http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 2, Pages 386–396 (2023)
DOI:10.31801/cfsuasmas.1160606
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: August 11, 2022; Accepted: December 20, 2022

ADVANCED REFINEMENTS OF BEREZIN NUMBER INEQUALITIES

Mehmet GÜRDAL[1] and Hamdullah BAŞARAN[2]

[1,2]Depatment of Mathematics, Süleyman Demirel University, Isparta, TÜRKİYE

Abstract. For a bounded linear operator A on a functional Hilbert space $\mathcal{H}(\Omega)$, with normalized reproducing kernel $\widehat{k}_\eta := \frac{k_\eta}{\|k_\eta\|_{\mathcal{H}}}$, the Berezin symbol and Berezin number are defined respectively by $\widetilde{A}(\eta) := \left\langle A\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_{\mathcal{H}}$ and $\mathrm{ber}(A) := \sup_{\eta\in\Omega} \left|\widetilde{A}(\eta)\right|$. A simple comparison of these properties produces the inequality $\mathrm{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\mathrm{ber}} + \left\|A^2\right\|_{\mathrm{ber}}^{1/2}\right)$ (see [17]). In this paper, we prove further inequalities relating them, and also establish some inequalities for the Berezin number of operators on functional Hilbert spaces.

1. Introduction

In almost every field of engineering and research, mathematical inequalities are the most effective instrument for identifying and describing solutions to practical issues. The restrictions of the many types of operators covered in analysis courses, including mathematical and functional analysis, must be carefully considered while creating theory and applications. Numerous mathematicians and scientists have been affected by the Berezin transformation of an operator defined on the kernel generating the Hilbert space. Many scholar have investigated the Berezin radius inequality in-depth, and it is covered in (3) (see [12, 13]). Researchers are really motivated to make changes and additions to this difference [5, 7, 14, 27, 28]. The Berezin transform is used in this paper to create a number of inequalities, notably Kittaneh's inequalities (see, [21, 22]), for operators in the functional Hilbert space. A number of additional inequalities for the Berezin norm and the radius of the Berezin operators were also illustrated using the modifications discussed before.

2020 *Mathematics Subject Classification.* 47A30, 47A63.
Keywords. Berezin symbol, Berezin number, functional Hilbert space, positive operator.
[1] gurdalmehmet@sdu.edu.tr; 0000-0003-0866-1869
[2] 07hamdullahbasaran@gmail.com-Corresponding author; 0000-0002-9864-9515.

Related findings can be found in [23]. The foundational ideas required to move on with the research's findings will now be presented.

Assume that $\mathbb{L}(\mathcal{H})$ stands for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$. The numerical range $W(T)$ is the representation of the sphere of $\mathcal{H}$ under the corresponding quadratic form $x \to \langle Tx, x\rangle$ for a bounded linear operator T on a Hilbert space $\mathcal{H}$. In more specific terms, $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}$. Additionally, the numerical radius is specified as

$$w(T) = \sup_{\lambda \in W(T)} |\lambda| = \sup_{\|x\|=1} |\langle Tx, x\rangle| .$$

We recall that the usual operator norm of an operator T is defined to be

$$\|T\| = \sup\{\|Tx\| : x \in \mathcal{H},\ \|x\| = 1\} .$$

Recall that if $A \in \mathbb{L}(\mathcal{H})$ and if f is a nonnegative increase on $[0, \infty]$, then $\|f(|A|)\| = f(\|A\|)$. Here $|A|$ represents positive operator $(A^*A)^{\frac{1}{2}}$.

Let Ω be a subset of a topological space X such that boundary $\partial\Omega$ is nonempty. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an infinite-dimensional Hilbert space of functions defined on Ω. We say that $\mathcal{H}$ is a functional Hilbert space (FHS) or reproducing kernel Hilbert space if the following two conditions are satisfied:

(i) for any $\eta \in \Omega$, the functionals $f \to f(\eta)$ are continuous on $\mathcal{H}$;

(ii) for any $\eta \in \Omega$, there exists $f_\eta \in \mathcal{H}$ such that $f_\eta(\eta)$:

According to the classical Riesz representation theorem, the assumption (i) implies that for any $\eta \in \Omega$ there exists $k_\eta \in \mathcal{H}$ such that

$$f(\eta) = \langle f, k_\eta\rangle,\ f \in \mathcal{H}.$$

The function k_η is called the reproducing kernel of $\mathcal{H}$ at point η. Note that by (ii), we surely have $k_\eta \neq 0$ and we denote $\widehat{k}_\eta$ as the normalized reproducing kernel, that is $\widehat{k}_\eta = \frac{k_\eta}{\|k_\eta\|}$.

