)\right|. \tag{14}$$

The dispersion of the angles $\theta_1, \theta_2, \ldots, \theta_n$ about the angle α can be calculated by taking as $\alpha=\bar{\theta}_{vectorial}$ and $\alpha=\bar{\theta}_{angular}$, respectively.

4.2. **Performance Criteria for Circular Variance.** The linear sample variance is defined as below

$$V=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2. \tag{15}$$

Equation (16) can be obtained by subtracting the right side of this equation from both sides of equation (15).

$$\left(n-1\right)V-\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2=0 \tag{16}$$

If the equation (16) is regulated, the equation (17) is obtained as

$$\sum_{i=1}^{n}\left[(x_i-\bar{x})^2-\frac{(n-1)}{n}V\right]=0. \tag{17}$$

If this equation is applied to circular data, equation (18) is obtained as

$$\sum_{i=1}^{n}\left[Wrap_{\pi}^{2}\left(\theta_i-\bar{\theta}\right)-\frac{n-1}{n}V_a\right]=0. \tag{18}$$

The fact that this is not zero shows that an error is to be occurred. The absolute error for variance is defined as follows

$$E_{var}=\left|\sum_{i=1}^{n}\left[Wrap_{\pi}^{2}\left(\theta_i-\bar{\theta}\right)-\frac{n-1}{n}V_a\right]\right|.$$

4.3. **Simulation of Performances.** In this section, four different examples were selected to compare the proposed method with the conventional method to measure performances of angular mean and angular variance.

Example 1. *It was discussed that discontinuity of circular data may lead to failure results in the previous section. Thus, the first example is selected from the range* $[0,\pi]$ *where data are continuous. For comparing performance of these methods, we consider the simulation data from the uniform distribution by generating a hundred data points. This process repeated a thousand times and some of the obtained results are shown in Table 1, Table 2 and Table 3.*

The obtained some results are shown in Table 1 and Table 2 which is included the absolute error and dispersion measure to prove the performance of the proposed approach. The linear sample mean, and angular mean give the same results in Table 1, because of the data are continuous in the range of $[0,\pi]$.

According to these results, since the circular data generated in the $[0,\pi]$ *interval show continuity, the linear mean* $\left(\bar{\theta}_l\right)$ *and the proposed angular mean* $\left(\bar{\theta}_a\right)$ *give the same results, while the vectorial mean* $\left(\bar{\theta}_v\right)$ *proposed by Mardia gives almost close results. Although the vectorial absolute error* $(E_{mean}^{(v)})$ *is quite low for continuous circular data, the angular absolute error* $(E_{mean}^{(a)})$ *of the proposed angular mean* $(\bar{\theta}_a)$ *using the wrapping operator is equal zero for all iterations. Because the data are continuous in the range* $[0,\pi]$, *the linear mean and the proposed angular mean are equal. Therefore, it can be concluded that the proposed angular mean performs more consistent and proper than the vectorial mean. Thus, the average errors of vectorial and angular mean are calculated as 1.5225 and 0.000, respectively.*

When the results are examined, it is seen that the $d_0(\bar{\theta}_v)$ *and* $d_0(\bar{\theta}_a)$, *which depend on the minor arc length, give almost the same results. In addition, the Table 2 reveals that the values of* $d_v(\bar{\theta}_v)$ *and* $d_v(\bar{\theta}_a)$ *depending on the vectorial distance give close results. When the* $d_a(\bar{\theta}_v)$ *and* $d_v(\bar{\theta}_a)$ *values depending on the*

TABLE 1. Comparison of circular means according to the absolute errors.

i	$\bar{\theta}_l$	$\bar{\theta}_v$	$\bar{\theta}_a$	$E^{(v)}_{mean}$	$E^{(a)}_{mean}$
1	1.528	1.524	1.528	0.377	0.000
100	1.516	1.524	1.516	0.793	0.000
200	1.743	1.770	1.743	2.747	0.000
300	1.606	1.617	1.606	1.051	0.000
400	1.590	1.601	1.590	1.107	0.000
500	1.490	1.489	1.490	0.142	0.000
600	1.568	1.576	1.568	0.805	0.000
700	1.653	1.656	1.653	0.266	0.000
800	1.541	1.533	1.541	0.798	0.000
900	1.491	1.464	1.491	2.729	0.000
1000	1.596	1.616	1.596	2.002	0.000

