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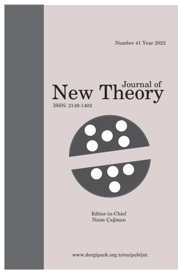

New Inequalities for Hyperbolic Lucas Functions

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Abstract — This article introduces the classic Wilker's, Wu-Srivastava, Hugyen's, Cusa-Hugyen's, and Wilker's-Anglesio type inequalities for hyperbolic Lucas functions with some new refinements.

Keywords — *Hyperbolic Lucas functions, Wilker's inequality, Cusa-Huygen's inequality, Wu-Srivastava inequality*

Mathematics Subject Classification (2020) — 26D05, 33B10

1. Introduction

As it is known, the theory of inequality is one of the most important branches of mathematics. Especially in functional analysis, differential equations and mathematical analysis, inequalities have a great impact. Fundamental research in this area belongs to the great mathematicians such as Hardy, Cauchy, Hölder, Littlewood, Minkowski and others. One of the curious topics of the theory of inequality are inequalities related to trigonometric and hyperbolic functions. The most famous studies on this subject belong to mathematicians such as Wilker, Huygen's, Mitrinovic, Wu, Srivastava, Adamovic, and Cusa. In this study, we will give analogues and some new improvements of these inequalities for hyperbolic Lucas functions. Hyperbolic Lucas functions are defined by inspiring the Binet formula for Lucas numbers, which are interesting in number theory and on which many studies have been made. The reason that makes these functions special is that they are related to the golden ratio. Because the golden ratio has many incredible applications in nature. Therefore, it would be interesting to give analogues of theorems related to classical hyperbolic and trigonometric functions for hyperbolic Lucas functions.

Now we will give some famous inequalities:

i. The Wilker's inequality is given as (see [1–15]).

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2 \tag{1}$$

ii. The Huygens inequality is given as (see [3, 4, 11, 12]).

$$\frac{2\sin x}{x}+\frac{\tan x}{x}>3 \tag{2}$$

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iii. The Cusa-Huygens inequality is given as (see [12, 16]).

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3} \tag{3}$$

iv. The Wu-Srivastava inequality is given as (see [9]).

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \tag{4}$$

v. The Wilker's-Anglesio inequality is given as (see [5] [17]).

$$\left(\frac{\sinh(x)}{x}\right)^2 + \frac{\tanh(x)}{x} > 2 + \frac{8}{45}x^3 \tanh(x) \tag{5}$$

Inequalities (1), (2), (3), (4) and (5) are satisfied for $x \in \left(0, \frac{\pi}{2}\right)$.

2. Preliminaries

This section provides some of the basic notions needed for the following sections. The classical hyperbolic functions are as follows.

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \sinh(x) = \frac{e^x - e^{-x}}{2}, \text{ and } \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{6}$$

Similarly, Stakhov and Rozin described hyperbolic Lucas functions in 2005 (see [8, 18, 19]).

Definition 2.1. The symmetrical hyperbolic Lucas sine, cosine, and tangent functions are defined as follows, respectively.

$$sLh(x) = \alpha^x - \alpha^{-x}, cLh(x) = \alpha^x + \alpha^{-x}, \text{ and } tLh(x) = \frac{\alpha^x - \alpha^{-x}}{\alpha^x + \alpha^{-x}} \text{ for all } x \in R \tag{7}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Definition 2.2. [20, 21] The generalized hyperbolic sine, cosine, and tangent functions are defined as follows, respectively.

i. $\sinh_\varphi(x) = \dfrac{\varphi^x - \varphi^{-x}}{2}$

ii. $\tanh_\varphi(x) = \dfrac{\varphi^x - \varphi^{-x}}{\varphi^x + \varphi^{-x}}$

iii. $\cosh_\varphi(x) = \dfrac{\varphi^x + \varphi^{-x}}{2}$

Some basic properties of hyperbolic Lucas functions are as follows:

i. $cLh(x) = cLh(-x)$

ii. $sLh(x) = -sLh(-x)$

iii. $tLh(x) = -tLh(-x)$

iv. $sLh'(x) = cLh(x)\ln(\alpha)$

v. $cLh'(x) = sLh(x)\ln(\alpha)$

vi. $tLh^{'}(x) = \dfrac{4\ln(\alpha)}{cLh^2(x)}$

Lemma 2.3. If $x \in [0, \infty)$, then the following inequalities hold:

i. $sLh(x) \geq 2x\ln(\alpha)$

ii. $x\ln(\alpha) \geq tLh(x)$

PROOF. *i*) Let $f : R^+ \to R$ be a function defined by

$$f(x) = sLh(x) - 2x\ln(\alpha)$$

The derivative of $f(x)$ is

$$f^{'}(x) = \ln(\alpha)(cLh(x) - 2) \geq 0$$

Because $cLh(x) \geq 2$ .Then we obtain $f(x)$ is an increasing function on the interval $[0, \infty)$, this means that $f(x) \geq f(0) = 0$ Therefore

$$sLh(x) \geq 2x\ln(\alpha)$$

Similarly, we can proof *ii*. □

Lemma 2.4. [8] If $x \neq 0$, then the following inequality holds:

$$cLh(x) < \frac{1}{4(\ln(\alpha))^3}\left(\frac{sLh(x)}{x}\right)^3 \tag{8}$$

PROOF. From the properties of hyperbolic Lucas functions, it is clear that it is sufficient to prove the theorem for $x > 0$.
Let $f : R^+ \to R$ be a function defined by $f(x) = \dfrac{sLh^3(x)}{x^3 cLh(x)}$
The derivative of $f(x)$ is

$$f^{'}(x) = \frac{sLh^2(x)}{x^4 cLh^2(x)}\left[2xcLh^2(x)\ln(\alpha) + 4x\ln(\alpha) - 3cLh(x)sLh(x)\right]$$

Now let $g : R^+ \to R$ be a function defined by

$$g(x) = 2xcLh^2(x)\ln(\alpha) + 4x\ln(\alpha) - 3cLh(x)sLh(x)$$

The derivative of $g(x)$ is

$$g^{'}(x) = 2\ln(\alpha)[2xsLh(2x) - 3cLh(2x)) + cLh^2(x) + 2]$$

Now let $h : R^+ \to R$ be a function defined by

$$h(x) = 2xsLh(2x) - 3cLh(2x)) + cLh^2(x) + 2$$

The derivative of $h(x)$ is

$$h^{'}(x) = 2sLh(2x)[1 - 2\ln(\alpha)] + 4xcLh(2x)$$

this show $h^{'}(x) > 0$ Then we obtain $h(x), g(x)$ are increasing and positive functions on $(0, \infty)$. Hence, we get $f(x)$ is an increasing on $(0, \infty)$, by using $\lim_{x\to 0^+} f(x) = 4(\ln(\alpha))^3$. We conclude that

$$f(x) > 4(\ln(\alpha))^3$$

□

Lemma 2.5. [22,23] If $x, y > 0$, and $\mu \in [0, 1]$, then

$$\mu x + (1 - \mu)y \geq x^{\mu}y^{1-\mu}$$

Lemma 2.6. [22, 23] (Cauchy-Schwarz inequality) If $x_i, y_i > 0$, then

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$

Lemma 2.7. If $x_i, y_i > 0$, $i = 1, 2, ..., n$, then

$$\left(\sum_{i=1}^{n}(x_i + y_i)\right)^2 \geq 4\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right) \tag{9}$$

Proof. We know that: $4xy \leq (x+y)^2, \forall x, y > 0$

$$4\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right) \leq \left[\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right) + \left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right)\right]^2$$
$$= \left[\sum_{i=1}^{n}\left(\sqrt{x_i y_i} + \sqrt{\frac{x_i^2 + y_i^2}{2}}\right)\right]^2$$

And by Lemma 2.6, we get

$$\left[\sum_{i=1}^{n}\left(\sqrt{x_i y_i} + \sqrt{\frac{x_i^2 + y_i^2}{2}}\right)\right]^2 \leq \left[\sum_{i=1}^{n}\sqrt{(1+1)\left(x_i y_i + \frac{x_i^2 + y_i^2}{2}\right)}\right]^2 = \left(\sum_{i=1}^{n}(x_i + y_i)\right)^2$$

□

Lemma 2.8. If $x, y > 0$, $x \geq y$ and $\mu \in [\frac{1}{2}, 1]$, then the following inequality holds:

$$\mu x + (1-\mu)y \geq x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \geq x^{\mu}y^{1-\mu} \tag{10}$$

Proof. We obtain the first part of the inequality directly from Lemma 2.5

$$\begin{aligned} \mu x + (1-\mu)y &= (2\mu - 1)(x - y) + (1-\mu)x + \mu y \\ &\geq x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \end{aligned}$$