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions.

Definition 1. *Let $\mathcal{H}$ be a FHS on a set Ω and let T be a bounded linear operator on $\mathcal{H}$.*

(i) For $\eta \in \Omega$, the Berezin transform of T at η (or Berezin symbol of T) is

$$\widetilde{T}(\eta) := \left\langle T\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_{\mathcal{H}} .$$

(ii) The Berezin range of T (or Berezin set of T) is

$$\mathrm{Ber}(T) := \mathrm{Range}(\widetilde{T}) = \left\{\widetilde{T}(\eta) : \eta \in \Omega\right\} .$$

(iii) The Berezin radius of T (or Berezin number of T) is

$$\mathrm{ber}(T) := \sup_{\eta \in \Omega} \left|\widetilde{T}(\eta)\right| .$$

We also define the following so-called Berezin norm of operators $T \in \mathcal{B}(\mathcal{H})$:

$$\|T\|_{\text{Ber}} := \sup_{\eta \in \Omega} \left\| T\widehat{k}_\eta \right\|.$$

It is easy to see that actually $\|T\|_{\text{Ber}}$ determines a new operator norm in $\mathbb{L}(\mathcal{H}(\Omega))$ (since the set of reproducing kernels $\{k_\eta : \eta \in \Omega\}$ span the space $\mathcal{H}(\Omega)$). It is also trivial that $\operatorname{ber}(T) \leq \|T\|_{\text{Ber}} \leq \|T\|$ (for more facts about functional Hilbert spaces and Berezin symbol, see, Aronzajn [3], Berezin [9] and Chalendar et al. [11]).

The Berezin transform $\widetilde{T}$ is a bounded real-analytic function on Ω for each bounded operator T on $\mathcal{H}$. The Berezin transform $\widetilde{T}$ frequently reflects the characteristics of the operator T. Since F. Berezin first proposed the Berezin transform in [9], it has become an essential tool in operator theory due to the fact that the Berezin transforms of many significant operators include information about their fundamental characteristics. It is said that Karaev initially explicitly introduced the Berezin set and Berezin number in [19], also denoted as $\operatorname{Ber}(T)$ and $\operatorname{ber}(T)$, respectively.

In a FHS, the Berezin range of an operator T is a subset of the numerical range of T. Hence $\operatorname{ber}(T) \leq w(T)$. An operator's numeric range has a number of intriguing characteristics. For instance, it is common knowledge that an operator's numerical range's closure contains the operator's spectrum. We refer to [1,2,16,24–26] for the fundamental attributes of the numerical radius.

For example, is it true, or under which additional conditions the following are true:

(i) $\operatorname{ber}(A) \geq \frac{1}{2}\|A\|$;

(ii)

$$\operatorname{ber}(A^n) \leq \operatorname{ber}(A)^n \tag{1}$$

for any integer $n \geq 1$; more generally, if A is not nilpotent, then

$$C_1 \operatorname{ber}(A)^n \leq \operatorname{ber}(A^n) \leq C_2 \operatorname{ber}(A)^n$$

for some constants $C_1, C_2 > 0$;

(iii) $\operatorname{ber}(AB) \leq \operatorname{ber}(A)\operatorname{ber}(B)$, where $A, B \in \mathcal{B}(\mathcal{H})$.

If $A = cI$ with $c \neq 0$, then obviously $\operatorname{ber}(A) = |c| > \dfrac{|c|}{2} = \dfrac{\|A\|}{2}$. However, it is known that in general the above inequality (i) is not satisfied (see Karaev [20]).

It is well-known that

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\| \tag{2}$$

and

$$\operatorname{ber}(A) \leq w(A) \leq \|A\| \tag{3}$$

for any $A \in \mathbb{L}(\mathcal{H})$. Additionally, Karaev introduced additional numerical properties of operators on the FHS in [19], including Berezin range and Berezin radius. See [4, 6, 13, 15, 20] for the fundamental characteristics and information on these novel concepts.

