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# The Form of Solutions and Periodic Nature for Some System of Difference Equations

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**Abstract**

In this paper, we study the form of the solution of the following systems of difference equations of order two

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}},\quad s_{n+1}=\frac{s_n w_{n-1}}{\pm s_n \pm w_{n-1}},$$

with nonzero real numbers initial conditions.

## 1. Introduction

Difference equations naturally occur as discrete analogs and numerical solutions to differential and delay differential equations that have applications in biology, ecology, economy, physics, and other fields. Thus, there has recently been an increase in interest in the study of qualitative analysis of systems of difference equations and rational difference equations. Although the form of difference equations is quite straightforward, it is extremely challenging to fully comprehend the behaviors of their solutions, see [1]-[7].

The periodicity of the solutions of the system of difference equations

$$w_{n+1}=\frac{m}{s_n},\quad s_{n+1}=\frac{p s_n}{w_{n-1}s_{n-1}},$$

was studied by Cinar in [8].
El-Dessoky and Elsayed [9] have analyzed the form of the solutions and the periodicity character of the following systems :

$$w_{n+1}=\frac{w_n s_{n-1}}{s_{n-1}\pm s_n},\quad s_{n+1}=\frac{s_n w_{n-1}}{w_{n-1}\pm w_n}.$$

Kurbanli et al. [10] discussed the periodicity of solutions of the system of difference equations

$$w_{n+1}=\frac{w_{n-1}+s_n}{w_{n-1}s_n-1},\quad s_{n+1}=\frac{s_{n-1}+w_n}{s_{n-1}w_n-1}.$$

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El-Dessoky [11] investigated the form of the solutions and the periodicity character of the following systems :

$$w_{n+1} = \frac{s_{n-1}s_{n-2}}{w_n(\pm 1 \pm s_{n-1}s_{n-2})}, \quad s_{n+1} = \frac{w_{n-1}w_{n-2}}{s_n(\pm 1 \pm w_{n-1}w_{n-2})}.$$

Touafek and Elsayed [12] have investigated the periodicity and determined the shape of the solutions of the systems of difference equations of order two:

$$w_{n+1} = \frac{s_n w_{n-1}}{\pm w_{n-1} \pm s_n}, \quad s_{n+1} = \frac{w_n s_{n-1}}{\pm w_n \pm w_{n-1}}.$$

Yalcınkaya [13] has found the sufficient conditions for the global asymptotic stability of the system of difference equations

$$w_{n+1} = \frac{w_n + s_{n-1}}{w_n s_{n-1} - 1}, \quad s_{n+1} = \frac{s_n + w_{n-1}}{s_n w_{n-1} - 1}.$$

Elsayed et al. [14] foud the form of the solutions of the systems of difference equations

$$w_{n+1} = \frac{s_n(w_{n-3} + s_{n-4})}{s_{n-4} + w_{n-3} - s_n}, \quad s_{n+1} = \frac{w_{n-2}(w_{n-2} + s_{n-3})}{2w_{n-2} + s_{n-3}}.$$

$$w_{n+1} = \frac{(s_{n-4} - w_{n-3})s_n}{s_{n-4} - w_{n-3} + s_n}, \quad s_{n+1} = \frac{(s_{n-3} - w_{n-2})w_{n-2}}{s_{n-3}}.$$

Yang et al. [15] has studied the behavior of the solutions of the systems

$$w_n = \frac{a}{s_{n-p}}, \quad s_n = \frac{b s_{n-p}}{w_{n-q} s_{n-q}}.$$

Touafek et al. [16] examined periodicity and provided the form of the solutions of the systems of nonlinear difference equations

$$w_{n+1} = \frac{w_{n-3}}{\pm 1 \pm w_{n-3}s_{n-1}}, \quad s_{n+1} = \frac{s_{n-3}}{\pm 1 \pm s_{n-3}w_{n-1}}.$$

Turki et al. [17] studied the dynamics of the twelfth-order difference equations

$$w_{n+1} = a w_{n-5} - \frac{b w_{n-5}}{c w_{n-5} - d w_{n-11}}.$$

Similarly, difference equations and nonlinear systems of the rational difference equations were investigated see [18]-[25]. This paper's main goal is to consider the systems of difference equations below

$$w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}, \quad s_{n+1} = \frac{s_n w_{n-1}}{\pm s_n \pm w_{n-1}},$$

where the initial conditions are $w_0$, $w_{-1}$, $s_0$ and $s_{-1}$ arbitrary positive real numbers.