TABLE 2. Comparison of circular means according to the dispersions.

i	$d_0(\bar{\theta}_v)$	$d_0(\bar{\theta}_a)$	$d_v(\bar{\theta}_v)$	$d_v(\bar{\theta}_a)$	$d_a(\bar{\theta}_v)$	$d_a(\bar{\theta}_a)$
1	0.816	0.817	0.381	0.381	0.004	0.000
100	0.888	0.888	0.431	0.431	0.008	0.000
200	0.734	0.735	0.335	0.335	0.027	0.000
300	0.764	0.764	0.348	0.348	0.011	0.000
400	0.844	0.844	0.408	0.408	0.011	0.000
500	0.761	0.761	0.346	0.346	0.001	0.000
600	0.803	0.804	0.366	0.366	0.008	0.000
700	0.723	0.723	0.320	0.320	0.003	0.000
800	0.710	0.711	0.311	0.311	0.008	0.000
900	0.767	0.767	0.361	0.362	0.027	0.000
1000	0.786	0.786	0.353	0.354	0.020	0.000

proposed wrapping operator are examined, the dispersion of the $\bar{\theta}_a$ is equal to zero for all repetition. Accordingly, it can be said that the proposed method gives more consistent results than the vectorial method.

The some of the obtained results are shown in Table 3 which is obtained from the comparison of the angular variance and the linear variance. The linear variance and angular variance give the same results in Table 3, because of the data are continuous in the range of $[0, \pi]$.

According to Table 3, since the circular data generated in the $[0, \pi]$ interval show continuity, while the linear variance (V_l) and the proposed angular variance (V_a)

TABLE 3. Comparison of the linear and angular variance according to absolute errors.

i	V_l	V_v	V_a	$E_{var}^{(v)}$	$E_{var}^{(v)}$
1	0.872	0.381	0.872	48.585	0.000
100	0.995	0.431	0.995	55.794	0.000
200	0.771	0.335	0.771	43.146	0.000
300	0.790	0.348	0.790	43.792	0.000
400	0.941	0.408	0.941	52.786	0.000
500	0.791	0.346	0.791	44.084	0.000
600	0.829	0.366	0.829	45.862	0.000
700	0.729	0.320	0.729	40.449	0.000
800	0.705	0.311	0.705	39.056	0.000
900	0.835	0.361	0.835	46.838	0.000
1000	0.802	0.353	0.802	44.404	0.000

give the same results, the vectorial variance (V_v) *proposed by Mardia gives almost close results with them. The angular absolute error* ($E_{var}^{(a)}$) *of the proposed angular variance* (V_a) *using the wrapping operator is equal zero for all iterations. Because the data are contiuous in the range* $[0, \pi]$*, the linear variance and proposed angular variance are equal. Therefore, it can be concluded that the proposed angular variance performs more consistent and proper than vectorial variance. Thus, the average errors of vectorial and angular variance are calculated as 46.157 and 0.000, respectively.*

Example 2. *For comparing performances of these methods, we consider the simulation data from the uniform distribution by generating a hundred data points in the range* $[-\pi, \pi]$*. This process repeated a thousand times. The linear mean is not used due to discontinuity in Example 2. In this case, angular mean and vectorial mean are compared by using absolute error and measure of dispersion, obtained results are shown in Table 4 and Table 5.*

In this example, the generated circular data shows discontinuity as it is defined in the range $[-\pi, \pi]$*. Therefore, the linear mean value cannot be calculated for discontinuous circular data and the vectorial mean* $\left(\bar{\theta}_v\right)$ *and angular mean* $\left(\bar{\theta}_a\right)$ *values are used, and absolute error value* (E_{mean}) *and dispersion of angles* ($d(\alpha)$) *are used as performance criterias in this example. According to these results, the mean absolute errors of vectorial and angular mean are calculated as 1.5948 and 0.000, respectively. Table 4 shows that the absolute errors for all iterations are equal zero. It can be inferred that the proposed angular mean provides better results than the vectorial mean.*

When the results are examined, angular and vectorial dispersion measures give different results due to discontinuity. When the $d_a(\bar{\theta}_v)$ *and* $d_a(\bar{\theta}_a)$ *values depending*

TABLE 4. Comparison of circular means according to the absolute errors.