Huban et al. [18, Theorem 3.1] have showed the following result:

$$\operatorname{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\operatorname{ber}} + \left\|A^2\right\|_{\operatorname{ber}}^{\frac{1}{2}}\right) \tag{4}$$

They also proved the following inequality as stronger than (4),

$$\operatorname{ber}(A) \leq \frac{1}{2}\left\||A| + |A^*|\right\|_{\operatorname{ber}}. \tag{5}$$

Another refinement has been established by same authors (see, [17, 18])

$$\operatorname{ber}^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|_{\operatorname{ber}}. \tag{6}$$

Also, in the same paper, they showed that

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\operatorname{ber}} \leq \operatorname{ber}^2(A). \tag{7}$$

In Section 2, we present an inequality that refines (4) and (6). Furthermore, we establish a refinement of the inequality (7).

In the paper, we need the following lemmas, which is important. The first lemma was introduced by Kittaneh in [22, Inequality 19].

Lemma 1. *If $A, B \in \mathbb{L}(\mathcal{H})$ is a positive operators, then we have*

$$\|A+B\| \leq \frac{1}{2}\left(\|A\| + \|B\| \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right).$$

In particular

$$\left\||A|^2 + |A^*|^2\right\| \leq \left\|A^2\right\| + \left\|A^2\right\|$$

for any $A \in \mathbb{L}(\mathcal{H})$.

Second lemma is known in the literature as the generalized mixed Schwarz inequality (see, e.g., [21]).

Lemma 2. *Let $A, B \in \mathbb{L}(\mathcal{H})$ and let $x, y \in \mathcal{H}$ be any vector. If f, g are nonnegative continuous functions on $[0, \infty]$ satisfying $f(t).g(t) = t$, $(t \geq 0)$ then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\| \, \|g(|A^*|)y\|.$$

In particular

$$|\langle Ax, y\rangle| \leq \sqrt{\left\langle |A|^{2(1-v)} x, x\right\rangle \left\langle |A|^{2v} y, y\right\rangle}, \ (0 \leq v \leq 1).$$

Lemma 3. *([10, Theorem IX.2.1])If $A, B \in \mathbb{L}(\mathcal{H})$ are positive operators, then we have*

$$\left\|A^t B^t\right\| \leq \|AB\|^t, \ (0 \leq t \leq 1).$$

2. Main Results

Using the same arguments as in [8, Theorem 1], we have the first result, a refinement of inequality (5).

Theorem 1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS. If $A \in \mathbb{L}(\mathcal{H})$, then we have*

$$\operatorname{ber}(A) \leq \frac{1}{2} \min_{0 \leq \upsilon \leq 1} \left\| |A|^{2(1-\upsilon)} + |A^*|^{2\upsilon} \right\|_{\operatorname{ber}}. \tag{8}$$

Proof. Let $\widehat{k}_\eta$ be an arbitrary. By utilizing Lemma 2 and the AM-GM inequality, we get

$$\begin{aligned} \left| \left\langle A\widehat{k}_\eta, \widehat{k}_\eta \right\rangle \right| &\leq \sqrt{\left\langle |A|^{2(1-\upsilon)} \widehat{k}_\eta, \widehat{k}_\eta \right\rangle \left\langle |A|^{2\upsilon} \widehat{k}_\eta, \widehat{k}_\eta \right\rangle} \\ &\leq \frac{1}{2} \left(\left\langle |A|^{2(1-\upsilon)} \widehat{k}_\eta, \widehat{k}_\eta \right\rangle + \left\langle |A|^{2\upsilon} \widehat{k}_\eta, \widehat{k}_\eta \right\rangle \right) \\ &\leq \frac{1}{2} \left\langle \left(|A|^{2(1-\upsilon)} + |A|^{2\upsilon} \right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle. \end{aligned}$$

Thus,

$$\left| \left\langle A\widehat{k}_\eta, \widehat{k}_\eta \right\rangle \right| \leq \frac{1}{2} \left\langle \left(|A|^{2(1-\upsilon)} + |A|^{2\upsilon} \right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle.$$

Now, taking the supremum over $\eta \in \Omega$ in the above inequality we reach

$$\operatorname{ber}(A) \leq \frac{1}{2} \left\| |A|^{2(1-\upsilon)} + |A^*|^{2\upsilon} \right\|_{\operatorname{ber}}. \tag{9}$$

Taking the minimum over all $\upsilon \in [0, 1]$, we have

$$\operatorname{ber}(A) \leq \frac{1}{2} \min_{0 \leq \upsilon \leq 1} \left\| |A|^{2(1-\upsilon)} + |A^*|^{2\upsilon} \right\|_{\operatorname{ber}}.$$

This completes the proof. □

The Theorem 1 accepts the following result.