## 2. The system $w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}$, $s_{n+1} = \frac{s_n w_{n-1}}{s_n + w_{n-1}}$

In this section, we examine the solutions of the system of the difference equations in the form :

$$w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}, \quad s_{n+1} = \frac{s_n w_{n-1}}{s_n + w_{n-1}}. \tag{2.1}$$

**Theorem 2.1.** *Assume that $\{w_n, s_n\}$ is a solution of system (2.1). Then for $n = 1, 2, \ldots$, we have*

$$w_{2n} = \frac{abcd}{cd((\xi_n - \eta_{n-1})a + \eta_n b) + ab((\eta_n - \xi_{n-1})c + \xi_n d)},$$

$$w_{2n+1} = \frac{abcd}{cd(\xi_n a + (\eta_{n+1} - \xi_n)b) + ab(\eta_n c + (\xi_{n+1} - \eta_n)d)},$$

$$s_{2n}=\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_nb)+ab((\xi_n-\eta_{n-1})c+\eta_nd)},$$

$$s_{2n+1}=\frac{abcd}{cd(\eta_na+(\xi_{n+1}-\eta_n)b)+ab(\xi_nc+(\eta_{n+1}-\xi_n)d)},$$

*where* $\{\eta_n\}_{n=1}^{\infty}=\{1,2,7,17,44,...\}$, $\eta_0=1$, $\eta_{-1}=0,\eta_{-2}=-1$, $\{\xi_n\}_{n=1}^{\infty}=\{1,3,6,17,...\}$, $\xi_0=0$, $\xi_n=\eta_{n+1}-\eta_n-\eta_{n-1}-\eta_{n-2}-\eta_{n-3}$, $w_0=a$, $w_{-1}=b$, $s_0=c$ *and* $s_{-1}=d$.

*Proof.* For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. That is

$$w_{2n-2}=\frac{abcd}{cd((\xi_{n-1}-\eta_{n-2})a+\eta_{n-1}b)+ab((\eta_{n-1}-\xi_{n-2})c+\xi_{n-1}d)},$$

$$w_{2n-1}=\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)},$$

$$s_{2n-2}=\frac{abcd}{cd((\eta_{n-1}-\xi_{n-2})a+\xi_{n-1}b)+ab((\xi_{n-1}-\eta_{n-2})c+\eta_{n-1}d)},$$

$$s_{2n-1}=\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}.$$

Now, it follows from Eq.(2.1) that

$$w_{2n}=\frac{w_{2n-1}s_{2n-2}}{w_{2n-1}+s_{2n-2}}$$

$$=\frac{\left(\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}\right)\left(\frac{abcd}{cd((\eta_{n-1}-\xi_{n-2})a+\xi_{n-1}b)+ab((\xi_{n-1}-\eta_{n-2})c+\eta_{n-1}d)}\right)}{\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}+\frac{abcd}{cd((\eta_{n-1}-\xi_{n-2})a+\xi_{n-1}b)+ab((\xi_{n-1}-\eta_{n-2})c+\eta_{n-1}d)}}$$

$$=\frac{abcd}{cd((\xi_{n-1}+\eta_{n-1}-\xi_{n-2})a+(\eta_n-\xi_{n-1}+\xi_{n-1})b)+ab((\eta_{n-1}+\xi_{n-1}-\eta_{n-2})c+(\xi_n-\eta_{n-1}+\eta_{n-1})d)}$$

$$=\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_nb)+ab((\eta_n-\xi_{n-1})c+\xi_nd)}.$$

