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A New Sequence of Bernstein-Durrmeyer Operators and Their L_p-Approximation Behaviour

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Abstract
The purpose of the present manuscript is to present a new sequence of Bernstein-Durrmeyer operators. First, we investigate approximation behaviour for these sequences of operators in Lebesgue Measurable space. Further, we discuss rate of convergence and order of approximation with the aid of Korovkin theorem, modulus of continuity and Peetre K-functional in l_p space. Moreover, Voronovskaja type theorem is introduced to approximate a class of functions which has first and second order continuous derivatives. In the last section, numerical and graphical analysis are investigated to show better approximation behaviour for these sequences of operators.

Keywords: *Rate of convergence; order of approximation; modulus of continuity; Bernstein-Durrmeyer operators.*
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1. Introduction

Operators theory is a fascinating field of research for the last two decades due to the advent of computer. It contributes important role in applied and pure mathematics, viz, fixed point theory, numerical analysis etc. In computational aspects of mathematics and shape of geometric objects, CAGD (Computer-aided Geometric design) plays an interesting role with the mathematical description. It focuses on mathematics which is compatible with computers in shape designing. To investigate the behavior of parametric surfaces and curves, control nets and control points has a significant role respectively. CAGD is widely used as an application in applied mathematics and industries. It has several applications in other branches of sciences, e.g., approximation theory, computer graphics, data structures, numerical analysis, computer algebra etc. In 1912, Bernstein [1] was the first who introduced a sequence of polynomials to present a smallest and easiest proof of celebrated theorem named as Weierstrass

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approximation theorem with the aid of binomial distribution as follows:

$$B_l(g;x) = \sum_{\nu=0}^{l} g\left(\frac{\nu}{l}\right) \left(\begin{array}{c} l \\ \nu \end{array} \right) \mu^{\nu}(1-\mu)^{l-\nu}, \quad \mu \in [0,1], \tag{1.1}$$

where g is a bounded function defined on $[0,1]$. The basis $\binom{l}{\nu}\mu^{\nu}(1-\mu)^{l-\nu}$ of Bernstein polynomials (1.1) has significant role in preserving the shape of the surfaces or curves (see [2]-[4]). Graphic design programs, viz, photoshop inkspaces and Adobe's illustrator deals with Bernstein polynomials in the form of Bèzier curves. To preserve the shape of the parametric surface or curve, it depends on basis $\binom{l}{\nu}\mu^{\nu}(1-\mu)^{l-\nu}$ which is used to design the curves.

In 1962, Schurer [5] presented the following modification of Bernstein operators (1.1) is denoted as $B_{m,l}$: $C[0,1+l] \to C[0,1]$ and given by:

$$B_{m,l}(g;\mu) = \sum_{i=0}^{m+l} g\left(\frac{j}{m}\right) \left(\begin{array}{c} m+l \\ j \end{array} \right) \mu^{k}(1-\mu)^{m+l-j}, \mu \in [0,1],$$

for $l \in \mathbb{N} \cup \{0\}$ and $g \in C[0,1+l]$. In the recent past, Several modifications have studied in various functional spaces to achieve better approximation results (see Acar et al. [6], Acu et al. [7], Braha et al. ([8], [9]), Cai et al. [10], Cetin et al. [11], Kajla et al. [12], Mohiuddine et al. [13]). Izgi [14] introduced a new sequence of Bernstein polynomials as:

$$A_n(h;u) = \sum_{k=0}^{n} q_{n,k,a,b}(u) h\left(\frac{k}{n}\frac{n+a}{n+b}\right),$$

where $q_{n,k,a,b}(u) = \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} u^k \left(\frac{n+a}{n+b} - u\right)^{n-k}$, $0 \le a \le b$, $u \in \left[0, \frac{n+a}{n+b}\right]$ and $h \in C\,[0,1]$. Further, he constructed two dimentional sequences of operators to approximate a class of bivariate continuous functions on square and triangular domain. Moreover, he investigated rate of convergence and order of approximation in different functional spaces with the aid of modulus of continuity, In the last, he presented another variant of these sequences to approximate a wider class, i.e., Lebesgue measurable class as:

$$T_n\,(h;u) = \frac{(n+b)(n+1)}{n+a} \sum_{k=0}^{n} q_{n,k}^{\sim}(u) \int_0^{\frac{n+a}{n+b}} q_{n,k}^{\sim}(t) h(t) dt, \tag{1.2}$$

where $q_{n,k}^{\sim}(u) =: q_{n,k,a,b}(u)$ and $I_n = \left[0, \frac{n+a}{n+b}\right]$.

and

$$T_n^*\,(h;u) = \begin{cases} T_n\,(h;u) & u \in I_n \\ h(u) & u \in [0,1]\ /\ I_n \end{cases} \tag{1.3}$$

Remark 1.1. Bernstein-Durrmeyer type operators defined by (1.2) are linear and positive.

2. Preliminaries

In this section, let's calculate the values of our operator $1, t, t^2, t^3$ and t^4 to examine the convergence states and show that our operator satisfies the Korovkin conditions. After that, with the help of these values, let's calculate their central moments.