Corollary 1. *If $A \in \mathbb{L}(\mathcal{H})$ and $0 \leq \upsilon \leq 1$, then we have*

$$\operatorname{ber}(A) \leq \frac{1}{4} \left(\|A\|_{\operatorname{ber}}^{2(1-\upsilon)} + \|A^*\|_{\operatorname{ber}}^{2\upsilon} + \sqrt{\left(\|A\|_{\operatorname{ber}}^{2(1-\upsilon)} - \|A\|_{\operatorname{ber}}^{2\upsilon} \right)^2 + 4 \left\| |A|^{(1-\upsilon)} |A^*|^{\upsilon} \right\|_{\operatorname{ber}}^2} \right).$$

Proof. Let $\widehat{k}_\eta$ be an arbitrary and $0 \leq \upsilon \leq 1$. We get

$$\begin{aligned} &\left\| |A|^{2(1-\upsilon)} + |A^*|^{2\upsilon} \right\|_{\operatorname{ber}} \\ &\leq \frac{1}{2} \Bigg(\left\| |A|^{2(1-\upsilon)} \right\|_{\operatorname{ber}} + \left\| |A^*|^{2\upsilon} \right\|_{\operatorname{ber}} \\ &+ \sqrt{\left(\left\| |A|^{2(1-\upsilon)} \right\|_{\operatorname{ber}} - \left\| |A^*|^{2\upsilon} \right\|_{\operatorname{ber}} \right)^2 + 4 \left\| |A|^{(1-\upsilon)} |A^*|^{\upsilon} \right\|_{\operatorname{ber}}^2} \Bigg) \\ &\text{(by Lemma 1)} \end{aligned}$$

$$= \frac{1}{2}\left(\|A\|_{\text{ber}}^{2(1-\upsilon)} + \|A^*\|_{\text{ber}}^{2\upsilon} + \sqrt{\left(\|A\|_{\text{ber}}^{2(1-\upsilon)} - \|A\|_{\text{ber}}^{2\upsilon}\right)^2 + 4\left\||A|^{(1-\upsilon)}|A^*|^{\upsilon}\right\|_{\text{ber}}^2}\right).$$

Thus, by (9),

$$\text{ber}(A) \le \tfrac{1}{4}\left(\|A\|_{\text{ber}}^{2(1-\upsilon)} + \||A^*|\|_{\text{ber}}^{2\upsilon} + \sqrt{\left(\|A\|_{\text{ber}}^{2(1-\upsilon)} - \|A\|_{\text{ber}}^{2\upsilon}\right)^2 + 4\left\||A|^{(1-\upsilon)}|A^*|^{\upsilon}\right\|_{\text{ber}}^2}\right).$$

□

Remark 1. *It follows from the Lemma 3 that*

$$\begin{aligned}\text{ber}(A) &\le \frac{1}{2}\left(\|A\|_{\text{ber}} + \left\||A|^{\frac{1}{2}}|A^*|^{\frac{1}{2}}\right\|_{\text{ber}}\right) \le \frac{1}{2}\left(\|A\|_{\text{ber}} + \||A||A^*|\|_{\text{ber}}^{\frac{1}{2}}\right)\\ &= \frac{1}{2}\left(\|A\|_{\text{ber}} + \left\|A^2\right\|_{\text{ber}}^{\frac{1}{2}}\right). \qquad (10)\end{aligned}$$

Our second result interprets as follows. This result includes refinement of the inequality (6).