And

$$s_{2n}=\frac{s_{2n-1}w_{2n-2}}{s_{2n-1}+w_{2n-2}}$$

$$=\frac{\left(\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}\right)\left(\frac{abcd}{cd((\xi_{n-1}-\eta_{n-2})a+\eta_{n-1}b)+ab((\eta_{n-1}-\xi_{n-2})c+\xi_{n-1}d)}\right)}{\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}+\frac{abcd}{cd((\xi_{n-1}-\eta_{n-2})a+\eta_{n-1}b)+ab((\eta_{n-1}-\xi_{n-2})c+\xi_{n-1}d)}}$$

$$=\frac{abcd}{cd((\eta_{n-1}+\xi_{n-1}-\eta_{n-2})a+(\xi_n-\eta_{n-1}+\eta_{n-1})b)+ab((\xi_{n-1}+\eta_{n-1}-\xi_{n-2})c+(\eta_n-\xi_{n-1}+\xi_{n-1})d)}$$

$$=\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_nb)+ab((\xi_n-\eta_{n-1})c+\eta_nd)}.$$

Also,

$$w_{2n+1} = \frac{w_{2n}s_{2n-1}}{w_{2n}+s_{2n-1}}$$

$$= \frac{\left(\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_n b)+ab((\eta_n-\xi_{n-1})c+\xi_n)d)}\right)\left(\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}\right)}{\frac{abcd}{cd((\xi_n-\eta_{n-1})a+\eta_n b)+ab((\eta_n-\xi_{n-1})c+\xi_n d)}+\frac{abcd}{cd(\eta_{n-1}a+(\xi_n-\eta_{n-1})b)+ab(\xi_{n-1}c+(\eta_n-\xi_{n-1})d)}}$$

$$= \frac{abcd}{cd((\xi_n-\eta_{n-1}+\eta_{n-1})a+(\eta_n+\xi_n-\eta_{n-1})b+ab((\eta_n-\xi_{n-1}+\xi_{n-1})c+(\xi_n+\eta_n-\xi_{n-1})d)}$$

$$= \frac{abcd}{cd(\xi_n a+(\eta_{n+1}-\xi_n)b)+ab(\eta_n c+(\xi_{n+1}-\eta_n)d)}.$$

And

$$s_{2n+1} = \frac{s_{2n}w_{2n-1}}{s_{2n}+w_{2n-1}}$$

$$= \frac{\left(\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_n b)+ab((\xi_n-\eta_{n-1})c+\eta_n d)}\right)\left(\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}\right)}{\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_n b)+ab((\xi_n-\eta_{n-1})c+\eta_n d)}+\frac{abcd}{cd(\xi_{n-1}a+(\eta_n-\xi_{n-1})b)+ab(\eta_{n-1}c+(\xi_n-\eta_{n-1})d)}}$$

$$= \frac{abcd}{cd((\eta_n-\xi_{n-1}+\xi_{n-1})a+(\xi_n+\eta_n-\xi_{n-1})b+ab((\xi_n-\eta_{n-1}+\eta_{n-1})c+(\eta_n+\xi_n-\eta_{n-1})d)}$$

$$= \frac{abcd}{cd(\eta_n a+(\xi_{n+1}-\eta_n)b)+ab(\xi_n c+(\eta_{n+1}-\xi_n)d)}.$$

□

**Example 2.2.** *Figure 2.1 demonstrates the behavior of the solutions of the system of difference equations (2.1) with* $w_{-1} = 4,\ w_0 = 1,\ s_{-1} = -2$ *and* $s_0 = 2$.

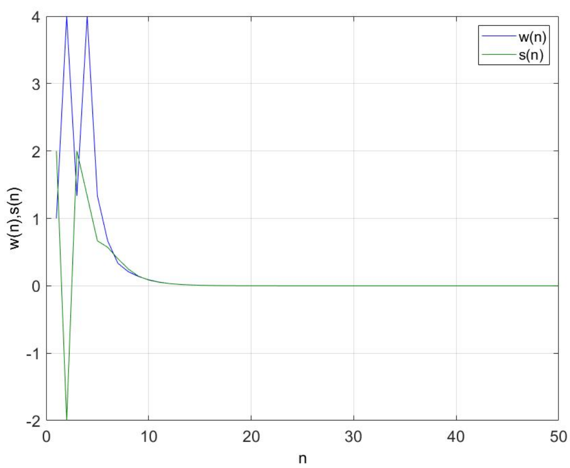


**Figure 2.1**

## 3. The system $w_{n+1}=\frac{w_ns_{n-1}}{w_n+s_{n-1}}$, $s_{n+1}=\frac{s_nw_{n-1}}{s_n-w_{n-1}}$

In this section, we investigate the solutions of the following system of the difference equations :

$$w_{n+1}=\frac{w_ns_{n-1}}{w_n+s_{n-1}},\quad s_{n+1}=\frac{s_nw_{n-1}}{s_n-w_{n-1}}. \tag{3.1}$$