Lemma 2.1. *Let $f_p(t) = t^p, p \in N \bigcup \{0\}$ be the test functions. Then, we have*

$$T_n\,(t^p;u) = \frac{(n+1)!}{(n+p+1)!} \sum_{s=0}^{p} \binom{p}{s} \frac{p!n!}{s!\,(n-s)!} \left(\frac{n+a}{n+b}\right)^{p-s} u^s.$$

Proof. We know

$$\begin{aligned}\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)t^p dt &= \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} \int_0^{\frac{n+a}{n+b}} t^k \left(\frac{n+a}{n+b}-t\right)^{n-k} t^p dt \\ &= \left(\frac{n+b}{n+a}\right)^n \binom{n}{k} \int_0^1 \left(\frac{n+a}{n+b}\right)^{n+p+1} x^{k+p}(1-x)^{n-k} dx \\ &= \left(\frac{n+a}{n+b}\right)^{p+1} \frac{n!}{k!(n-k)!}\frac{(k+p)!}{(n+p+1)!}.\end{aligned}$$

In view of (1.2), we have

$$\begin{aligned}T_n\left(t^p;u\right) &= \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u) \int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)t^p dt \\ &= \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\left(\frac{n+a}{n+b}\right)^{p+1}\frac{n!}{k!(n-k)!}\frac{(k+p)!}{(n+p+1)!} \\ &= \left(\frac{n+a}{n+b}\right)^{p+1}\frac{(n+1)!}{(n+p+1)!}\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\frac{k!}{(k+p)!}.\end{aligned}$$

Now, the p^{th} order derivative of the $x^p(x+y)^n$ expression is as:

$$\begin{aligned}\frac{\partial^p}{\partial u^p}\left[u^p(u+v)^n\right] &= \frac{\partial^p}{\partial u^p}\sum_{k=0}^{n}\binom{n}{k}u^{k+p}v^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{(k+p)!}{k!}u^k v^{n-k}, \qquad (2.1)\end{aligned}$$

or

$$\frac{\partial^p}{\partial u^p}\left[u^p(u+v)^n\right] = \sum_{s=0}^{p}\binom{p}{s}\frac{p!n!}{s!(n-s)!}u^s(u+v)^{n-s}. \qquad (2.2)$$

Combining equation (2.1) and equation (2.2), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(k+p)!}{k!}u^k v^{n-k} = \sum_{s=0}^{p}\binom{p}{s}\frac{p!n!}{s!(n-s)!}u^s(u+v)^{n-s}.$$

Choosing $u+v=\frac{n+a}{n+b}$ and multiply both the sides in the above equation with $\left(\frac{n+b}{n+a}\right)^n$, we have

$$\left(\frac{n+b}{n+a}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(k+p)!}{k!}u^k\left(\frac{n+a}{n+b}-u\right)^{n-k} = \left(\frac{n+b}{n+a}\right)^n\sum_{s=0}^{p}\binom{p}{s}\frac{p!n!}{s!(n-s)!}u^s\left(\frac{n+a}{n+b}\right)^{n-s}. \qquad (2.3)$$

In the light of equation (2.1) and (2.3), we arrive at the required result. □

Lemma 2.2. *Let $f_p(t) = t^p, p \in \{0,1,2,3,4\}$ be the test function. Then*

$$
\begin{aligned}
T_n(1;u) &= 1,\\
T_n(t;u) &= u - \frac{2}{n+2}u + \frac{n+a}{(n+2)(n+b)},\\
T_n(t^2;u) &= u^2 - \frac{6(n+1)}{(n+2)(n+3)}u^2 + \frac{4n(n+a)}{(n+2)(n+3)(n+b)}u + \frac{2}{(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2,\\
T_n(t^3;u) &= u^3 - 12\frac{(n^2+2n+2)}{(n+2)(n+3)(n+4)}u^3 + \frac{9n(n-1)(n+a)}{(n+2)(n+3)(n+4)(n+b)}u^2\\
&\quad + \frac{18n}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2 u + \frac{6}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3,\\
T_n(t^4;u) &= u^4 - 20\frac{(n^3+3n^2+8n+6)}{(n+2)(n+3)(n+4)(n+5)}u^4 + \frac{16n(n-1)(n-2)(n+a)}{(n+2)(n+3)(n+4)(n+5)(n+b)}u^3\\
&\quad + \frac{72n(n-1)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^2 u^2 + \frac{96}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^3 u\\
&\quad + \frac{24}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^4.
\end{aligned}
$$

Proof. In the direction of Lemma 2.1, one can easily arrive at the proof of Lemma 2.2. □

Consider $\delta_{n,p}(u) = T_n((t-u)^p; u),\ p \in \{0,1,2...\}$. Then, we obtain the central moments in the following Lemma 2.3:

Lemma 2.3. *For the operators given by $\delta_{n,p}(u)$, we have*

$$
\begin{aligned}
\delta_{n,0}(u) &= 1,\\
\delta_{n,1}(u) &= -\frac{2}{n+2}u + \frac{2}{n+2}\frac{n+a}{n+b},\\
\delta_{n,2}(u) &= \frac{2(n-3)}{(n+2)(n+3)}u\left(\frac{n+a}{n+b} - u\right) + \frac{2}{(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2,\\
\delta_{n,3}(u) &= \frac{24(n-1)}{(n+2)(n+3)(n+4)}u^3 - \frac{36(n-1)}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)u^2\\
&\quad + \frac{12(n-2)}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2 u + \frac{6}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3,\\
\delta_{n,4}(u) &= \frac{12(n^2-21n+10)}{(n+2)(n+3)(n+4)(n+5)}u^4 - \frac{2(5n^3-3n^2-242n+120)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)u^3\\
&\quad + \frac{12(n^2-27n+20)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^2 u^2 - \frac{24(n+1)}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^3 u\\
&\quad + \frac{24}{(n+2)(n+3)(n+4)(n+5)}\left(\frac{n+a}{n+b}\right)^4.
\end{aligned}
$$

Proof. In view of Lemma 2.2, we can easily proof Lemma 2.3. □

Now, we consider

$$
\delta = \max_{0\le u\le \frac{n+a}{n+b}} \delta_{n,2}(u) = \frac{(n+1)}{2(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2 \le \frac{1}{2(n+2)} < \frac{1}{n} \tag{2.4}
$$

and

$$
\mu = \max_{0\le u\le \frac{n+a}{n+b}} \delta_{n,4}(u) \le \frac{24}{(n+2)(n+3)} \text{ and } \mu < \frac{1}{n} \text{ for } n > 20.
$$

Let $C_{M_u}[0,1] = \left\{h \in C[0,1] : |h(u)| \leq M\left(1+u^2\right) \text{ for all } u \in \mathbb{R},\, M > 0\right\}$ and for $1 \leq p < \infty$,

$$L_p[0,1] = \left\{ h \text{ is measurable} : \int\limits_0^1 |h(u)|^p \, du < \infty \right\}.$$

Lemma 2.4. *For $h \in C_{M_u}[0,1]$ endowed with the norm $\|h(u)\|_\infty = \sup_{u\in[0,1]} |h(u)|$, we have*

$$\|T_n(h)\|_\infty \leq \|h\|_\infty,$$

i.e., the operator given by (1.2) is bounded.

Proof. In terms of the definition (1.2) and Lemma 2.2, we get

$$\begin{aligned} \|T_n(h)\|_\infty &\leq \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q_{n,k}^{\sim}(u) \int\limits_0^{\frac{n+a}{n+b}} q_{n,k}^{\sim}(t)\,|h(t)|\,dt \\ &\leq \|h\|_\infty T_n(1;u) \\ &= \|h\|_\infty . \end{aligned}$$

Since the operators introduced by (1.2) is linear and bounded. Therefore, it is continuous.

Let

$$W_n(u,t) = \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q_{n,k}^{\sim}(u) q_{n,k}^{\sim}(t),$$

then we can write (1.2) as:

$$T_n(h;u) = \int\limits_{I_n} W_n(u,t)h(t)dt.$$

It is easy to see that

$$\int\limits_{I_n} W_n(u,t)dt = 1 < \infty,$$

$$\int\limits_{I_n} W_n(u,t)du = 1 < \infty,$$

for all $n = 0,1,2...$ (see [15], page 31-32), for $h \in L_p(I_n)$, $T_n(h;u)$ exist for almost all u and belongs to $L_p(I_n)$. Due to Orlicz theorem, there exist a $K > 0$ such that

$$\int\limits_{I_n} |T_n(h;u)|^p \, du \leq K\,\|h\|_\infty . \tag{2.5}$$

□

3. Direct approximation results

Theorem 3.1. *Let $h \in C_{M_u}[0,1]$. Then, one has*

$$\lim_{n\to\infty} T_n(h;u) = h(u),$$

uniformly on $[0,1]$.

Proof. In view of Lemma 2.2, it is easy to check

$$\lim_{n\to\infty} T_n(f_p(t);u) = f_p(u),$$

for $p = 0,1,2$ uniformly on $[0,1]$. Applying Bohman-Korovkin Theorem, the result follows. □

The first modulus of continuity is given by

$$\omega_1(h,\delta)=\sup_{\substack{|t-u|<\delta\\ t,u\in[0,1]}}|h(t)-h(u)|.$$

Theorem 3.2. *Let $h\in C_{M_u}[0,1]$. Then, we have*

$$|T_n(h;u)-h(u)|\le 2\omega_1\left(h,\frac{1}{\sqrt{n}}\right).$$

Proof. In view of Lemma 2.3, (2.4) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}|T_n(h;u)-h(u)| &\le T_n(|h(t)-h(u)|;u)\\ &\le T_n(\left(1+\frac{|t-u|}{\delta}\right)\omega_1(h,\delta);u)\\ &= \omega_1(h,\delta)\left[1+\frac{1}{\delta}T_n(|t-u|;u)\right]\\ &\le \omega_1(h,\delta)\left[1+\frac{1}{\delta}\sqrt{T_n((t-u)^2;u)}\right]\\ &\le \omega_1(h,\delta)\left[1+\frac{1}{\delta}\sqrt{\frac{1}{n}}\right].\end{aligned}$$