Theorem 2. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS. Assume that $A \in \mathbb{L}(\mathcal{H})$ and f, g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$, $(t \ge 0)$. Then we have*

$$\begin{aligned}\text{ber}^2(A) &\le \frac{1}{4}\left\|f^4(|A|) + g^4(|A^*|)\right\|_{\text{ber}} + \frac{1}{4}\left\|f^2(|A|)g^2(|A^*|) + g^2(|A^*|)f^2(|A|)\right\|_{\text{ber}}\\ &\le \frac{1}{4}\left\|f^4(|A|) + g^4(|A^*|)\right\|_{\text{ber}}. \qquad (11)\end{aligned}$$

Proof. Let $\widehat{k}_\eta$ be a normalized reproducing kernel. Using the Lemma 2, AM-GM inequality and Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|^2 &\le \left\|f(|A|)\widehat{k}_\eta\right\|\left\|g(|A^*|)\widehat{k}_\eta\right\|\\ &\le \left\langle f^2(|A|)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\left\langle g^2(|A^*|)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\\ &\le \left(\frac{\left\langle f^2(|A|)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle + \left\langle g^2(|A^*|)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle}{2}\right)^2\\ &\le \frac{1}{4}\left\langle\left(f^2(|A|) + g^2(|A^*|)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^2\\ &\le \frac{1}{4}\left\langle\left(f^2(|A|) + g^2(|A^*|)\right)^2\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\\ &= \frac{1}{4}\left\langle\left(f^4(|A|) + g^4(|A^*|) + f^2(|A|)g^2(|A^*|) + g^2(|A^*|)f^2(|A|)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\\ &= \frac{1}{4}\left\langle\left(f^4(|A|) + g^4(|A^*|)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\\ &+ \frac{1}{4}\left\langle\left(f^2(|A|)g^2(|A^*|) + g^2(|A^*|)f^2(|A|)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle.\end{aligned}$$

Hence,

$$\left|\left\langle A\widehat{k}_\eta,\widehat{k}_\eta\right\rangle\right|^2$$
$$\le \frac{1}{4}\left\langle \left(f^4\left(|A|\right)+g^4\left(|A^*|\right)\right)\widehat{k}_\eta,\widehat{k}_\eta\right\rangle+\frac{1}{4}\left\langle \left(f^2\left(|A|\right)g^2\left(|A^*|\right)+g^2\left(|A^*|\right)f^2\left(|A|\right)\right)\widehat{k}_\eta,\widehat{k}_\eta\right\rangle.$$

Now, we taking the supremum over $\eta\in\Omega$ in the above inequality we have

$$\operatorname{ber}^2(A)\le \frac{1}{4}\left\|f^4\left(|A|\right)+g^4\left(|A^*|\right)\right\|_{\text{ber}}+\frac{1}{4}\left\|f^2\left(|A|\right)g^2\left(|A^*|\right)+g^2\left(|A^*|\right)f^2\left(|A|\right)\right\|_{\text{ber}}. \tag{12}$$

Also, by [22, Corollary 1], we have

$$\begin{aligned}
&\left\|f^2\left(|A|\right)g^2\left(|A^*|\right)+g^2\left(|A^*|\right)f^2\left(|A|\right)\right\|_{\text{ber}}\\
&=\frac{1}{2}\left\|\left(f^2\left(|A|\right)+g^2\left(|A^*|\right)\right)^2-\left(f^2\left(|A|\right)-g^2\left(|A^*|\right)\right)^2\right\|_{\text{ber}}\\
&\le\frac{1}{2}\left\|\left(f^2\left(|A|\right)+g^2\left(|A^*|\right)\right)^2+\left(f^2\left(|A|\right)-g^2\left(|A^*|\right)\right)^2\right\|_{\text{ber}}\\
&=\left\|f^4\left(|A|\right)+g^4\left(|A^*|\right)\right\|_{\text{ber}}.
\end{aligned} \tag{13}$$

When we combine the relations (12) and (13), we obtain (11). □

Corollary 2. *In the Theorem 2, if we accept $f(t)=t^{1-\upsilon}$ and $g(t)=t^{\upsilon}$ with $0\le\upsilon\le1$, we obtain*

$$\begin{aligned}
\operatorname{ber}^2(A)&\le\frac{1}{4}\left\||A|^{4(1-\upsilon)}+|A^*|^{4\upsilon}\right\|_{\text{ber}}+\frac{1}{4}\left\||A|^{2(1-\upsilon)}|A^*|^{2\upsilon}+|A^*|^{2\upsilon}|A|^{2(1-\upsilon)}\right\|_{\text{ber}}\\
&\le\frac{1}{2}\left\||A|^{4(1-\upsilon)}+|A^*|^{4\upsilon}\right\|_{\text{ber}}.
\end{aligned}$$

In particular,

$$\operatorname{ber}^2(A)\le\frac{1}{4}\left\||A|^2+|A^*|^2\right\|_{\text{ber}}+\frac{1}{4}\left\||A|\,|A^*|+|A^*|\,|A|\right\|_{\text{ber}}\le\frac{1}{2}\left\||A|^2+|A^*|^2\right\|_{\text{ber}}.$$

We obtain on both sides of

$$\operatorname{ber}^2(A)\le\frac{1}{4}\left(\left\||A|^2+|A^*|^2\right\|_{\text{ber}}+\left\||A|\,|A^*|+|A^*|\,|A|\right\|_{\text{ber}}\right) \tag{14}$$

for normal operator A.