**Theorem 3.1.** *Assume that $\{w_n,s_n\}$ is a solution of system (3.1). Then for $n=1,2,...,$ we have*

$$w_{2n-1}=\frac{abd}{a(f_{n+2}b+f_{n+1}d)+f_{n+2}bd},\quad w_{2n}=\frac{acd}{a(f_{n+2}d+f_{n+2}c)+f_{n+3}cd},$$

$$s_{2n-1}=\frac{bcd}{c(f_{n+2}d+f_{n+1}b)-f_{n+2}bd},\quad s_{2n}=\frac{abc}{a(f_{n+3}b-f_{n+2}c)+f_{n+2}bc},$$

*where $\{f_m\}_{m=0}^{\infty}=\{0,1,0,1,1,2,3,...\}$.*

*Proof.* For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$ and $n-2$. That is

$$w_{2n-3}=\frac{abd}{a(f_{n+1}b+f_nd)+f_{n+1}bd},\quad w_{2n-2}=\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd},$$

$$w_{2n-5}=\frac{abd}{a(f_nb+f_{n-1}d)+f_nbd},\quad w_{2n-4}=\frac{acd}{a(f_nd+f_nc)+f_{n+1}cd},$$

$$s_{2n-3}=\frac{bcd}{c(f_{n+1}d+f_nb)-f_{n+1}bd},\quad s_{2n-2}=\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc},$$

$$s_{2n-5}=\frac{bcd}{c(f_nd+f_{n-1}b)-f_nbd},\quad s_{2n-4}=\frac{abc}{a(f_{n+1}b-f_nc)+f_nbc}.$$

Now, from Eq.(3.1) we get :

$$w_{2n}=\frac{w_{2n-1}s_{2n-2}}{w_{2n-1}+s_{2n-2}}$$

$$=\frac{\left(\frac{abd}{a(f_{n+2}b+f_{n+1}d)+f_{n+2}bd}\right)\left(\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}\right)}{\frac{abd}{a(f_{n+2}b+f_{n+1}d)+f_{n+2}bd}+\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}}$$

$$=\frac{a^2b^2cd}{ab((f_{n+2}b-f_{n+1}c)ad+f_{n+1}bcd+(f_{n+2}b+f_{n+1}d)ac+f_{n+2}bcd)}$$

$$=\frac{abcd}{f_{n+2}bad-f_{n+1}cad+f_{n+1}bcd+f_{n+2}bac+f_{n+1}dac+f_{n+2}bcd}$$

$$=\frac{abcd}{b(f_{n+2}ad+(f_{n+1}+f_{n+2})cd+f_{n+2}ac)}$$

$$=\frac{acd}{(f_{n+2}d+f_{n+2}c)a+f_{n+3}cd}.$$

And

$$s_{2n}=\frac{s_{2n-1}w_{2n-2}}{s_{2n-1}-w_{2n-2}}$$

$$= \frac{\left(\frac{bcd}{c(f_{n+2}d+f_{n+1}b)-f_{n+2}bd}\right)\left(\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}\right)}{\frac{bcd}{c(f_{n+2}d+f_{n+1}b)-f_{n+2}bd}-\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd})}$$

$$= \frac{abc^2d^2}{cd((f_{n+1}d+f_{n+1}c)ad+f_{n+2}bcd-(f_{n+2}d+f_{n+1}b)ac+f_{n+2}abd)}$$

$$= \frac{abcd}{f_{n+1}bad+f_{n+1}cab+f_{n+2}bcd-f_{n+2}dac-f_{n+1}bac+f_{n+2}abd}$$