Choosing $\delta=\frac{1}{\sqrt{n}}$, we arrive at the desired result. □

For each $0\le\alpha\le 1$ and $M>0$, let $Lip_m\alpha$ denote the set of all functions h on $[0,1]$ such that

$$|h(u)-h(v)|\le M|u-v|^{\alpha}. \tag{3.1}$$

Theorem 3.3. *If h satisfy condition (3.1), then we have*

$$|T_n(h;u)-h(u)|\le M\left(\frac{1}{n}\right)^{\frac{\alpha}{2}}.$$

Proof. Use Cauchy-Schwartz inequality, (2.4) and (3.1), we have

$$\begin{aligned}|T_n(h;u)-h(u)| &= \left|\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\tilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\tilde{q_{n,k}}(t)h(t)dt-h(u)\right|\\ &\le \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\tilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\tilde{q_{n,k}}(t)\,|h(t)-h(u)|\,dt\\ &\le \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\tilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\tilde{q_{n,k}}(t)M\,|u-v|^{\alpha}\,dt\\ &\le M\left(\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n}\tilde{q_{n,k}}(u)\int_0^{\frac{n+a}{n+b}}\tilde{q_{n,k}}(t)\,(u-v)^2\,dt\right)^{\frac{\alpha}{2}}\\ &\le M\left(\frac{1}{n}\right)^{\frac{\alpha}{2}}\end{aligned}$$

the proof is completed. □

Theorem 3.4. *If* $h \in L_1[0,1]$, $u \in (0,1)$ *and* h *endowed with a continuous derivative on the interval* $[0,1]$, *then*

$$|T_n(h;u) - h(u)| \leq \left| -\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b} \right| + 2\sqrt{\delta_{n,2}(u)}\omega_1\left(h', \sqrt{\delta_{n,2}(u)}\right).$$

Proof. Since h is differentiable on $[0,1]$ therefore by mean value theorem of differential calculus we have

$$h(t) - h(u) = (t-u)h'(\theta) = (t-u)h'(u) + (t-u)\left(h'(\theta) - h'(u)\right), \tag{3.2}$$

where $\theta := \theta(u,t)$ belongs to the interval obtained by u and t. Then, on combining (1.3) to (3.2), we have

$$\begin{aligned}
T_n(h;u) - h(u) &= \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)(t-u)h'(u)dt \\
&\quad + \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)(t-u)\left(h'(\theta) - h'(u)\right)dt \\
&= h'(u)T_n((t-u);u) \\
&\quad + \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)(t-u)\left(h'(\theta) - h'(u)\right)dt,
\end{aligned}$$

$$\begin{aligned}
|T_n(h;u) - h(u)| &\leq \left| -\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b} \right| \\
&\quad + \frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\,|t-u|\left|h'(\theta) - h'(u)\right|dt.
\end{aligned}$$

Now, we use properties of modulus of continuity

$$\begin{aligned}
\left|h'(\theta) - h'(u)\right| &\leq \omega_1\left(h', |\theta - u|\right) \leq \left(1 + \frac{|\theta - u|}{\beta}\right)\omega\left(h', \beta\right) \\
&\leq \left(1 + \frac{|t-u|}{\beta}\right)\omega\left(h', \beta\right).
\end{aligned}$$

Since $u \leq \theta \leq t$. Therefore, we have

$$\begin{aligned}
|T_n(h;u) - h(u)| &\leq \left| -\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b} \right| \\
&\quad + \omega\left(h', \beta\right)\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\,|t-u|\left(1 + \frac{|t-u|}{\beta}\right)dt.
\end{aligned}$$

Let's examine the last term of last inequality;

$$\begin{aligned}
&\omega\left(h', \beta\right)\frac{n+b}{n+a}(n+1)\sum_{k=0}^{n} q^{\sim}_{n,k}(u)\int_0^{\frac{n+a}{n+b}} q^{\sim}_{n,k}(t)\,|t-u|\left(1 + \frac{|t-u|}{\beta}\right)dt \\
&= \omega\left(h', \beta\right)\left(T_n(|t-u|;u) + \frac{1}{\beta}T_n((t-u)^2;u)\right) \\
&= \omega\left(h', \beta\right)\left(\sqrt{T_n((t-u)^2;u)} + \frac{1}{\beta}T_n((t-u)^2;u)\right) \\
&= \omega\left(h', \beta\right)\left(\sqrt{\delta_{n,2}(u)}\left[1 + \frac{1}{\beta}\delta_{n,2}(u)\right]\right).
\end{aligned}$$

Then, on choosing $\beta = \sqrt{\delta_{n,2}(u)}$, we prove the desired result. □

4. Voronovskaya-type theorem

In this section, we prove Voronvoskaya-type asymptotic theorem for the operators $T_n(h;u)$ to approximate a class of functions which has first and second order continuous derivatives.