The following consequence demonstrates that inequality (14) is likewise sharper than inequality (4).

Corollary 3. *If $A\in\mathbb{L}(\mathcal{H})$, then we have*

$$\operatorname{ber}(A)\le\frac{1}{2}\sqrt{\left\||A|^2+|A^*|^2\right\|_{\text{ber}}+\left\||A|\,|A^*|+|A^*|\,|A|\right\|_{\text{ber}}}\le\frac{1}{2}\left(\|A\|_{\text{ber}}+\left\|A^2\right\|_{\text{ber}}^{\frac{1}{2}}\right). \tag{15}$$

Proof. Since $\||A|\,|A^*|\| = \|A^2\|$ and $|A| = |A^*|$ for any $A \in \mathbb{L}(\mathcal{H})$, it is clear that

$$\begin{aligned} \||A|\,|A^*| + |A^*|\,|A|\|_{\text{ber}} &\leq \||A|\,|A^*|\|_{\text{ber}} + \||A^*|\,|A|\|_{\text{ber}} \\ &= \||A|\,|A^*|\|_{\text{ber}} + \left\|(|A|\,|A^*|)^*\right\|_{\text{ber}} \qquad (16) \\ &= 2\,\||A|\,|A^*|\|_{\text{ber}} = 2\left\|A^2\right\|_{\text{ber}}. \end{aligned}$$

So, since $\|A\|^2 = \left\|A^2\right\|^{\frac{1}{2}}\left\|A^2\right\|^{\frac{1}{2}} \leq \|A\|\left\|A^2\right\|^{\frac{1}{2}}$, we get

$$\begin{aligned} \operatorname{ber}^2(A) &\leq \frac{1}{4}\left(\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} + \||A|\,|A^*| + |A^*|\,|A|\|_{\text{ber}}\right) \\ &\leq \frac{1}{4}\left(\left\|A^2\right\|_{\text{ber}} + \|A\|_{\text{ber}}^2\right) + \frac{1}{4}\||A|\,|A^*| + |A^*|\,|A|\|_{\text{ber}} \text{ (by Lemma 1)} \\ &\leq \frac{1}{4}\left(\left\|A^2\right\|_{\text{ber}} + \|A\|_{\text{ber}}^2\right) + \frac{1}{2}\left\|A^2\right\|_{\text{ber}} \\ &\text{(by the inequality (16))} \\ &= \frac{1}{4}\left(\left\|A^2\right\|_{\text{ber}} + 3\|A\|_{\text{ber}}^2\right) \\ &\leq \frac{1}{4}\left(\left\|A^2\right\|_{\text{ber}} + 2\|A\|_{\text{ber}}\left\|A^2\right\|_{\text{ber}}^{\frac{1}{2}} + \|A\|_{\text{ber}}^2\right) \\ &= \frac{1}{4}\left(\left\|A^2\right\|_{\text{ber}} + \left\|A^2\right\|_{\text{ber}}^{\frac{1}{2}}\right)^2. \end{aligned}$$

This gives the desired result. □

Remark 2. *According to the study in [17],*

$$\operatorname{ber}^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} + \frac{1}{2}\operatorname{ber}(|A|\,|A^*|). \qquad (17)$$

It is obvious that inequality (14) improves the inequality (17).

We need the following lemma (see, [23]).

Lemma 4. *If $A, B \in \mathbb{L}(\mathcal{H})$, then we have*

$$\begin{aligned} \|A+B\|_{\text{ber}} &\leq \sqrt{\left\||A|^2 + |B|^2\right\|_{\text{ber}} + \|A^*B + B^*A\|_{\text{ber}}} \\ &\leq \sqrt{\|A\|_{\text{ber}}^2 + \|B\|_{\text{ber}}^2 + 2\|A^*B\|_{\text{ber}}} \qquad (18) \\ &\leq \|A\|_{\text{ber}} + \|B\|_{\text{ber}}. \end{aligned}$$

Our refinement of the inequality (7) is offered in the following theorem.