$$= \frac{abcd}{d(f_{n+2}cb+(f_{n+1}+f_{n+2})ba-f_{n+2}ac)}$$

$$= \frac{acb}{(f_{n+3}b-f_{n+2}c)a+f_{n+2}cb}.$$

Also,

$$w_{2n-1} = \frac{w_{2n-2}s_{2n-3}}{w_{2n-2}+s_{2n-3}}$$

$$= \frac{\left(\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}\right)\left(\frac{bcd}{c(f_{n+1}d+f_nb)-f_{n+1}bd}\right)}{\frac{acd}{a(f_{n+1}d+f_{n+1}c)+f_{n+2}cd}+\frac{bcd}{c(f_{n+1}d+f_nb)-f_{n+1}bd}}$$

$$= \frac{abc^2d^2}{cd((f_{n+1}d+f_nb)ac-f_{n+1}bad+(f_{n+1}d+f_{n+1}c)ab+f_{n+2}bcd)}$$

$$= \frac{abcd}{f_{n+1}adc+f_ncab-f_{n+1}bad+f_{n+1}bad+f_{n+1}bac+f_{n+2}bcd}$$

$$= \frac{abcd}{c(f_{n+1}ad+(f_n+f_{n+1})ba+f_{n+2}bd)}$$

$$= \frac{abd}{(f_{n+2}b+f_{n+1}d)a+f_{n+2}bd}.$$

And

$$s_{2n-1} = \frac{s_{2n-2}w_{2n-3}}{s_{2n-2}-w_{2n-3}}$$

$$= \frac{\left(\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}\right)\left(\frac{abd}{a(f_{n+1}b+f_nd)+f_{n+1}bd}\right)}{\frac{abc}{a(f_{n+2}b-f_{n+1}c)+f_{n+1}bc}-\frac{abd}{a(f_{n+1}b+f_nd)+f_{n+1}bd}}$$

$$= \frac{a^2b^2cd}{ab((f_{n+1}b+f_nd)ac+f_{n+1}bcd-(f_{n+2}b-f_{n+1}c)ad-f_{n+1}cbd)}$$

$$= \frac{abcd}{f_{n+1}bac+f_ncad+f_{n+1}bcd-f_{n+2}dab+f_{n+1}dac-f_{n+1}cbd}$$

$$= \frac{abcd}{a(f_{n+1}cb+(f_n+f_{n+1})dc-f_{n+2}bd)}$$

$$= \frac{bcb}{(f_{n+2}d+f_{n+1}b)c-f_{n+2}bd}.$$

□

**Example 3.2.** *We assume a numerical example for Eq.(3.1) where* $w_{-1}=3,\ w_0=0.5,\ s_{-1}=-4$ *and* $s_0=1$. *See Fig. 3.1.*

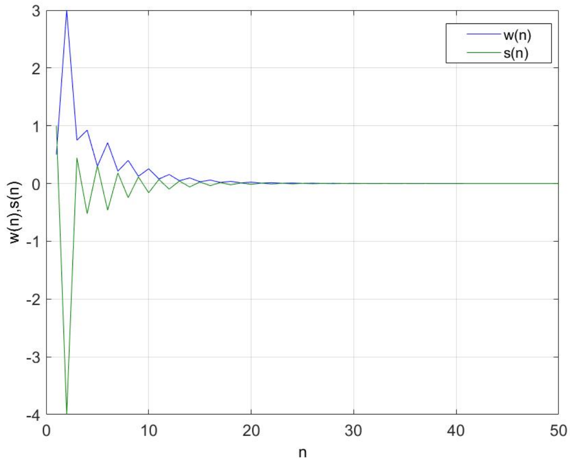


**Figure 3.1**

## 4. The system $w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}},\ s_{n+1}=\frac{s_n w_{n-1}}{-s_n+w_{n-1}}$

In this section, we give a specific form the solutions of the system of the difference equation in the form:

$$w_{n+1}=\frac{w_n s_{n-1}}{w_n+s_{n-1}},\quad s_{n+1}=\frac{s_n w_{n-1}}{-s_n+w_{n-1}}. \tag{4.1}$$

**Theorem 4.1.** *Assume that $\{w_n,s_n\}$ is a solution of system (4.1). Then for $n=1,2,...$, we have*

$$w_{2n}=\frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)},$$

$$w_{2n+1}=\frac{abcd}{cd((\xi_{n+1}-\eta_{n+1})a+\eta_{n+1}b)+ab(\xi_n c+(\xi_{n+1}-\eta_{n+2})d)},$$

$$s_{2n}=\frac{abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)},$$

$$s_{2n+1}=\frac{abcd}{cd(-\xi_n a+(\eta_{n+2}-\xi_{n+1})b)+ab((\xi_{n+1}-\eta_{n+1})c+\eta_{n+1}d)},$$