i	$\bar{\theta}_v$	$\bar{\theta}_a$	$E^{(v)}_{mean}$	$E^{(a)}_{mean}$
1	-3.139	-0.463	2.557	0.000
100	2.837	2.479	1.938	0.000
200	1.009	1.289	2.893	0.000
300	1.040	0.574	2.700	0.000
400	2.971	-2.918	1.759	0.000
500	0.025	-0.783	0.816	0.000
600	2.479	2.312	2.165	0.000
700	0.098	0.355	0.653	0.000
800	-0.238	2.521	0.547	0.000
900	-1.390	2.228	2.688	0.000
1000	1.523	2.438	2.775	0.000

TABLE 5. Comparison of circular means according to the dispersions.

i	$d_0(\bar{\theta}_v)$	$d_0(\bar{\theta}_a)$	$d_v(\bar{\theta}_v)$	$d_v(\bar{\theta}_a)$	$d_a(\bar{\theta}_v)$	$d_a(\bar{\theta}_a)$
1	1.522	1.598	0.969	1.028	0.026	0.000
100	1.352	1.379	0.842	0.852	0.019	0.000
200	1.430	1.447	0.889	0.893	0.029	0.000
300	1.530	1.507	0.955	0.960	0.027	0.000
400	1.460	1.458	0.905	0.912	0.018	0.000
500	1.523	1.522	0.957	0.971	0.008	0.000
600	1.520	1.514	0.955	0.956	0.022	0.000
700	1.439	1.447	0.900	0.903	0.007	0.000
800	1.419	1.713	0.882	1.110	0.005	0.000
900	1.462	1.669	0.914	1.077	0.027	0.000
1000	1.482	1.511	0.931	0.958	0.028	0.000

on the proposed wrapping operator are examined, the dispersion of the $\bar{\theta}_a$ *is equal to zero for all repetition. It can be said that the proposed method gives more consistent results than the vectorial method. Therefore, it is seen that the proposed method gives more consistent results than the vectorial method.*

The angular variance and vectorial variance are compared by using absolute error and the obtained results are shown in Table 6.

In this example, the generated circular data shows discontinuity as it is defined in the range $[-\pi, \pi]$*. Therefore, the linear variance cannot be calculated for discontinuous circular data, the vectorial variance* (V_v) *and angular variance* (V_a) *are*

TABLE 6. Comparison of the angular and vectorial variance according to absolute errors.

i	$\bar{\theta}_v$	$\bar{\theta}_a$	$E_{var}^{(v)}$	$E_{var}^{(a)}$
1	0.969	3.434	244.038	0.000
100	0.842	2.557	169.831	0.000
200	0.889	2.767	185.994	0.000
300	0.955	3.112	213.511	0.000
400	0.905	2.987	206.140	0.000
500	0.957	3.073	209.457	0.000
600	0.955	3.088	211.124	0.000
700	0.900	2.807	188.807	0.000
800	0.882	3.763	285.193	0.000
900	0.914	3.616	267.541	0.000
1000	0.931	3.122	216.904	0.000

used, and absolute error (E_{var}) is used as performance criteria in this example. According to these results, the mean absolute errors of vectorial and angular variance are calculated as 218.639 and 0.000, respectively. Table 6 shows that the absolute errors for all iterations are equal zero. It can be inferred that the proposed angular variance provides better results than the vectorial variance.

Example 3. *For comparing performance of these methods, we consider the simulation data from the von Mises distribution $\left(vM\left(\mu = \frac{\pi}{2}, \kappa = 10\right)\right)$ by generating a hundred data points. This process repeated a thousand times. The high values of concentration parameter (κ) reduce to discontinuity of circular data. For this reason, the differences between vectorial mean $\left(\bar{\theta}_v\right)$ and angular mean $\left(\bar{\theta}_a\right)$ decrease. The some of the obtained results are shown in Table 7 which is acquired from the comparison of the circular means and the linear mean. The linear mean $\left(\bar{\theta}_l\right)$ and angular mean $\left(\bar{\theta}_a\right)$ give the same results in Table 7 due to the high values of κ.*

The high values of κ reduce to discontinuity of circular data. For this reason, the differences between vectorial and angular mean decrease. The linear mean $\left(\bar{\theta}_l\right)$ and the proposed angular mean $\left(\bar{\theta}_a\right)$ give the same results, while the vectorial mean $\left(\bar{\theta}_v\right)$ proposed by Mardia ([3]) gives almost close results. Although the vectorial absolute error ($E_{mean}^{(v)}$) is quite low for continuous circular data, the angular absolute error ($E_{mean}^{(a)}$) of the proposed angular mean $\left(\bar{\theta}_a\right)$ using the wrapping operator is equal zero for all iterations. Because of the high values of κ, the data are contiuous and so the linear mean and the proposed angular mean are equal. Therefore, it proves that the proposed angular mean performs more consistent and proper than the vectorial mean. According to these results, the average errors of vectorial and angular mean are calculated as 0.114 and 0.000, respectively.