Theorem 3. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS. If $A \in \mathbb{L}(\mathcal{H})$, then*

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} \leq \frac{1}{2}\sqrt{2\operatorname{ber}^4(A) + \frac{1}{8}\left\|(A+A^*)^2(A-A^*)^2\right\|_{\text{ber}}} \leq \operatorname{ber}^2(A). \qquad (19)$$

Proof. Let $\widehat{k}_\eta$ be a normalized reproducing kernel. Let $A = B + iC$ be the Cartesian decomposition of A. Then B and C are self-adjoint and

$$\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right|^2 = \left\langle B\widehat{k}_\eta, \widehat{k}_\eta \right\rangle^2 + \left\langle C\widehat{k}_\eta, \widehat{k}_\eta \right\rangle^2.$$

Using a little calculation, we have

$$\|B\|_{\text{ber}} \leq \text{ber}(A) \text{ and } \|C\|_{\text{ber}} \leq \text{ber}(A). \tag{20}$$

Now, by using Lemma 4 and the submultiplicative property of usual operator norm, we get

$$\begin{aligned}
\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} &= \frac{1}{2}\left\|B^2 + C^2\right\|_{\text{ber}} \\
&\leq \frac{1}{2}\sqrt{\|B\|_{\text{ber}}^4 + \|C\|_{\text{ber}}^4 + 2\left\|B^2C^2\right\|_{\text{ber}}} \\
&\leq \frac{1}{2}\sqrt{2\text{ber}^4(A) + 2\left\|B^2C^2\right\|_{\text{ber}}} \\
&\text{(by the inequality (20))} \\
&\leq \frac{1}{2}\sqrt{2\text{ber}^4(A) + 2\|B\|_{\text{ber}}^2\|C\|_{\text{ber}}^2} \\
&\leq \text{ber}^2(A).
\end{aligned}$$

Thus, from the above inequalities we get

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} \leq \frac{1}{2}\sqrt{2\left(\text{ber}^4(A) + \left\|B^2C^2\right\|_{\text{ber}}\right)} \leq \text{ber}^2(A),$$

hence, we get (19) as required. □

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement This work's abstract was given at the "4th International Conference on Pure and Applied Mathematics (ICPAM - VAN 2022)" on June 22–23, 2022 in Van, Turkey.

References

[1] Alomari, M. W., Refinements of some numerical radius inequalities for Hilbert space operators, *Linear Multilinear Algebra*, 69(7) (2021), 1208-1223. https://doi.org/10.1080/03081087.2019.1624682

[2] Alomari, M. W., Improvements of some numerical radius inequalities, *Azerb. J. Math.*, 12(1) (2022), 124-137.

[3] Aronzajn, N., Theory of reproducing kernels, *Trans. Amer. Math. Soc.*, 68 (1950), 337-404. https://doi.org/10.1090/S0002-9947-1950-0051437-7

[4] Bakherad, M., Some Berezin number inequalities for operator matrices, *Czechoslovak Math. J.*, 68(143:4) (2018), 997-1009. https://doi.org/10.21136/CMJ.2018.0048-17

[5] Bakherad, M., Garayev, M. T., Berezin number inequalities for operators, *Concr. Oper.*, 6(1) (2019), 33-43. http://doi.org/10.1515/conop-2019-0003

[6] Bakherad, M., Hajmohamadi, M., Lashkaripour R., Sahoo, S., Some extensions of Berezin number inequalities on operators, *Rocky Mountain J. Math.*, 51(6) (2021), 1941-1951. https://doi.org/10.1216/rmj.2021.51.1941

[7] Başaran, H., Gürdal, M., Berezin number inequalities via inequality, *Honam Math. J.,* 43(3) (2021)-523-537. https://doi.org/10.5831/HMJ.2021.43.3.523

[8] Başaran, H., Huban, M. B., Gürdal, M., Inequalities related to Berezin norm and Berezin number of operators, *Bull. Math. Anal. Appl.*, 14(2) (2022), 1-11. https://doi.org/10.54671/bmaa-2022-2-1

[9] Berezin, F. A., Covariant and contravariant symbols for operators, *Math. USSR-Izv.*, 6 (1972), 1117-1151. https://doi.org/10.1070/IM1972v006n05ABEH001913

[10] Bhatia, R., Matrix Analysis, New York, Springer-Verlag, 1997.

