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ON AUTOMORPHISMS OF LIE ALGEBRA OF SYMMETRIC POLYNOMIALS

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Abstract. Let L_n be the free Lie algebra of rank n over a field K of characteristic zero, $L_{n,c} = L_n/(L_n''+\gamma_{c+1}(L_n))$ be the free metabelian nilpotent of class c Lie algebra, and $F_n = L_n/L_n''$ be the free metabelian Lie algebra generated by $x_1,\ldots,x_n$ over a field K of characteristic zero. We call a polynomial $p(X_n)$ in these Lie algebras *symmetric* if $p(x_1,\ldots,x_n) = p(x_{\pi(1)},\ldots,x_{\pi(n)})$ for each element of the symmetric group S_n. The sets $L_n^{S_n}$, $F_n^{S_n}$, and $L_{n,c}^{S_n}$ of symmetric polynomials coincides with the algebras of invariants of the group S_n in L_n, F_n, and $L_{n,c}$, respectively. We determine the groups $\mathrm{Inn}(L_{n,c}^{S_n}) \cap \mathrm{Inn}(L_{n,c})$ and $\mathrm{Inn}(F_n^{S_n})\cap\mathrm{Inn}(F_n)$ of inner automorphisms of the algebras $L_{n,c}^{S_n}$ and $F_n^{S_n}$ in the groups $\mathrm{Inn}(L_{n,c})$ and $\mathrm{Inn}(F_n)$, respectively. In particular, we obtain the descriptions of the groups $\mathrm{Aut}(L_2^{S_2})\cap\mathrm{Aut}(L_2)$ and $\mathrm{Aut}(F_2^{S_2})\cap\mathrm{Aut}(F_2)$ of automorphisms of the algebras $L_2^{S_2}$ and $F_2^{S_2}$ in the groups $\mathrm{Aut}(L_2)$ and $\mathrm{Aut}(F_2)$, respectively.

1. Introduction

Let A_n be the free algebra of rank n over a field K of characteristic zero in a variety of algebras generated by $X_n = \{x_1,\ldots,x_n\}$. A polynomial $p(X_n) \in A_n$ is said to be symmetric if $p(x_1,\ldots,x_n) = p(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all $\pi \in S_n$. The set of such polynomials is equal to the algebra $A_n^{S_n}$ of invariants of the symmetric group S_n. The algebra $A_n^{S_n}$ is well known, when $A_n = K[X_n]$ is the commutative associative unitary algebra by the fundamental theorem on symmetric polynomials: $K[X_n]^{S_n} = K[\sigma_1,\ldots,\sigma_n]$, $\sigma_i = x_1^i + \cdots + x_n^i$. For the case $A_n = K\langle X_n\rangle$, the associative algebra of rank n, see e.g. [6].

Now let $A_n = F_n$ be the free metabelian Lie algebra of rank n over K. It is well known, see e.g. [3], that the algebra $F_n^{S_n}$ of symmetric polynomials is not finitely generated. Recently the authors [5] have provided an infinite set of generators for $F_2^{S_2}$, later the result was generalized in [4].

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One may consider the group $\mathrm{Aut}(F_n^{S_n})$ of automorphisms preserving the algebra $F_n^{S_n}$. The group $\mathrm{Aut}(F_n)$ is a semidirect product of the general linear group $\mathrm{GL}_n(K)$ and the group $\mathrm{IAut}(F_n)$ of automorphisms which are equivalent to the identity map modulo the commutator ideal F_n'. Hence it is natural to work in $\mathrm{IAut}(F_n)$ approaching the group $\mathrm{Aut}(F_n^{S_n}) \cap \mathrm{IAut}(F_n)$. Additionally, the cases $\mathrm{Aut}(F_n^{S_n}) \cap \mathrm{Aut}(F_n)$ and $\mathrm{Aut}(L_{n,c}^{S_n}) \cap \mathrm{Aut}(L_{n,c})$ are of the interest in the present study, where L_n and $L_{n,c}$ are the free Lie algebra and the free metabelian Lie algebra of nilpotency class c, respectively.

In this study, we determine the inner automorphisms of $F_n^{S_n}$ and of $L_{n,c}^{S_n}$, which are inner automorphisms of F_n and of $L_{n,c}$. Later we describe the groups $\mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$, $\mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$ and $\mathrm{Aut}(L_{2,c}^{S_2}) \cap \mathrm{Aut}(L_{2,c})$.