TABLE 7. Comparison of circular means according to the absolute errors.

i	$\bar{\theta}_l$	$\bar{\theta}_v$	$\bar{\theta}_a$	$E_{mean}^{(v)}$	$E_{mean}^{(a)}$
1	1.593	1.593	1.593	0.027	0.000
100	1.591	1.588	1.591	0.285	0.000
200	1.557	1.558	1.557	0.116	0.000
300	1.554	1.554	1.554	0.052	0.000
400	1.609	1.609	1.609	0.053	0.000
500	1.576	1.573	1.576	0.235	0.000
600	1.563	1.564	1.563	0.148	0.000
700	1.527	1.528	1.527	0.056	0.000
800	1.554	1.553	1.554	0.086	0.000
900	1.592	1.591	1.592	0.159	0.000
1000	1.623	1.621	1.623	0.140	0.000

TABLE 8. Comparison of circular means according to the dispersions.

i	$d_0(\bar{\theta}_v)$	$d_0(\bar{\theta}_a)$	$d_v(\bar{\theta}_v)$	$d_v(\bar{\theta}_a)$	$d_a(\bar{\theta}_v)$	$d_a(\bar{\theta}_a)$
1	0.251	0.251	0.047	0.047	0.000	0.000
100	0.280	0.281	0.059	0.059	0.003	0.000
200	0.249	0.249	0.048	0.048	0.001	0.000
300	0.243	0.243	0.049	0.049	0.001	0.000
400	0.259	0.259	0.048	0.048	0.001	0.000
500	0.254	0.254	0.052	0.052	0.002	0.000
600	0.259	0.259	0.051	0.051	0.001	0.000
700	0.243	0.243	0.043	0.043	0.001	0.000
800	0.265	0.265	0.053	0.053	0.001	0.000
900	0.267	0.267	0.058	0.058	0.002	0.000
1000	0.285	0.285	0.061	0.061	0.001	0.000

When the results are examined, it is seen that the $d_0(\bar{\theta}_v)$ and $d_0(\bar{\theta}_a)$, which depend on the minor arc length, give almost the same results. In addition, the values of $d_v(\bar{\theta}_v)$ and $d_v(\bar{\theta}_a)$ depending on the vectorial distance give close results. When $d_a(\bar{\theta}_v)$ and $d_a(\bar{\theta}_a)$ values depending on the proposed wrapping operator are examined, the dispersion of the $\bar{\theta}_a$ is equal to zero for all repetition. Accordingly, it can be said that the proposed method gives more consistent results than the vectorial method.

In the same way, the some of the obtained results from the comparison of the angular and vectorial variance are shown in Table 9. The linear variance and angular variance give the same results in Table 9.

TABLE 9. Comparison of the angular and vectorial variance according to absolute errors.

i	V_l	V_v	V_a	$E_{var}^{(v)}$	$E_{var}^{(a)}$
1	0.097	0.047	0.097	4.942	0.000
100	0.123	0.059	0.123	6.339	0.000
200	0.100	0.048	0.100	5.123	0.000
300	0.101	0.049	0.101	5.184	0.000
400	0.098	0.048	0.098	4.996	0.000
500	0.108	0.052	0.108	5.529	0.000
600	0.105	0.051	0.105	5.386	0.000
700	0.089	0.043	0.089	4.529	0.000
800	0.111	0.053	0.111	5.654	0.000
900	0.121	0.058	0.121	6.266	0.000
1000	0.127	0.061	0.127	6.522	0.000

The high values of κ reduce to discontinuity of circular data. For this reason, the differences between vectorial and angular variance decrease. While the linear variance (V_l) and the proposed angular variance (V_a) give the same results, the vectorial variance (V_v) proposed by Mardia ([3]) gives almost close results. Although the vectorial absolute error $E_{var}^{(v)}$ is quite low for continuous circular data, the angular absolute error $E_{var}^{(a)}$ of the proposed angular variance (V_a) using wrapping operator is equal zero for all iterations. Because of the high values of κ, the data are contiuous and so the linear variance and the proposed angular variance are equal. Therefore, it proves that the proposed angular variance performs more consistent and proper than the vectorial variance. According to these results, the average errors of vectorial and angular variance are calculated as 5.416 and 0.000, respectively.