Now, we have to illustrate that the below inequality holds :

$$x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \geq x^{\mu}y^{1-\mu}$$

For this let's define a function $f : [1, \infty) \to R$

$$f(t) = t^{1-\mu} + (2\mu - 1)(t - 1) - t^{\mu}$$
$$f'(t) = (1-\mu)t^{-\mu} + (2\mu - 1) - \mu t^{\mu - 1}$$
$$f''(t) = (1-\mu)(-\mu)t^{-\mu-1} - \mu(\mu - 1)t^{\mu-2} = \mu(\mu - 1)\left[\frac{1}{t^{\mu+1}} - \frac{1}{t^{2-\mu}}\right] \geq 0, \forall t \geq 1$$

then we obtain :

$$\forall t \geq 1, f'(t) \geq f'(1) = 0$$

thus $f(t)$ is an increasing and positive function for all $t \geq 1$. If we take $t = \dfrac{x}{y}$ and multiply both sides of the inequality by y, then we obtain :

$$\left(\frac{x}{y}\right)^{1-\mu} y + (2\mu - 1)(x - y) \geq \left(\frac{x}{y}\right)^{\mu} y$$

or

$$x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \geq x^{\mu}y^{1-\mu}$$

□

Lemma 2.9. If $x, y > 0$, $x \geq y$ and $\mu \in [\frac{1}{2}, \frac{3}{4}]$, then the following inequality is satisfied

$$\mu x + (1-\mu)y \geq x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{x-y}{2} \geq x^{\mu} y^{1-\mu} \tag{11}$$

PROOF. By Lemma 2.5, we obtain

$$\mu x + (1-\mu)y = (\mu - \frac{1}{2})x + (\frac{3}{2} - \mu)y + \frac{x-y}{2} \geq x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{x-y}{2}$$

Now we have to demonstrate that the following inequality is satisfied:

$$x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{x-y}{2} \geq x^{\mu} y^{1-\mu} \tag{12}$$

By Lemma 2.5, we obtain

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] \geq \left(\frac{x}{y}\right)^{\frac{1}{4}}$$

Also we know for all $\mu \in [\frac{1}{2}, \frac{3}{4}]$

$$\left(\frac{x}{y}\right)^{\frac{3}{4}-\mu} \geq 1$$

or

$$\left(\frac{x}{y}\right)^{\frac{1}{4}} \geq \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}$$

is true. Then we get:

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] \geq \left(\frac{x}{y}\right)^{\frac{1}{4}} \geq \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}} \tag{13}$$

It is clear that the inequality (12) is equivalent to the following inequality:

$$\frac{x-y}{2} \geq x^{\mu-\frac{1}{2}} y^{1-\mu} \left[\sqrt{x} - \sqrt{y}\right]$$

If $x = y$, the inequality is trivial. So let's assume $x > y$ and divide both side of the inequality by $\sqrt{y}(\sqrt{x} - \sqrt{y})$ then we get the following inequality:

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] > \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}$$

This inequality is true according to the (13). □

Ibrahimov [24] proved the below inequalities for generalized hyperbolic functions

Theorem 2.10. If $x \geq 0$ and $s > f > 1$ then the following inequalities are satisfied:

i. $\sinh_s(x) \ln f \geq \sinh_f(x) \ln s$

ii. $\tanh_s(x) \ln f \leq \tanh_f(x) \ln s$

iii. $\cosh_s(x) \ln f \geq \cosh_f(x) \ln s$

3. Main Results

Theorem 3.1. (Wu-Srivastava type inequality) If x nonzero real number then the following inequality holds:

$$\left(\frac{x}{sLh(x)}\right)^2 + \frac{x}{tLh(x)} > \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right) \tag{14}$$

Proof. From the properties of hyperbolic Lucas functions, obviously that it is sufficient to prove the theorem for $x > 0$.
Let $f : R^+ \to R$ be a function defined by

$$f(x) = x^2 + xsLh(2x) - \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right) sLh^2(x)$$