2. Preliminaries

Let L_n be the free Lie algebra of rank $n \geq 2$ generated by $X_n = \{x_1, \ldots, x_n\}$ over a field K of characteristic zero. We denote by $F_n = L_n/L_n''$ the free metabelian Lie algebra, and $L_{n,c} = L_n/(L_n'' + \gamma^{c+1}(L_n))$ the free metabelian nilpotent Lie algebra of nilpotency class c, where $\gamma^1(L_n) = [L_n, L_n] = L_n'$ is the commutator ideal of L_n, $\gamma^k(L_n) = [\gamma^{k-1}(L_n), L_n]$, $k \geq 2$, and $L_n'' = [L_n', L_n']$. We assume that the algebras F_n and $L_{n,c}$ of rank n are generated by the same set X_n.

The commutator ideal F_n' of the free metabelian Lie algebra F_n is of a natural $K[X_n]$-module structure as a consequence of the metabelian identity

$$[[z_1, z_2], [z_3, z_4]] = 0, \quad z_1, z_2, z_3, z_4 \in F_n,$$

with action:

$$f(X_n)g(x_1, \ldots, x_n) = f(X_n)g(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_n), \quad f(X_n) \in F_n', \quad g(X_n) \in K[X_n],$$

where $K[X_n]$ is the (commutative, associative, unitary) polynomial algebra, and the adjoint action is defined as $z_1 \mathrm{ad} z_2 = [z_1, z_2]$, for $z_1, z_2 \in F_n$. One may define a similar action on the free metabelian nilpotent Lie algebra $L_{n,c}$. It is well known by Bahturin [1] that the monomials $[x_{k_1}, x_{k_2}]x_{k_3} \cdots x_{k_l}$, $k_1 > k_2 \leq k_3 \leq k_l$, forms a basis for F_n', which is a basis for $L_{n,c}'$ when $l \leq c - 2$.

An element $s(X_n)$ in L_n, F_n, or $L_{n,c}$ is called symmetric if

$$s(x_1, \ldots, x_n) = s(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \pi s(x_1, \ldots, x_n)$$

for each permutation π in the symmetric group S_n. The sets $L_n^{S_n}$, $F_n^{S_n}$, and $L_{n,c}^{S_n}$ of symmetric polynomials coincide with the algebras of invariants of the group S_n. See the work [5] for $F_2^{S_2}$, and its generalization [4] for the full description of the algebra $F_n^{S_n}$. The decription of the algebra $L_{n,c}^{S_n}$ is a direct consequence of the known results on the algebra $F_n^{S_n}$.

Let $A_n = L_n, F_n, L_{n,c}$. It is well known that the automorphism group $\mathrm{Aut}(A_n)$ is a semidirect product of the general linear group $\mathrm{GL}_n(K)$ and the group $\mathrm{IAut}(A_n)$ of automorphisms which are equivalent to the identity map modulo the commutator ideal A_n'. Hence it is natural to work in $\mathrm{IAut}(A_n)$ when determining the whole group $\mathrm{Aut}(A_n)$. Now consider the group $\mathrm{Aut}(A_n^{S_n})$ of automorphisms consisting of automorphisms of A_n preserving each symmetric polynomial in the algebra $A_n^{S_n}$.

In the next section, approaching the group $\mathrm{IAut}(A_n^{S_n})$, we describe the inner automorphism group $\mathrm{Inn}(A_n^{S_n}) \cap \mathrm{Inn}(A_n)$ in the group $\mathrm{Inn}(A_n)$ for $A_n = F_n$, and $A_n = L_{n,c}$. The case $\mathrm{Inn}(L_n^{S_n}) \cap \mathrm{Inn}(L_n)$ is skipped since the free Lie algebra

L_n does not have nonidentical inner automorphisms. Later, we obtain the groups $\mathrm{Aut}(L_2^{S_2})$, $\mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$, and $\mathrm{Aut}(L_{2,c}^{S_2}) \cap \mathrm{Aut}(L_{2,c})$ as a consequence of the results obtained.