Example 4. *For comparing performance of these methods, we consider the simulation data from the von Mises distribution $\left(vM\left(\mu=\frac{\pi}{2},\kappa=2\right)\right)$ by generating a hundred data points. This process repeated a thousand times. The low values of κ increase discontinuity of circular data. For this reason, the difference between vectorial mean and angular mean increase. The linear mean is not used due to the discontinuity in Example 4.*

The angular mean and vectorial mean are compared by using absolute error and measure of dispersion, the obtained results are shown in Table 10 and Table 11.

The low values of κ increase discontinuity of circular data. For this reason, the difference between vectorial mean and angular mean increase. Therefore, the linear

TABLE 10. Comparison of circular means according to the absolute errors.

i	$\bar{\theta}_v$	$\bar{\theta}_a$	$E^{(v)}_{mean}$	$E^{(a)}_{mean}$
1	1.573	1.558	1.507	0.000
100	1.688	1.700	1.277	0.000
200	1.458	1.437	2.052	0.000
300	1.576	1.581	0.525	0.000
400	1.652	1.650	0.208	0.000
500	1.699	1.688	1.079	0.000
600	1.581	1.544	2.620	0.000
700	1.424	1.456	3.042	0.000
800	1.607	1.603	0.389	0.000
900	1.524	1.529	0.584	0.000
1000	1.552	1.535	1.679	0.000

mean $\left(\bar{\theta}_l\right)$ cannot be calculated for discontinuous circular data, the vectorial mean $\left(\bar{\theta}_v\right)$ and angular mean $\left(\bar{\theta}_a\right)$ are used, and absolute error value (E_{mean}) is used as performance criterion in this example. According to these results, the mean absolute errors of vectorial and angular mean are calculated as 1.596 and 0.000, respectively. Table 10 shows that the absolute errors for all iterations are equal zero. It can be inferred that the proposed angular mean provides better results than the vectorial mean.

TABLE 11. Comparison of circular means according to the dispersion.

i	$d_0(\bar{\theta}_v)$	$d_0(\bar{\theta}_a)$	$d_v(\bar{\theta}_v)$	$d_v(\bar{\theta}_a)$	$d_a(\bar{\theta}_v)$	$d_a(\bar{\theta}_a)$
1	0.594	0.595	0.242	0.242	0.015	0.000
100	0.715	0.715	0.331	0.331	0.013	0.000
200	0.676	0.678	0.307	0.308	0.021	0.000
300	0.622	0.623	0.282	0.282	0.005	0.000
400	0.735	0.735	0.344	0.344	0.002	0.000
500	0.634	0.635	0.270	0.270	0.011	0.000
600	0.569	0.568	0.233	0.234	0.026	0.000
700	0.713	0.710	0.330	0.331	0.030	0.000
800	0.643	0.643	0.273	0.273	0.004	0.000
900	0.705	0.705	0.325	0.325	0.006	0.000
1000	0.576	0.576	0.243	0.243	0.017	0.000

When the results are examined, it is seen that the $d_0(\bar{\theta}_v)$ *and* $d_0(\bar{\theta}_a)$*, which depend on the minor arc length, give almost the same results. In addition, the values of* $d_v(\bar{\theta}_v)$ *and* $d_v(\bar{\theta}_a)$ *depending on the vectorial distance give close results. When* $d_a(\bar{\theta}_v)$ *and* $d_a(\bar{\theta}_a)$ *values depending on the proposed wrapping operator are examined, the dispersion of the* $\bar{\theta}_a$ *is equal to zero for all repetition. Accordingly, it can be said that the proposed method gives more consistent results than the vectorial method.*

In the same way, the angular variance and vectorial variance are compared by using absolute error and the obtained results are shown in Table 12.