The derivatives of $f(x)$ are

$$f'(x) = 2x - \left(1+\frac{1}{2\ln\alpha}\right) sLh(2x) + 2xcLh(2x)\ln\alpha$$

$$f''(x) = (2 - cLh(2x) + 4xsLh(2x))(\ln(\alpha))^2$$

$$f'''(x) = -2sLh(2x)\ln(\alpha) + 4sLh(2x)(\ln(\alpha))^2 + 8xcLh(2x)(\ln(\alpha))^3$$

$$\begin{aligned} f^{(4)}(x) &= -4cLh(2x)(\ln(\alpha))^2 + 16cLh(2x)(\ln(\alpha))^3 + 16xsLh(2x)(\ln(\alpha))^4 \\ &= 4cLh(2x)(\ln(\alpha))^2(4\ln(\alpha)-1) + 16xsLh(2x)(\ln(\alpha))^4 \end{aligned}$$

This means that $f^{(4)}(x) \geq f^{(4)}(0) = 8(\ln(\alpha))^2(4\ln(\alpha)-1) > 0$, $f'''(0) = 0$, $f''(0) = 0$, $f'(0) = 0$. Thus $f'''(x)$, $f''(x)$, $f'(x)$ and $f(x)$ are increasing and positive functions on the interval $[0, \infty)$, this means that $f(x) \geq f(0) = 0$, for all $x \geq 0$. Therefore

$$x^2 + xsLh(2x) \geq \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right) sLh^2(x)$$

By dividing both sides of the inequality by $sLh^2(x)$ for $x > 0$, we obtain

$$\left(\frac{x}{sLh(x)}\right)^2 + \frac{x}{tLh(x)} > \frac{1}{\ln\alpha}\left(\frac{1}{4\ln\alpha}+1\right)$$

□

In addition we give Cusa-Huygens type inequality for hyperbolic Lucas functions.

Theorem 3.2. If $x \neq 0$, then the following inequality is satisfied:

$$\frac{sLh(x)}{x} < \left(\frac{cLh(x)}{2}+1\right)\ln(\alpha) \tag{15}$$

Proof. From the properties of hyperbolic Lucas functions, obviously that it is sufficient to prove the theorem for $x > 0$.

Let $f : R^+ \to R$ be a function defined by

$$f(x) = \left(\frac{cLh(x)}{2}+1\right)\ln(\alpha) - \frac{sLh(x)}{x}$$

The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \frac{sLh(x)}{2x}(\ln(\alpha))^2 - \frac{xcLh(x)\ln(\alpha) - sLh(x)}{x^2} \\ &= \frac{x^2 sLh(x)(\ln(\alpha))^2 - 2xcLh(x)\ln(\alpha) + 2sLh(x)}{2x^2} \end{aligned}$$

In addition, let $g : R^+ \to R$ be a function defined by

$$g(x) = x^2 sLh(x)(\ln(\alpha))^2 - 2xcLh(x)\ln(\alpha) + 2sLh(x)$$

The derivative of $g(x)$ is

$$g'(x) = x^2 cLh(x)(\ln(\alpha))^3 > 0$$

Then we obtain $g(x)$ is an increasing function on $(0, \infty)$. This means that $g(x) > g(0) = 0$. Hence, we get $f(x)$ is an increasing on $(0, \infty)$, by using

$$\lim_{x\to 0^+}\left[\left(\frac{cLh(x)}{2}+1\right)\ln(\alpha) - \frac{sLh(x)}{x}\right] = 0$$

We conclude that $f(x) > 0$. □

Furthermore, we give Huygens type inequality for hyperbolic Lucas functions.

Theorem 3.3. If $x \neq 0$, then the following inequality is satisfied:

$$2\frac{sLh(x)}{x} + \frac{tLh(x)}{x} > 3\,(4)^{\frac{1}{3}}\ln(\alpha) \tag{16}$$

Proof. By Lemmas 2.4, 2.5, we get

$$\frac{2}{3}\frac{sLh(x)}{x} + \frac{1}{3}\frac{tLh(x)}{x} > \left(\frac{sLh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tLh(x)}{x}\right)^{\frac{1}{3}} = \frac{sLh(x)}{x}\frac{1}{\sqrt[3]{cLh(x)}} > (4)^{\frac{1}{3}}\ln(\alpha)$$

□

Besides, we give two Refinements of Huygens inequality for hyperbolic Lucas functions.