*where $\{\xi_n\}_{n=1}^{\infty}=\{1,0,-5,-13,-12,...\},\ \xi_0=1,\ \xi_{-1}=0,\ \eta_n=((\xi_{n-2}+\xi_{n-1}+\xi_n)\div 2)$ and $\{\eta_n\}_{n=1}^{\infty}=\{1,1,-2,-9,-15,...\}$.*

*Proof.* For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. That is

$$w_{2n-2}=\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_{n-1}b)+ab(\eta_{n-1}c+(\eta_n-\xi_n)d)},$$

$$w_{2n-1}=\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)},$$

$$s_{2n-2}=\frac{abcd}{cd(-\eta_{n-1}a+(\xi_n-\eta_n)b)+ab((\eta_n-\xi_{n-1})c+\xi_{n-1}d)},$$

$$s_{2n-1}=\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}.$$

Now, it follows from Eq.(4.1) that

$$
\begin{aligned}
w_{2n} &= \frac{w_{2n-1}s_{2n-2}}{w_{2n-1}+s_{2n-2}} \\
&= \frac{\left(\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}\right)\left(\frac{abcd}{cd(-\eta_{n-1}a+(\xi_n-\eta_n)b)+ab((\eta_n-\xi_{n-1})c+\xi_{n-1}d)}\right)}{\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}+\frac{abcd}{cd(-\eta_{n-1}a+(\xi_n-\eta_n)b)+ab((\eta_n-\xi_{n-1})c+\xi_{n-1}d)}} \\
&= \frac{abcd}{cd((\xi_n-\eta_n-\eta_{n-1})a+(\eta_n+\xi_n-\eta_n)b)+ab((\xi_{n-1}+\eta_n-\xi_{n-1})c+(\xi_n-\eta_{n+1}+\xi_{n-1})d)} \\
&= \frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)}.
\end{aligned}
$$

And

$$
\begin{aligned}
s_{2n} &= \frac{s_{2n-1}w_{2n-2}}{-s_{2n-1}+w_{2n-2}} \\
&= \frac{\left(\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}\right)\left(\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_{n-1}b)+ab(\eta_{n-1}c+(\eta_n-\xi_n)d)}\right)}{\frac{-abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}+\frac{abcd}{cd((\eta_n-\xi_{n-1})a+\xi_{n-1}b)+ab(\eta_{n-1}c+(\eta_n-\xi_n)d)}} \\
&= \frac{abcd}{cd((-\xi_{n-1}-\eta_n+\xi_{n-1})a+(\eta_{n+1}-\xi_n-\xi_{n-1})b)+ab((\xi_n-\eta_n-\eta_{n-1})c+(\eta_n-\eta_n+\xi_n)d} \\
&= \frac{abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
w_{2n+1} &= \frac{w_{2n}s_{2n-1}}{w_{2n}+s_{2n-1}} \\
&= \frac{\left(\frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)}\right)\left(\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}\right)}{\frac{abcd}{cd((\eta_{n+1}-\xi_n)a+\xi_n b)+ab(\eta_n c+(\eta_{n+1}-\xi_{n+1})d)}+\frac{abcd}{cd(-\xi_{n-1}a+(\eta_{n+1}-\xi_n)b)+ab((\xi_n-\eta_n)c+\eta_n d)}} \\
&= \frac{abcd}{cd((\eta_{n+1}-\xi_n-\xi_{n-1})a+(\xi_n+\eta_{n+1}-\xi_n)b+ab((\eta_n+\xi_n-\eta_n)c+(\eta_{n+1}-\xi_{n+1}+\eta_n)d)} \\
&= \frac{abcd}{cd((\xi_{n+1}-\eta_{n+1})a+\eta_{n+1}b)+ab(\xi_n c+(\xi_{n+1}-\eta_{n+2})d)}.
\end{aligned}
$$