TABLE 12. Comparison of the angular and vectorial variance according to absolute errors.

i	V_v	V_a	$E_{var}^{(v)}$	$E_{var}^{(a)}$
1	0.242	0.586	34.060	0.000
100	0.331	0.826	49.001	0.000
200	0.307	0.835	52.245	0.000
300	0.282	0.750	46.319	0.000
400	0.344	0.889	53.981	0.000
500	0.270	0.660	38.643	0.000
600	0.233	0.607	37.024	0.000
700	0.330	0.849	51.323	0.000
800	0.273	0.726	44.830	0.000
900	0.325	0.817	48.687	0.000
1000	0.243	0.616	36.930	0.000

The low values of κ *increase discontinuity of circular data. For this reason, the difference between vectorial mean and angular mean increase. Therefore, the linear variance* (V_l) *cannot be calculated for discontinuous circular data, the vectorial variance* (V_v) *and angular variance* (V_a) *are used, and absolute error value* (E_{var}) *is used as performance criteria in this example. According to these results, the mean absolute errors of vectorial and angular variance are calculated as 46.299 and 0.000, respectively. Table 12 shows that the absolute errors for all iterations are equal zero. It can be inferred that the proposed angular variance provides better results than the vectorial variance.*

5. Real Data Applications

In order to compare the performance of the proposed method with the conventional methods and measure performance of angular mean and angular variance, it has been applied on three real dataset that using in environmental and ecological applications. These are movements of ants' dataset ([36]), movements of blue periwinkles' dataset ([37, 38]), and dance directions of bees' dataset ([39]).

5.1. **Movements of ants' dataset.** Route learning is the key to the survival of many ants. Ants show remarkable navigational ability, traveling long distances between profitable foraging areas and their nest. They have low resolution vision. For this reason, ants who travel along a particular route, produce pheromone trails secreted from their abdominal glands. Trail pheromone is used for route learning, and effects on route choice. In this example, we analyze the dataset that presents the orientation of the ants towards a black target when released in a round arena. The ants tended to run towards the target. This experiment was originally conducted by Jander ([36]) and later mentioned in Fisher ([4]). The data consists of 100 observations ([40]). For this data set, the vectorial sample mean and resultant direction are calculated 3.20 radians (183°) and 0.61, respectively ([4]). The directions of the ants are shown in Table 13.

Table 13. The directions of the ants.

330	290	60	200	200	180	280	220	190	180	140	40	300	80
180	160	280	180	170	190	180	140	150	150	210	200	170	200
160	200	190	250	180	30	200	180	200	350	210	190	160	170
200	180	120	200	210	130	30	210	200	230	180	140	360	150
180	160	210	190	180	230	50	150	210	180	110	270	180	200
190	210	220	200	60	260	110	180	170	200	220	160	70	190
10	220	180	210	170	90	160	180	170	200	120	150	300	190
160	180												

The circular histogram, the scatter and the vectorial and the proposed angular mean of the movements of the ants' dataset are given in Figure 3.

According to Figure 3, the vectorial and angular mean are calculated as 183.1385° and 184.7°, respectively. The absolute error for vectorial mean and angular mean are computed as 2.7253 and 0.0000, respectively. Since the absolute error for angular mean is equal 0.0000, the proposed angular mean method performs more consistent and proper than the other method. The vectorial variance is 0.3899 and the absolute error of it is 81.1333. The angular variance is calculated as 1.2095 and the absolute error of it is computed as 0.0000. Since the absolute error for angular variance is equal 0.0000, the proposed angular variance method performs more consistent and proper than the other method.

5.2. **Movements of blue periwinkles' dataset.** Blue periwinkles (Nodilittorina unifasciata) are very small blue shells that feed on microscopic algae. They live on the rocky shore in cluster of thousands and are able to survive a long time out of water. They travel up to 12 m in search of food. This dataset contains the directions of small blue periwinkles after they had been relocated down shore from the height at which they normally live. The original dataset not only contains the directions of the movement, but also contains the distances of the periwinkles after