Theorem 3.4. If $x \neq 0$, then the following inequality is satisfied:

$$2\frac{sLh(x)}{x} + \frac{tLh(x)}{x} > 3\left(\frac{sLh(x)}{x}\right)^{\frac{1}{3}}\left(\frac{tLh(x)}{x}\right)^{\frac{2}{3}} + \frac{sLh(x) - tLh(x)}{x} > 3(4)^{\frac{1}{3}}\ln(\alpha)$$

Proof. From the properties of hyperbolic Lucas functions, obviously that it is sufficient to prove the theorem for $x > 0$. By using Lemmas 2.4, 2.8 we get

$$\begin{aligned}\frac{2}{3}\frac{sLh(x)}{x} + \frac{1}{3}\frac{tLh(x)}{x} &> \left(\frac{sLh(x)}{x}\right)^{\frac{1}{3}}\left(\frac{tLh(x)}{x}\right)^{\frac{2}{3}} + \frac{1}{3}\left(\frac{sLh(x) - tLh(x)}{x}\right) > \\ &> \left(\frac{sLh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tLh(x)}{x}\right)^{\frac{1}{3}} = \frac{sLh(x)}{x}\frac{1}{\sqrt[3]{cLh(x)}} > (4)^{\frac{1}{3}}\ln(\alpha)\end{aligned}$$

□

Theorem 3.5. If $x \neq 0$, then the following inequality is satisfied:

$$2\frac{sLh(x)}{x} + \frac{tLh(x)}{x} > 3\left[\left(\frac{sLh(x)}{x}\right)^{\frac{1}{6}}\left(\frac{tLh(x)}{x}\right)^{\frac{5}{6}} + \frac{sLh(x) - tLh(x)}{2x}\right] > 3(4)^{\frac{1}{3}}\ln(\alpha)$$

Proof. From the properties of hyperbolic Lucas functions, clearly that it is sufficient to prove the theorem for $x > 0$. By Lemmas 2.4, 2.9 we get

$$\begin{aligned}\frac{2}{3}\frac{sLh(x)}{x} + \frac{1}{3}\frac{tLh(x)}{x} &> \left(\frac{sLh(x)}{x}\right)^{\frac{1}{6}}\left(\frac{tLh(x)}{x}\right)^{\frac{5}{6}} + \frac{sLh(x) - tLh(x)}{2x} > \\ &> \left(\frac{sLh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tLh(x)}{x}\right)^{\frac{1}{3}} = \frac{sLh(x)}{x}\frac{1}{\sqrt[3]{cLh(x)}} > (4)^{\frac{1}{3}}\ln(\alpha)\end{aligned}$$

□

Next, we give Wilker's inequality for hyperbolic Lucas functions.

Theorem 3.6. If $x \neq 0$ then the following inequality is satisfied:

$$\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x} > 4(\ln\alpha)^{\frac{3}{2}} \tag{17}$$

PROOF. From the properties of hyperbolic Lucas functions, clearly that it is sufficient to prove the theorem for $x > 0$. By Lemmas 2.4, 2.5 we get

$$\frac{1}{2}(\frac{sLh(x)}{x})^2 + \frac{1}{2}\frac{tLh(x)}{x} > \frac{sLh(x)}{x}\sqrt{\frac{tLh(x)}{x}} = \sqrt{\left(\frac{sLh(x)}{x}\right)^3 . \frac{1}{cLh(x)}} > 2(\ln\alpha)^{\frac{3}{2}}$$

□

Finally, we give Wilker's-Anglesio inequality for hyperbolic Lucas functions.

Theorem 3.7. If $x \neq 0$ then the following inequality is satisfied:

$$\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x} > 2\ln\alpha + \frac{8}{45}(\ln\alpha)^4 x^3 tLh(x) \tag{18}$$

PROOF. From the properties of hyperbolic Lucas functions, clearly that it is sufficient to prove the theorem for $x > 0$. Let $B : R^+ \to R$ be a function defined by

$$B(x) = \frac{\frac{1}{4(\ln\alpha)^2}\left(\frac{sLh(x)}{x}\right)^2 + \frac{1}{\ln\alpha}\frac{tLh(x)}{x} - 2}{x^3 tLh(x)}$$

Bahşi [8] proved that this function is increasing on $(0, \infty)$ and

$$\lim_{x\to 0+} B(x) = \frac{8(\ln\alpha)^3}{45} \tag{19}$$

This means that

$$\frac{1}{4(\ln\alpha)^2}\left(\frac{sLh(x)}{x}\right)^2 + \frac{1}{\ln\alpha}\frac{tLh(x)}{x} > 2 + \frac{8(\ln\alpha)^3}{45}x^3 tLh(x)$$