And

$$
\begin{aligned}
s_{2n+1} &= \frac{s_{2n}w_{2n-1}}{-s_{2n}+w_{2n-1}} \\
&= \frac{\left(\frac{abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)}\right)\left(\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}\right)}{\frac{-abcd}{cd(-\eta_n a+(\xi_{n+1}-\eta_{n+1})b)+ab((\eta_{n+1}-\xi_n)c+\xi_n d)}+\frac{abcd}{cd((\xi_n-\eta_n)a+\eta_n b)+ab(\xi_{n-1}c+(\xi_n-\eta_{n+1})d)}} \\
&= \frac{abcd}{cd((-\eta_n-\xi_n+\eta_n)a+(\xi_{n+1}-\eta_{n+1}-\eta_n)b)+ab((\eta_{n+1}-\xi_n-\xi_{n-1})c+(\xi_n-\xi_n+\eta_{n+1})d} \\
&= \frac{abcd}{cd(-\xi_n a+(\eta_{n+2}-\xi_{n+1})b)+ab((\xi_{n+1}-\eta_{n+1})c+\eta_{n+1}d)}.
\end{aligned}
$$

□

**Example 4.2.** *Consider the solutions of Eq.(4.1) when* $w_{-1}=3,\ w_0=-1,\ s_{-1}=4$ *and* $s_0=-1$. *See Fig. 4.1.*

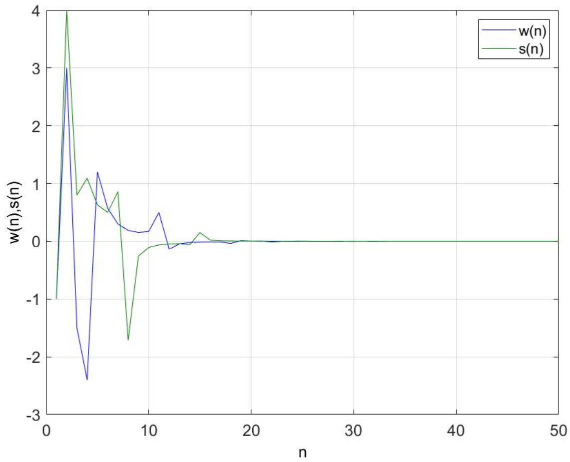


**Figure 4.1**

## 5. The system $w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}$, $s_{n+1} = \frac{s_n w_{n-1}}{-s_n - w_{n-1}}$

In this section, we obtain the solutions form for the system of two difference equations :

$$w_{n+1} = \frac{w_n s_{n-1}}{w_n + s_{n-1}}, \quad s_{n+1} = \frac{s_n w_{n-1}}{-s_n - w_{n-1}}. \tag{5.1}$$

**Theorem 5.1.** *Suppose that $\{w_n, s_n\}$ are solutions of system (5.1). Then every solutions of system (5.1) are periodic with period twelve and given by the following formulas for n=0,1,2,... ,*