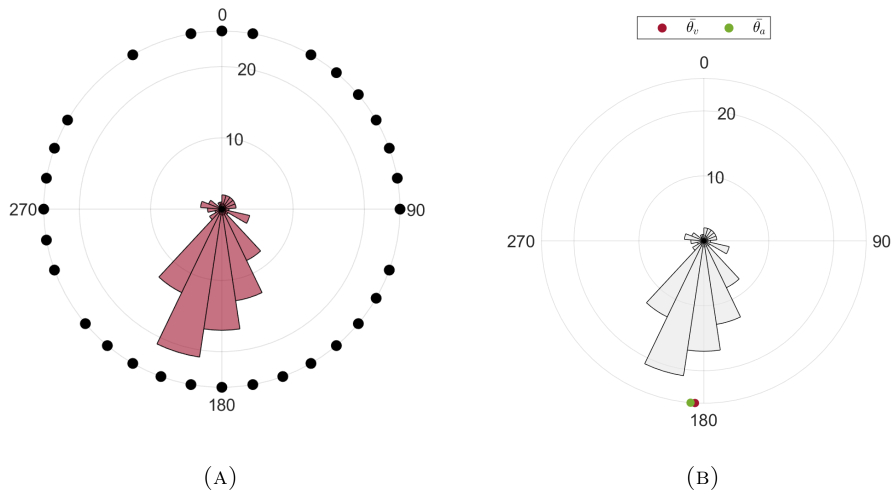


FIGURE 3. The circular histogram of the movements of the ants' dataset (A) scatter of the data; (B) the vectorial and angular mean of the data

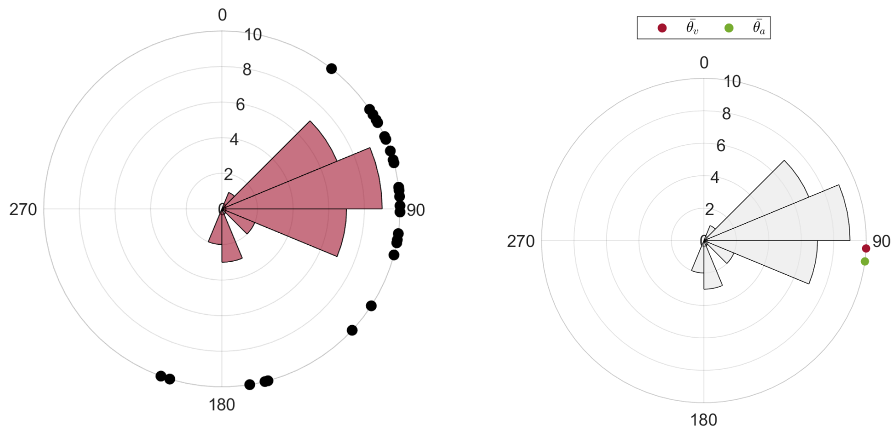
relocation. But in this paper, the distance measurement is omitted. Two different locations are combined for the purposes of this example. A total of 31 animals were involved in the study, 15 of which were measured one day after transplantation and the other 16 of which were measured four days after ([41]). The directions of the blue periwinkles are shown in Table 14.

TABLE 14. The directions of blue periwinkles.

67	66	74	61	58	60	100	89
171	166	98	60	197	98	86	123
165	133	101	105	71	84	75	98
83	71	74	91	38	200	56	

The circular histogram, the scatter and the vectorial and the proposed angular mean of the movements of the blue periwinkles' dataset are given in Figure 4.

According to Figure 4, the vectorial and angular mean are calculated as 92.7931° and 97.3871°, respectively. The absolute error for vectorial mean and angular mean are computed as 2.4856 and 0.0000, respectively. These results show that the proposed angular mean method more consistent and proper than the other method. The vectorial variance is 0.2251 and the absolute error of it is 9.5705. The angular variance is calculated as 0.5441 and the absolute error of it is computed as 0.0000.



(A) (B)

FIGURE 4. The circular histogram of the movements of the blue periwinkles' dataset (A) scatter of the data; (B) the vectorial and angular mean of the data

These results show that the proposed angular variance method more consistent and proper than the other method.

5.3. **Dance directions of bees' dataset.** How honeybees perceive polarized light from the sky was a longstanding problem in the literature ([42]). It has long been known that bees can use the pattern of polarized light in the sky (e-vector pattern) as a compass cue even if they can see only a small part of the whole pattern ([43]). Honeybees frequently dance with some view of the sky, orienting themselves to the sun or natural polarized skylight ([44]).