3. Main Results

3.1. **Inner automorphisms of $F_n^{S_n}$.** Let $u \in F_n'$ be an element from the commutator ideal of the free metabelian Lie algebra F_n. Then the adjoint operator $\mathrm{ad}u : v \to [v,u]$ is a nilpotent derivation of F_n, and $\psi_u = \exp(\mathrm{ad}u) = 1 + \mathrm{ad}u$ is an automorphism of the Lie algebra F_n. The inner automorphism group $\mathrm{Inn}(F_n)$ of F_n consisting of such automorphisms is abelian: $\psi_{u_1}\psi_{u_2} = \psi_{u_1+u_2}$, $\psi_u^{-1} = \psi_{-u}$.

In the following theorem we determine the group $\mathrm{Inn}(F_n^{S_n}) \cap \mathrm{Inn}(F_n)$ of inner automorphisms preserving the algebra $F_n^{S_n}$.

Theorem 3.1. *The automorphism $\psi_u \in \mathrm{Inn}(F_n^{S_n}) \cap \mathrm{Inn}(F_n)$ if and only if $u \in (F_n')^{S_n}$.*

Proof. If $u \in (F_n')^{S_n}$, then clearly $\psi_u(v) = v + [v,u] \in F_n^{S_n}$ for every $v \in F_n^{S_n}$. Conversely let $v \in F_n^{S_n}$ be a symmetric polynomial, and $u \in F_n'$ be an arbitrary element. We may assume that the linear (symmetric) summand v_l of v is nonzero, since ψ_u acts identically on the commutator ideal F_n' of the free metabelian Lie algebra F_n. Then $\psi_u(v) \in F_n^{S_n}$ implies that $[v,u] = [v_l,u] \in F_n^{S_n}$. For each $\pi \in S_n$, we have that

$$[v_l,u] = \pi[v_l,u] = [\pi v_l, \pi u] = [v_l, \pi u],$$

and $[v_l, u - \pi u] = 0$, which gives that $u - \pi u = 0$ or $u = \pi u$. □

3.2. **Inner automorphisms of $L_{n,c}^{S_n}$.** Let $u \in L_{n,c}$ be an element from the free metabelian Lie algebra $L_{n,c}$. Then the adjoint operator $\mathrm{ad}u(v) = [v,u]$ is a nilpotent derivation of $L_{n,c}$, since $\mathrm{ad}^c u = 0$ and

$$\varepsilon_u = \exp(\mathrm{ad}u) = 1 + \mathrm{ad}u + \frac{1}{2}\mathrm{ad}^2 u + \cdots + \frac{1}{(c-1)!}\mathrm{ad}^{c-1}u$$

is an inner automorphism of $L_{n,c}$. The set $\mathrm{Inn}(L_{n,c}) = \{\varepsilon_u \mid u \in L_{n,c}\}$ is the inner automorphism group of $L_{n,c}$. In this subsection, we investigate the group $\mathrm{Inn}(L_{n,c}^{S_n}) \cap \mathrm{Inn}(L_{n,c})$ of inner automorphisms of the algebra $L_{n,c}^{S_n}$.

Lemma 3.2. *Let $u = \sum_{i=1}^n \alpha_i x_i$ for some $\alpha_i \in K$, and $v = \sum_{i=1}^n x_i \in L_{n,c}^{S_n}$ such that $[u,v] \in L_{n,c}^{S_n}$. Then $u = \alpha v$ for some $\alpha \in K$.*

Proof. Let $\pi = (1k) \in S_n$ be a fixed transposition for $k = 2, \ldots, n$. Then

$$\pi u = \alpha_1 x_{\pi(1)} + \cdots + \alpha_n x_{\pi(n)} = \alpha_1 x_k + \alpha_k x_1 + \sum_{i \neq 1,k} \alpha_i x_i,$$

and $u - \pi u = \alpha_1(x_1 - x_k) + \alpha_k(x_k - x_1) = \alpha_{1k}(x_1 - x_k)$, where $\alpha_{1k} = \alpha_1 - \alpha_k$. Now $[u,v] \in L_{n,c}^{S_n}$ gives that $[u,v] = \pi[u,v] = [\pi u, \pi v] = [\pi u, v]$, and hence

$$\begin{aligned} 0 = [u - \pi u, v] &= [\alpha_{1k}(x_1 - x_k), x_1 + \cdots + x_n] \\ &= \alpha_{1k}\left(2[x_1,x_k] + \sum_{i \neq 1,k}[x_1,x_i] + \sum_{i \neq 1,k}[x_i,x_k]\right) \end{aligned}$$

where the elements in the paranthesis are basis elements of $L_{n,c}$, which implies that $\alpha_{1k} = 0$, $k \geq 2$. This completes the proof by the choice $\alpha = \alpha_1 = \cdots = \alpha_n$. □

Theorem 3.3. *The automorphism* $1 \neq \psi_u \in \mathrm{Inn}(L_{n,c}^{S_n}) \cap \mathrm{Inn}(L_{n,c})$ *if and only if* $u \in L_{n,c}^{S_n}$.