$$w_{12n-1} = b, \qquad w_{12n} = a,$$

$$w_{12n+1} = \frac{ad}{a+d}, \qquad w_{12n+2} = \frac{adc}{(a+d)c+ad},$$

$$w_{12n+3} = \frac{adb}{(a+d)b-ad}, \qquad w_{12n+4} = \frac{dc}{d+c},$$

$$w_{12n+5} = -b, \qquad w_{12n+6} = -a,$$

$$w_{12n+7} = \frac{-ad}{a+d}, \qquad w_{12n+8} = \frac{-adc}{(a+d)c+ad},$$

$$w_{12n+9} = \frac{-adb}{(a+d)b-ad}, \qquad w_{12n+10} = \frac{-dc}{d+c},$$

$$s_{12n-1} = d, \qquad s_{12n} = c,$$

$$s_{12n+1} = \frac{-cb}{c+b}, \qquad s_{12n+2} = \frac{-abc}{cb-(c+b)a},$$

$$s_{12n+3} = \frac{-cdb}{(b+d)c+bd}, \qquad s_{12n+4} = \frac{-ba}{b-a},$$

$$s_{12n+5} = -d, \qquad s_{12n+6} = -c,$$

$$s_{12n+7}=\frac{cb}{c+b}, \qquad s_{12n+8}=\frac{abc}{cb-(c+b)a},$$

$$s_{12n+9}=\frac{cdb}{(b+d)c+bd}, \qquad s_{12n+10}=\frac{ba}{b-a}.$$

*Proof.* For n=0, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. That is

$$w_{12n-13}=b, \qquad w_{12n-12}=a,$$

$$w_{12n-11}=\frac{ad}{a+d}, \qquad w_{12n-10}=\frac{adc}{(a+d)c+ad},$$

$$w_{12n-9}=\frac{adb}{(a+d)b-ad}, \qquad w_{12n-8}=\frac{dc}{d+c},$$

$$w_{12n-7}=-b, \qquad w_{12n-6}=-a,$$

$$w_{12n-5}=\frac{-ad}{a+d}, \qquad w_{12n-4}=\frac{-adc}{(a+d)c+ad},$$

$$w_{12n-3}=\frac{-adb}{(a+d)b-ad}, \qquad w_{12n-2}=\frac{-dc}{d+c},$$

$$s_{12n-13}=d, \qquad s_{12n-12}=c,$$

$$s_{12n-11}=\frac{-cb}{c+b}, \qquad s_{12n-10}=\frac{-abc}{cb-(c+b)a},$$

$$s_{12n-9}=\frac{-cdb}{(b+d)c+bd}, \qquad s_{12n-8}=\frac{-ba}{b-a},$$

$$s_{12n-7}=-d, \qquad s_{12n-6}=-c,$$

$$s_{12n-5}=\frac{cb}{c+b}, \qquad s_{12n-4}=\frac{abc}{cb-(c+b)a},$$

$$s_{12n-3}=\frac{cdb}{(b+d)c+bd}, \qquad s_{12n-2}=\frac{ba}{b-a}.$$

Now, from Eq.(5.1) that

$$w_{12n}=\frac{w_{12n-1}s_{12n-2}}{w_{12n-1}+s_{12n-2}}=\frac{\frac{b^2a}{b-a}}{b+\frac{ba}{b-a}}=\frac{b^2a}{b^2-ab+ab}=a.$$

$$s_{12n}=\frac{s_{12n-1}w_{12n-2}}{-s_{12n-1}-w_{12n-2}}=\frac{\frac{-d^2c}{c+d}}{-d+\frac{cd}{c+d}}=\frac{-d^2c}{-d^2-cd+cd}=c.$$

Also,

$$w_{12n+5}=\frac{w_{12n+4}s_{12n+3}}{w_{12n+4}+s_{12n+3}}=\frac{\left(\frac{dc}{d+c}\right)\left(\frac{-cbd}{(b+d)c+bd}\right)}{\frac{dc}{d+c}-\frac{cbd}{(b+d)c+bd}}=\frac{-c^2bd^2}{cd(bc+dc+bd-bd-cb)}=-b.$$

$$s_{12n+5}=\frac{s_{12n+4}w_{12n+3}}{-s_{12n+4}-w_{12n+3}}=\frac{\left(\frac{-ba}{b-a}\right)\left(\frac{abd}{(a+d)b-ad}\right)}{-\frac{-bd}{b-a}-\frac{abd}{(a+d)b-ad}}=\frac{-a^2b^2d}{ba(ab+db-ad-bd+ad)}=-d.$$

Similarly, obtaining the other relations is very simple. Thus, the proof is completed.

□

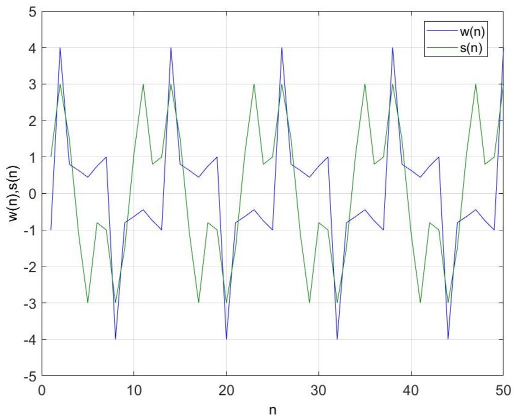


**Figure 5.1**

**Example 5.2.** *Figure 5.1 considers the solution of Eq.(5.1) with* $w_{-1} = 4,\ w_0 = -1,\ s_{-1} = 3$ *and* $s_0 = 1$.

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