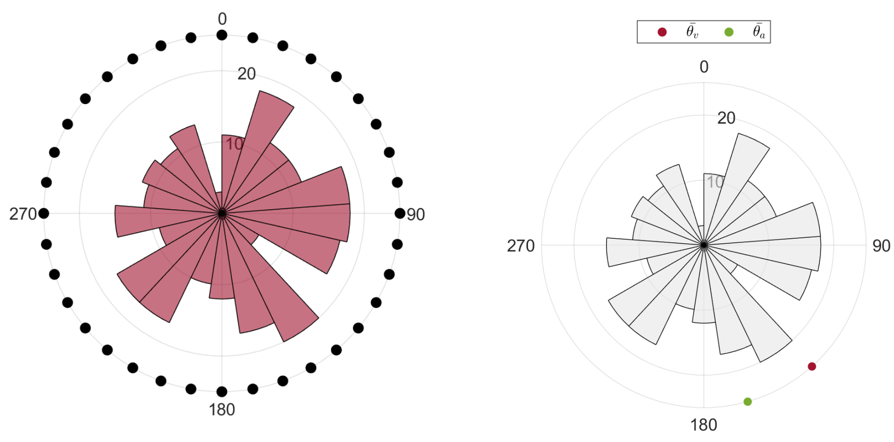
This dataset shows the dance directions of 279 honeybees viewing a zenith patch of artificially polarized light. This dataset was measured experimentally to prove that special receptors at the dorsal margin of the eye are required to detect polarized light and derive compass information in sky patterns ([39]). The waggle dances were recorded by a video and were analyzed later by measuring the directions of the individual waggle runs. The dance directions of the honeybees are shown in Table 15.

The circular histogram, the scatter and the vectorial and the proposed angular mean of the dance directions of the bees' dataset are given in Figure 5.

According to Figure 5, the vectorial and angular mean are calculated as 138.2749° and 164.3369°, respectively. The absolute error for vectorial mean and angular mean are computed as 1.2445 and 0.0000, respectively. Since the absolute error for

TABLE 15. Dance directions of bees.

Direction	0	10	20	30	40	50	60	70	80	90
Frequency	3	8	9	9	6	6	12	9	9	9
Direction	100	110	120	130	140	150	160	170	180	190
Frequency	9	12	5	6	8	12	8	9	12	5
Direction	200	210	220	230	240	250	260	270	280	290
Frequency	5	9	8	5	12	9	8	7	3	8
Direction	300	310	320	330	340	350				
Frequency	12	6	5	5	8	3				



(A) (B)

FIGURE 5. The circular histogram of the dance directions of the bees' dataset (A) scatter of the data; (B) the vectorial and angular mean of the data

angular mean is equal 0.0000, the proposed angular mean method performs more consistent and proper than the other method. The vectorial variance is 0.9223 and the absolute error of it is 562.7965. The angular variance is calculated as 2.9467 and the absolute error of it is computed as 0.0000. Since the absolute error for angular variance is equal 0.0000, the proposed angular variance method performs more consistent and proper than the other method.

6. CONCLUSION

In this study, a new approach was proposed for the calculation of the mean and variance of circular data. Circular data can be cut across interval boundaries as a result of some arithmetic operations. In this situation, the principal interval can be reduced to a symmetric principal interval by using the wrapping process. The proposed methods have been developed based on the wrapping process. These methods were compared with Mardia's methods ([3]) in the literature, using both artificial data and real datasets. The absolute error and absolute percentage error which is proposed by using the wrapping process were considered as the performance criteria in the comparisons. In the simulation study, four different artificial data were generated from the uniform and von Mises distribution according to the continuity and discontinuity of the data. The comparisons were performed by changing the interval range for the uniform distribution and altering the κ for the von mises distribution. The attained results showed that the proposed angular mean and variance methods outperform vectorial methods in the literature. In this study, the angular statistics (mean and variance) and the linear statistics (mean and variance) give the same result in the range of $[0, \pi]$. On the other hand, the low values of the κ increase discontinuity of the circular data which generated from von Mises distribution. For this reason, the difference between vectorial statistics and angular statistics increases. These methods give the same results in this situation. In order to compare the performance of the proposed method with the conventional methods and measure the performance of angular mean and angular variance, it has been applied on three real datasets that using in environmental and ecological applications. These are movements of ants' dataset, movements of blue periwinkles' dataset, and dance directions of bees' dataset. The proposed methods achieved more consistent results in the calculation of the mean and variance of the circular data when compared to the vectorial methods. Thus, it has been demonstrated that the proposed method can be easily applied to environmental and ecological data as well as artificial data. Consequently, the simulation study and applications show that the proposed angular methods based on the wrapping process are simple and consistent. It can be easily applied to different datasets in various fields. Also, it is useful for practitioners regarding the applicability.

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