Theorem 4.1. *Let $h \in C_{M_u}[0,1]$. If $h^{'}, h^{''}$ exists at a fixed point $u \in [0,1]$ then we have*

$$\lim_{n\to\infty} n\left\{T_n(h;u) - h(u)\right\} = (-2u+1)\, h^{'}(u) + u(1-u)h^{''}(u).$$

Proof. Let $u \in [0,1]$ be fixed. By Taylor's expansion of h, we can write

$$h(t) = h(u) + (t-u)h^{'}(u) + \frac{1}{2}(t-u)^2 h^{''}(u) + \varphi(t,u)(t-u)^2. \tag{4.1}$$

Where the function $\varphi(t,u)$ is the Peano form of remainder, $\varphi(t,u) \in C_{M_u}[0,1]$ and

$$\lim_{n\to\infty} \varphi(t,u) = 0.$$

Applying $T_n(h;u)$ both the sides of (4.1) and Lemma 2.3, we have

$$\begin{aligned} n\left\{T_n(h;u) - h(u)\right\} &= n\Bigg\{\left(-\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b}\right)h^{'}(u) \\ &\quad + \frac{1}{2}\left(\frac{2(n-3)}{(n+2)(n+3)}u\left(\frac{n+a}{n+b} - u\right) + \frac{2}{(n+2)(n+3)}\left(\frac{n+a}{n+b}\right)^2\right)h^{''}(u)\Bigg\} \\ &\quad + nT_n(\varphi(t,u)(t-u)^2;u). \end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$nT_n(\varphi(t,u)(t-u)^2;u) \le \left(T_n(\varphi^2(t,u);u)\right)^{\frac{1}{2}} \left(T_n((t-u)^4;u)\right)^{\frac{1}{2}}. \tag{4.2}$$

One can observe that $\varphi^2(u,u) = 0$ and $\varphi^2(.,u) \in C_{M_u}[0,1]$. Then, it follows that

$$\lim_{n\to\infty} T_n(\varphi^2(t,u);u) = \varphi^2(u,u) = 0. \tag{4.3}$$

Now, from (4.2) and (4.3), we obtain

$$\lim_{n\to\infty} nT_n(\varphi(t,u)(t-u)^2;u) = 0. \tag{4.4}$$

From (4.4), we get the required result. □

5. Local approximation

The K-functional is given by :

$$K_2(h,\delta) = \inf_{g\in W^2}\left\{\|h-g\|_\infty + \delta\left\|g^{''}\right\|\right\},$$

where $\delta > 0$, $W^2 = \left\{g : g^{'},\ g^{''} \in C[0,1]\right\}$ and by [13] there exists a positive constant $M > 0$ such that

$$K_2(h,\delta) \le M\,\omega_2(h,\delta).$$

Where the second order modulus of continuity for $h \in C_{M_u}[0,1]$ is defined as:

$$\omega_2(h,\delta) = \sup_{\substack{|t-u|<\delta \\ t,u\in[0,1]}} |h(t+2x) - 2h(u+x) + h(u)|.$$

Theorem 5.1. *For the operators introduced by* $T_n(.;.)$ *and* $h \in C_{M_u}[0,1]$, *we have*

$$\|T_n(h;u) - h(u)\|_\infty \leq 2K_2\left(h, \frac{\delta_n^1}{2}\right) + \delta_n^2 \left\|g'\right\|_\infty,$$

here $\delta_n^1 = \max_{u\in[0,1]}\left\{\frac{8u^3+u+1}{(n+3)(n+4)}\right\} = \frac{10}{(n+3)(n+4)}$ *and* $\delta_n^2 = \max\left\{\inf_{u\in[0,1]}\left\{\frac{|1-2u|}{n+2}\right\}\right\} = \frac{1}{(n+2)}$.

Proof. Let $g \in W^2$ and $t \in [0,1]$. By Taylor's expansion, we have

$$g(t) = g(u) + (t-u)g'(u) + \int_u^t (t-v)g''(v)dv.$$

Applying (1.2) on both the sides of above relation and using Lemma 2.3, we have

$$T_n(g;u) = g(u) + \left(-\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b}\right)g'(u) + T_n\left(\int_u^t (t-v)g''(v)dv; u\right).$$

Further

$$\begin{aligned} |T_n(g;u) - g(u)| &\leq \left|-\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b}\right|\left|g'(u)\right| + T_n\left(\int_u^t |t-v|\left|g''(v)\right|dv; u\right) \\ &\leq \left|-\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b}\right|\left\|g'\right\|_\infty + \left\|g''\right\|_\infty\left[T_n\left(\int_u^t (t-v)^2 dv; u\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{5.1}$$

In the light of Lemma 2.2

$$\begin{aligned} T_n\left(\int_u^t (t-v)^2 dv; u\right) &= T_n\left(\frac{1}{3}(t-u)^3; u\right) \\ &= 8\frac{n-1}{(n+2)(n+3)(n+4)}u^3 - 12\frac{n-1}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)u^2 \\ &\quad +4\frac{n-2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2 u \\ &\quad +\frac{2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3. \end{aligned} \tag{5.2}$$