Proof. If $u, v \in L_{n,c}^{S_n}$ then it is straighforward to see that

$$\varepsilon_u(v) = v + [v,u] + \cdots + (1/(c-1)!)[[\cdots[v,u],\ldots],u] \in L_{n,c}^{S_n}.$$

Conversely, let $u = u_l + u_0 \in L_{n,c}$ be an arbitrary element and $v = v_l + v_0 \in L_{n,c}^{S_n}$ be a symmetric polynomial such that $\varepsilon_u(v) \in L_{n,c}^{S_n}$, where u_l and v_l are the linear components of u and v, respectively. In the expression of $\varepsilon_u(v)$, the component of degree 2 is $[v_l, u_l]$ which is symmetric by the natural grading on the Lie algebra $L_{n,c}^{S_n}$. Hence $u_l = \alpha v_l$ for some $\alpha \in K$ by Lemma 3.2, and $[v_l, u_l] = 0$. Note that $[v_0, u_0] = 0$ by metabelian identity. The computations

$$\begin{aligned}\varepsilon_u(v) =& v + [v_l + v_0, u_l + u_0] \sum_{k=0}^{c-2} \frac{u_l^k}{(k+1)!} \\ =& v + [v_0, v_l] \sum_{k=0}^{c-3} \frac{\alpha^{k+1} v_l^k}{(k+1)!} + [v_l, u_0] \sum_{k=0}^{c-3} \frac{\alpha^k v_l^k}{(k+1)!}\end{aligned}$$

give that $[v_l, u_0] \sum_{k=0}^{c-3} \frac{\alpha^k v_l^k}{(k+1)!} \in L_{n,c}^{S_n}$, and that $u_0 \in L_{n,c}^{S_n}$ by Theorem 3.1. □

3.3. **On automorphisms of $L_2^{S_2}$, $F_2^{S_2}$, and $L_{2,c}^{S_2}$.** In the sequel, we fix the notation $x_1 = x$, $x_2 = y$, for the sake of simplicity. It is well known by [2] that each automorphism of L_2 is linear. The next theorem determines the automorphism group $\mathrm{Aut}(L_2^{S_2})$.

Theorem 3.4. *Let* $\xi \in \mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$. *Then* ξ *and its inverse* ξ^{-1} *are of the form*

$$\xi(x) = ax + by, \quad \xi(y) = bx + ay,$$
$$\xi^{-1}(x) = c^{-1}ax - c^{-1}by, \quad \xi^{-1}(y) = -c^{-1}bx + c^{-1}ay,$$

such that $c = a^2 - b^2 \neq 0$, $a, b \in K$.

Proof. Let ξ be of the form $\xi : x \to ax + by,\ y \to cx + dy$ such that $ad \neq bc$, where $a, b, c, d \in K$. Since $x + y \in L_2^{S_2}$, then $\xi(x + y) = (a + c)x + (b + d)y \in L_2^{S_2}$, which is contained in $K\{x + y\}$. Hence $a + c = b + d$. On the other hand $[[x, y], x] - [[x, y], y] \in L_2^{S_2}$, and

$$\xi([[x,y],x] - [[x,y],y]) = \beta\,((a-c)[[x,y],x] + (b-d)[[x,y],y])$$

where $\beta = ad - bc \neq 0$. The fact that $\xi \in \mathrm{Aut}(L_2^{S_2}) \cap \mathrm{Aut}(L_2)$ gives $a - c = -b + d$. Consequently, $a = d$ and $b = c$. Conversely, it is straightforward to show that the automorphism stated in the theorem preserves symmetric polynomials. □

It is well known, see the work by Drensky [3], that each automorphism of F_2 is a product of a linear automorphism and an inner automorphism of F_2. Using this fact, we obtain the following result as a consequence of Theorem 3.4 and Theorem 3.1.