It is obvious

$$\frac{1}{\ln\alpha}\left[\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x}\right] > \frac{1}{4(\ln\alpha)^2}\left(\frac{sLh(x)}{x}\right)^2 + \frac{1}{\ln\alpha}\frac{tLh(x)}{x} > 2 + \frac{8(\ln\alpha)^3}{45}x^3 tLh(x)$$

Hence

$$\left(\frac{sLh(x)}{x}\right)^2 + \frac{tLh(x)}{x} > 2\ln\alpha + \frac{8}{45}(\ln\alpha)^4 x^3 tLh(x) \tag{20}$$

□

Corollary 3.8. If $x \neq 0$, then the following inequalities are satisfied:

$$\frac{2}{x}sLh(x) + \frac{1}{x}tLh(x) > \frac{sLh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cLh^2(x)}{2cLh^3(x)}}\right) \tag{21}$$

$$\frac{sLh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cLh^2(x)}{2cLh^3(x)}}\right) > \frac{sLh(x)}{x}\left(1 + \frac{2}{\sqrt{cLh(x)}}\right) \tag{22}$$

$$\frac{sLh(x)}{x}\left(1 + \frac{2}{\sqrt{cLh(x)}}\right) > 3\sqrt[3]{4}\ln(\alpha) \tag{23}$$

PROOF. By Lemma 2.7 we obtain:

$$2sLh(x) + tLh(x) > sLh(x) + 2\sqrt[4]{\frac{sLh(x)tLh(x)}{2}\left(sLh^2(x) + tLh^2(x)\right)}$$

$$= sLh(x)\left(1 + 2\sqrt[4]{\frac{1 + cLh^2(x)}{2cLh^3(x)}}\right)$$

Hence, (21) is proved. By Lemma 2.5 we obtain:

$$1 + cLh^2(x) \geq 2cLh(x) \tag{24}$$

$$1 + \frac{2}{\sqrt{cLh(x)}} \geq \frac{3}{\sqrt[3]{cLh(x)}} \tag{25}$$

and by inequality (24), we get the inequality (22). Also by using Lemma 2.4 and inequality (25) we obtain the inequality (23). □

Now we calculate the limit using Theorem 3.2 and Lemma 2.4 without using L'Hôpital's rule.

Corollary 3.9.

$$\lim_{x \to 0} \frac{sLh(x)}{x} = 2\ln(\alpha) \tag{26}$$

PROOF. By Lemma 2.4 and Theorem 3.2, we obtain

$$\sqrt[3]{4cLh(x)}ln(\alpha) < \frac{sLh(x)}{x} < \left(\frac{cLh(x)}{2} + 1\right)\ln(\alpha)$$

Take $f(x) = \sqrt[3]{4cLh(x)}\ln(\alpha)$; $g(x) = \dfrac{sLh(x)}{x}$; $h(x) = \left(\dfrac{cLh(x)}{2} + 1\right)\ln(\alpha)$

Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt[3]{4cLh(x)}\ln(\alpha) = 2\ln(\alpha)$$

and

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \left(\frac{cLh(x)}{2} + 1\right)\ln(\alpha) = 2\ln(\alpha)$$

By Sandwich theorem,

$$\lim_{x \to 0} \frac{sLh(x)}{x} = 2\ln(\alpha)$$

□

Corollary 3.10. If $x \geq 0$, then the following inequality is satisfied:

$$2\sinh(x)(\ln\alpha)^2 + tLh(x)\ln\alpha \geq sLh(x) + \tanh(x) \tag{27}$$

PROOF. Let f be a function defined by

$$f(x) = 2\sinh(x)(\ln\alpha)^2 + tLh(x)\ln\alpha - sLh(x) - \tanh(x)$$

The derivative of $f(x)$ is

$$f^{'}(x) = \ln\alpha\left(2\cosh(x)\ln\alpha - clh(x)\right) + \left(\frac{4(\ln\alpha)^2}{(clh(x))^2} - \frac{1}{(\cosh(x))^2}\right) \geq 0$$

According to Theorem 2.10 $f(x)$ is an increasing function on $[0, \infty)$, this means that

$$f(x) \geq f(0) = 0$$

□

4. Conclusion

In this study, analogues of some important inequalities related to hyperbolic and trigonometric functions are obtained for hyperbolic Lucas functions. In addition, some modifications of Young's inequality have been proved and new results have been obtained for Lucas functions as a result of these modifications. In the future studies can investigate improvements and generalizations of these inequalities.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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