Combining equation (5.2) and (5.1), we obtain

$$\begin{aligned} |T_n(g;u) - g(u)| &\leq \left|-\frac{2}{n+2}u + \frac{1}{n+2}\frac{n+a}{n+b}\right|\left\|g'\right\|_\infty \\ &\quad + \left\|g''\right\|_\infty\left\{8\frac{n-1}{(n+2)(n+3)(n+4)}u^3 - 12\frac{n-1}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)u^2\right| \\ &\quad \left| +4\frac{n-2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^2 u + \frac{2}{(n+2)(n+3)(n+4)}\left(\frac{n+a}{n+b}\right)^3\right\}^{\frac{1}{2}} \\ &\leq \frac{|1-2u|}{n+2}\left\|g'\right\|_\infty + \left\|g''\right\|_\infty\left[\frac{8u^3+u+1}{(n+3)(n+4)}\right]^{\frac{1}{2}}. \end{aligned}$$

With the aid of Lemma 2.4

$$\begin{aligned} \left|T_n(h;u)-h(u)\right| &= \left|T_n(h;u)-T_n(g;u)+T_n(g;u)-g(u)+g(u)-h(u)\right| \\ &\leq \left|T_n(h-g;u)\right|+\left|T_n(g;u)-g(u)\right|+\left|g(u)-h(u)\right| \\ &\leq 2\left\|h-g\right\|_\infty+\frac{|1-2u|}{n+2}\left\|g^{'}\right\|_\infty+\left\|g^{''}\right\|_\infty\left[\frac{8u^3+u+1}{(n+3)(n+4)}\right]^{\frac{1}{2}}, \end{aligned}$$

for $u\in[0,1]$. If the right side of the last inequality is taken as the maximum. The proof is completed. □

Here. We introduce the direct estimate of the operators (1.2) with the aid of Lipschitz-type maximal function of order $\beta\in(0,1]$ defined by Lenze [16] as follows:

$$\omega_\beta^*(h,u)=\sup_{\substack{t\neq u\\ u,t\in[0,1]}}\frac{|h(t)-h(u)|}{|t-u|^\beta}. \tag{5.3}$$

Using (5.3), the following inequality is achieved.

$$|h(t)-h(u)|\leq\omega_\beta^*(h,u)\,|t-u|^\beta\,\omega_\beta^*(h,u)\,\delta^{\frac{\beta}{2}}. \tag{5.4}$$

Theorem 5.2. *Let $h\in C_{M_u}[0,1]$ and $\beta\in(0,1]$. Then, we have*

$$|T_n(h;u)-h(u)|\leq\omega_\beta^*(h,u)\,\delta^{\frac{\beta}{2}}.$$

Proof. If we use (5.4), (2.4) $\left(\delta=\max_{0\leq u\leq\frac{n+a}{n+b}}\delta\,(t-u)^2\right)$ and use Cauchy-Schwartz-Bunyakowsky inequlity, then by using the operators (1.2) we have

$$\begin{aligned} |T_n(h;u)-h(u)| &\leq T_n\left(|h(t)-h(u)|\,;u\right) \\ &\leq \omega_\beta^*(h,u)\,T_n\left(|t-u|^\beta\,;u\right) \\ &\leq \omega_\beta^*(h,u)\,T_n\left((t-u)^2\,;u\right)^{\frac{\beta}{2}} \\ &\leq \omega_\beta^*(h,u)\,\delta^{\frac{\beta}{2}}. \end{aligned}$$

□

6. L_p approximation

Theorem 6.1. *Let $h\in L_p[0,1]$ for $0\leq p<\infty$. Then*

$$\lim_{n\to\infty}\left\|T_n(h)-h\right\|_{L_p(I_n)}=0,$$

is available.

Proof. First, we need to show that there exist a $K>0$ such that $\|T_n\|_{L_p(I_n)}\leq K$ for any $n\in\mathbb{N}$. For this purpose, if we use (2.5) we have $\|T_n\|_{L_p(I_n)}\leq K$. We consider the operator (1.3).

Let's remember the Luzin theorem for a given $\varepsilon>0$, there exists $f\in C[0,1]$ such that

$$\|h-f\|_{L_p[0,1]}<\frac{\varepsilon}{2(K+1)}.$$

By using Theorem 3.1 for the same ε there exist n_0 such that for all $n>n_0$

$$\|T_n(f;u)-f(u)\|_{L_p(I_n)}\leq\frac{\varepsilon}{2}.$$

Based on this information, the following result is obtained

$$\begin{aligned} \|T_n(h)-h\|_{L_p(I_n)} &\leq \|T_n(h)-T_n(f)\|_{L_p(I_n)}+\|T_n(f)-f\|_{C(I_n)}+\|h-f\|_{L_p(I_n)} \\ &= (K+1)\,\|h-f\|_{L_p(I_n)}+\|T_n(f)-f\|_{C(I_n)} \\ &< \varepsilon. \end{aligned}$$