Theorem 3.5. *Let* $\varphi \in \mathrm{Aut}(F_2^{S_2}) \cap \mathrm{Aut}(F_2)$. *Then* φ *is a product of a linear automorphism* $\xi : x \to ax + by,\ y \to bx + ay$, *and an inner automorphism* ψ_u *where* $u \in (F_2')^{S_2}$.

Theorem 3.6. *Let* $\varphi \in \mathrm{Aut}(L_{2,c}^{S_2}) \cap \mathrm{Aut}(L_{2,c})$. *Then* φ *is a product of a linear automorphism* $\xi : x \to ax+by,\ y \to bx+ay$, *and an automorphism* $\phi \in \mathrm{IAut}(L_{2,c}^{S_2})$ *of the form*

$$\begin{aligned}\phi : & x \to x + [x,y]f(x,y) \\ & y \to y - [x,y]f(y,x).\end{aligned}$$

Proof. It is sufficient to show that an automorphism $\phi \in \mathrm{IAut}(L_{2,c})$ preserving symmetric polynomials satisfies the condition of the theorem. In general ϕ is of the form

$$\begin{aligned}\phi : & x \to x + [x,y]f(x,y) \\ & y \to y + [x,y]g(x,y).\end{aligned}$$

Since $x + y \in L_{2,c}^{S_2}$, then $\phi(x+y) = x + y + [x,y](f(x,y) + g(x,y))$ is symmetric, and hence

$$x + y + [x,y](f(x,y) + g(x,y)) = y + x - [x,y](f(y,x) + g(y,x)).$$

This gives that

$$f(x,y) + g(x,y) + f(y,x) + g(y,x) = 0 \tag{3.1}$$

in the commutator ideal $L'_{2,c}$ of $L_{2,c}$, which is a $K[x,y]$-module freely generated by $[x,y]$. Now by the symmetric polynomial $[x,y](x-y)$, we have that

$$[x + [x,y]f(x,y), y + [x,y]g(x,y)](x-y) = [x,y](x-y)(1 + f(x,y)y - xg(x,y))$$

is symmetric. Consequently $[x,y](x-y)(yf(x,y) - xg(x,y))$ is symmetric. The following computations complete the proof.

$$[x,y](x-y)(yf(x,y) - xg(x,y)) = [x,y](x-y)(xf(y,x) - yg(y,x)),$$

and thus using Equation 3.1 we have

$$\begin{aligned}0 &= yf(x,y) - xg(x,y) - xf(y,x) + yg(y,x) \\ &= yf(x,y) + xg(y,x) + xf(x,y) + yg(y,x) \\ &= (x+y)(f(x,y) + g(y,x)).\end{aligned}$$

□

4. Conclusion

In this study, inner automorphisms of algebras of symmetric polynomials of (relatively) free Lie algebras in the group of automorphisms of (relatively) free Lie algebras were determined. The next step might be sharpening the result by finding all inner automorphisms of those algebras. For this purpose, one needs to have generators for the algebras under consideration, and handle the automorphisms by their action on those generators.

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References

[1] Yu.A. Bahturin, Identical Relations in Lie Algebras (Russian), Nauka, Moscow, (1985). Translation: VNU Science Press, Utrecht, (1987).
[2] P.M. Cohn, Subalgebras of free associative algebras, Proc. London Math. Soc. Vol.14, No.3, pp. 618-632 (1964).
[3] V. Drensky, Fixed algebras of residually nilpotent Lie algebras, Proc. Amer. Math. Soc., Vol.120, No.4, pp. 1021-1028 (1994).
[4] V. Drensky, Ş. Fındık, N. Ş. Öğüşlü, Symmetric polynomials in the free metabelian Lie algebras, Mediterranean Journal of Mathematics, Vol.17, No.5, pp. 1-11 (2020).
[5] Ş. Fındık, N. Ş. Öğüşlü, Palindromes in the free metabelian Lie algebras, Internat. J. Algebra Comput., Vol.29, No.5, pp. 885-891 (2019).
[6] M.C. Wolf, Symmetric functions of non-commutative elements, Duke Math. J., Vol.2, No.4, pp. 626-637 (1936).

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