[11] Chalendar, I., Fricain, E., Gürdal, M., Karaev, M., Compactness and Berezin symbols, *Acta Sci. Math. (Szeged)*, 78(1-2) (2012), 315-329. https://doi.org/10.1007/BF03651352

[12] Garayev, M., Bouzeffour, F., Gürdal, M., Yangöz, C. M., Refinements of Kantorovich type, Schwarz and Berezin number inequalities, *Extracta Math.*, 35 (2020), 1-20. https://doi.org/10.17398/2605-5686.35.1.1

[13] Garayev, M. T., Guedri, H., Gürdal, M., Alsahli, G. M., On some problems for operators on the reproducing kernel Hilbert space, *Linear Multilinear Algebra*, 69(11) (2021), 2059-2077. https://doi.org/10.1080/03081087.2019.1659220

[14] Gürdal, M., Başaran, H., *A*-Berezin number of operators, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 48(1) (2022), 77-87. https://doi.org/10.30546/2409-4994.48.1.2022.77

[15] Hajmohamadi, M., Lashkaripour, R., Bakherad, M., Improvements of Berezin number inequalities, *Linear Multilinear Algebra*, 68(6) (2020), 1218-1229. https://doi.org/10.1080/03081087.2018.1538310

[16] Haydarbeygi, Z., Sababbeb, M., Moradi H. R., A convex treatment of numerical radius inequalities, *Czechoslovak Math. J.*, 72 (2022), 601–614. https://doi.org/10.21136/CMJ.2022.0068-21

[17] Huban, M. B., Başaran, H., Gürdal, M., New upper bounds related to the Berezin number inequalities, *J. Inequal. Spec. Funct.*, 12(3) (2021), 1-12.

[18] Huban, M. B., Başaran, H., Gürdal, M., Some new inequalities via Berezin numbers, *Turk. J. Math. Comput. Sci.*, 14(1) (2022), 129-137. https://doi.org/10.47000/tjmcs.1014841

[19] Karaev, M. T., Berezin symbol and invertibility of operators on the functional Hilbert spaces, *J. Funct. Anal.*, 238 (2006), 181-192. doi:10.1016/j.jfa.2006.04.030

[20] Karaev, M. T., Reproducing kernels and Berezin symbols techniques in various questions of operator theory, *Complex Anal. Oper. Theory*, 7 (2013), 983-1018. https://doi.org/10.1007/s11785-012-0232-z

[21] Kittaneh, F., Notes on some inequalities for Hilbert space operators, *Publ. Res. Ins. Math. Sci.*, 24 (1988), 283-293. https://doi.org/10.2977/prims/1195175202

[22] Kittaneh, F., Norm inequalities for sums and differences of positive operators, *Linear Algebra Appl.*, 383 (2004), 85-91. https://doi.org/10.1016/j.laa.2003.11.023

[23] Najafabadi, F. P., Moradi, H. R., Advanced refinements of numerical radius inequalities, *Int. J. Math. Model. Comput.*, 11(4) (2021), 1-10. https://doi.org/10.30495/IJM2C.2021.684828

[24] Omidvar, M. E., Moradi, H. R., Better bounds on the numerical radii of Hilbert space operators, *Linear Algebra Appl.*, 604 (2020) 265-277. https://doi.org/10.1016/j.laa.2020.06.021

[25] Omidvar, M. E., Moradi, H. R., Shebrawi, K., Sharpening some classical numerical radius inequalities, *Oper. Matrices.*, 12(2) (2018), 407-416. doi:10.7153/oam-2018-12-26

[26] Tafazoli, S., Moradi, H. R., Furuichi, S., Harikrishnan, P., Further inequalities for the numerical radius of Hilbert space operators, *J. Math. Inequal.*, 13(4) (2019), 955-967. doi:10.7153/jmi-2019-13-68

[27] Tapdigoglu, R., New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space, *Oper. Matrices*, 15(3) (2021), 1445-1460. https://doi.org/10.7153/oam-2021-15-64

[28] Yamancı, U., Tunç, R., Gürdal, M., Berezin number, Grüss-type inequalities and their applications, *Bull. Malays. Math. Sci. Soc.*, 43(3) (2020), 2287-2296. https://doi.org/10.1007/s40840-019-00804-x