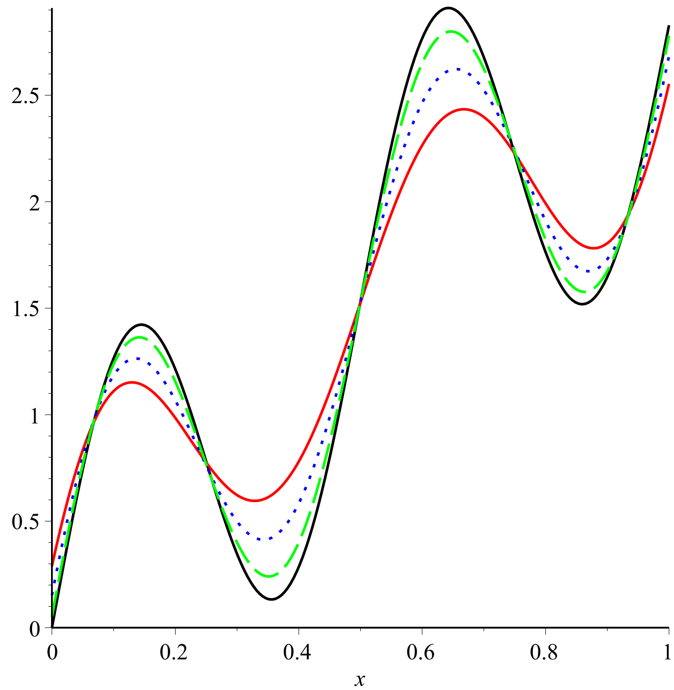
Then, the prof is completed. □

7. Some plots

In this section, we discuss the approximation behaviour of the sequence of operator defined by (1.2) for different functions with the help of graphs. In addition, margins of error is shown with tables of numerical values.

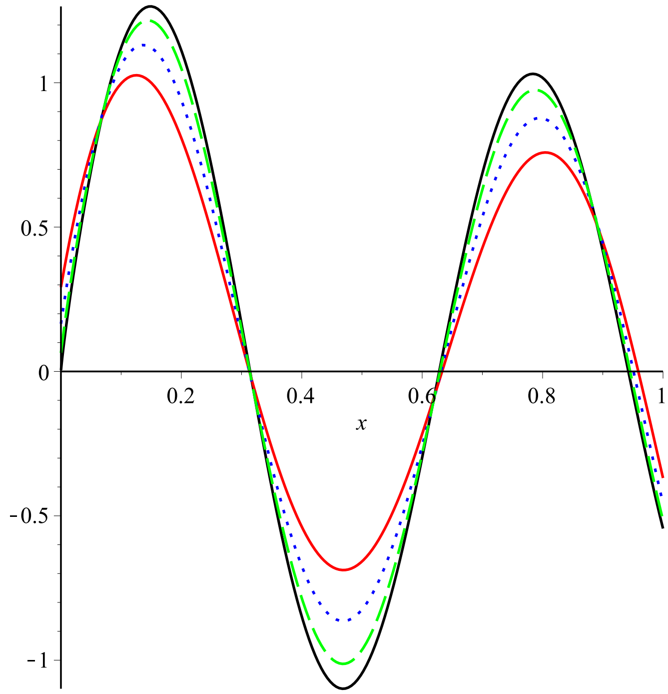
Example 7.1. Let $a = 0.4$, $b = 0.5$ and $h(u) = \sin(4\pi u) + 4\sin(\frac{1}{4}\pi u)$. Fig. 1 shows the $T_n(h;u)$ operator's approximation to the $h(u)$ (black) function for the values $n = 50$ (red), $n = 100$ (blue) and $n = 300$ (green).

Figure 1. $T_n(h;u)$ Operator's approximation to the function $h(u) = \sin(4\pi u) + 4\sin(\frac{1}{4}\pi u)$ for different n values.



Example 7.2. Let be $a = 0.9$, $b = 0.8$ and $h(u) = u^{\frac{-1}{8}}\sin(10u)$. Fig. 2 shows the $T_n(h;u)$ operator's approximation to the $h(u)$ (black) function for the values $n = 50$ (red), $n = 100$ (blue) and $n = 300$ (green).

Figure 2. $T_n\,(h;u)$ Operator's approximation to the function $h(u) = u^{\frac{-1}{8}}\sin(10u)$ for different n values.



Now let's compare the classical Bernstein -Durrmeyer operator defined below with our operator defined in (1.2) with a graph;

$$S_n(h;u) = (n+1)\sum_{k=0}^{n}\varphi_{n,k}(u)\int_0^1 \varphi_{n,k}(t)h(t)dt$$

here $\varphi_{n,k}(u) = \binom{n}{k}u^k\,(1-u)^{n-k}$, $h \in C\,[0,1]\,,\, u \in [0,1]\,.$

Example 7.3. Let be $a = 10$, $b = 50$ and $h(u) = u^{\frac{-1}{8}}\sin(10u)$. Fig. 3 shows the $T_n\,(h;u)$ (blue) and $S_n(h;u)$ (red) operators are approximation to the $h(u)$ (black) function for the value $n = 100$.

Table 1 shows the numerical values obtained with the maximum value of the statement $|T_n\,(h;u) - h(u)|$, in order to examine how the $T_n\,(h;u)$ operator approximation the function $h(u) = \sin(4\pi u) + 4\sin(\frac{1}{4}\pi u)$ for $a = 0.4$, $b = 0.5$ and different n, u values.

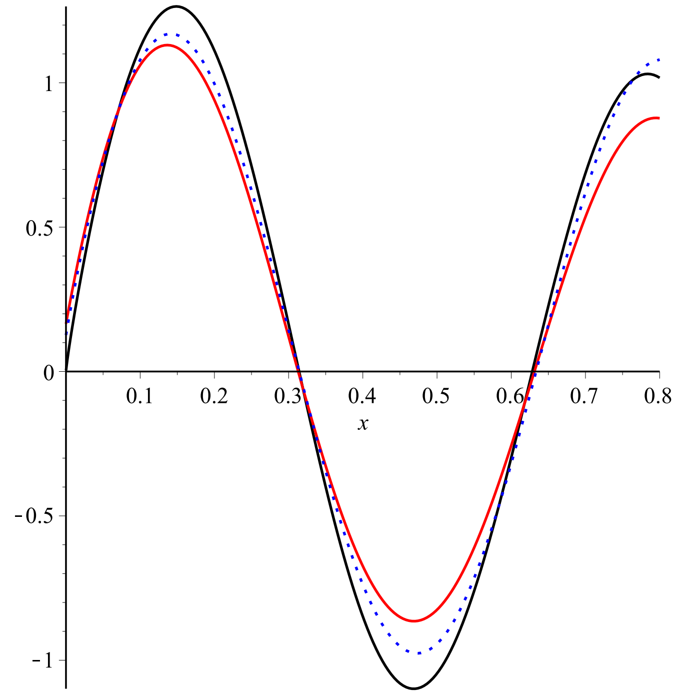
Table 1. Error margins between the $T_n\,(h;u)$ operator and $h(u) = \sin(4\pi u) + 4\sin(\frac{1}{4}\pi u)$

n	$u = 0.2$	$u = 0.4$	$u = 0.6$	$u = 0.8$
150	0.1065373586	0.213548394	0.216177741	0.106684568
250	0.0686472096	0.135810327	0.137416949	0.068763059
500	0.0362438876	0.071056815	0.071872379	0.036314182
1000	0.0186274446	0.036364660	0.036775374	0.018666478

The definition Izgi[14] provided to compare the approaches of different operators can be given as it comprises statements that can be simplified as the numerator and the denominator. L_n and T_n are operators defined in the same range:

$$\lim_{n\to\infty}\frac{\sup_{0\le u\le\frac{n+a}{n+b}}|L_n(h;u)-h(u)|}{\sup_{0\le u\le\frac{n+a}{n+b}}|T_n(h;u)-h(u)|} = \begin{cases} 0, & T_n,\ \textit{faster} \\ \infty, & L_n,\ \textit{faster} \\ c(\text{constant}), & \textit{equally fast} \end{cases}$$

Figure 3. $T_n\,(h;u)$ and $S_n(h;u)$ Operators are approximation to the function $h(u) = u^{\frac{-1}{8}}\sin(10u)$ for $n = 100$.



Based on this definition, it is possible to examine the rate of approximation of the operators defined by

$$E_n(h;u) = \frac{\sup_{0\le u\le \frac{n+a}{n+b}} |T_n(h;u) - h(u)|}{\sup_{0\le u\le \frac{n+a}{n+b}} |S_n(h;u) - h(u)|}.$$

The E_n operator was defined by Aydın Izgi [17] in 2013 and this ratio is used as a measurement in many articles.

As shown in Fig 3, the operators will be compared in Table 2 for $a = 0.1$, $b = 0.8$ and different n values in order to use the u points where the difference is seen more clearly.

Table 2. Error margins between the $E_n\,(h;u)$ operator and $h(u) = u^{\frac{-1}{8}}\sin(10u)$

n	$u = 0.15$	$u = 0.45$	$u = 0.6$	$u = 0.75$
150	0.9946389397	0.9924182950	0.9755127166	0.9850499535
250	0.9967498131	0.9953122198	0.9842312426	0.9907327967
500	0.9983634502	0.9976010256	0.9916588067	0.9952461550
1000	0.9991709201	0.9987821715	0.9956978930	0.9975866425

Table 2, shows that $T_n\,(h;u)$ operators approximation the $h(u) = u^{\frac{-1}{8}}\sin(10u)$ function better than the operators of $S_n\,(h;u)$.

8. Conclusion

In general, a new sequence of Bernstein-Durrmeyer operators was defined in our study. First, the approximation behaviors for the defined operator sequences in the Lebesgue Measurable space were investigated in the article. Then, with the help of Korovkin's theorem, the modulus of continuity and the Peetre K-function on the space l_p, the convergence rate and the order of approximation are discussed. Also, the Voronovskaja type theorem was proved to approximate a class of functions with continuous derivatives of the first and second order. Finally, numerical and graphical analyses were examined to show better approximation behavior for these operator sequences, and it was seen that the operator we have just defined works more efficiently than the Bernstein-Durrmeyer operator defined earlier.

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