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Research Article

Fibonacci 3-Parameter Generalized Quaternions

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Abstract

There are many studies on Fibonacci quaternions and their generalizations. Recently, Şentürk and Ünal (2022) introduced 3-parameter generalized quaternions. The goal of this study is to introduce Fibonacci and Lucas 3-parameter generalized quaternions and to investigate their properties. After obtaining Binet formulas for these quaternions, generalizations of some well-known identities are presented.

Keywords: Fibonacci quaternions, Lucas quaternions, 3-parameter generalized quaternions.

Genelleştirilmiş 3-Parametreli Fibonacci Kuaterniyonları

Öz

Fibonacci kuaterniyonları ve bu kuaterniyonların genelleştirmeleri hakkında birçok çalışma göze çarpmaktadır. Geçtiğimiz günlerde Şentürk and Ünal (2022), 3-parametreli genellştirilmiş kuaterniyonları tanıtmışlardır. Bu çalışmada genelleştirilmiş 3-parametreli Fibonacci ve Lucas kuaterniyonları tanıtılmış ve özellikleri araştırılmıştır. Bu kuaterniyonlar için Binet formülleri elde edildikten sonra iyi bilinen bazı özdeşliklerin genelleştirmeleri sunulmuştur.

Anahtar Kelimeler: Fibonacci kuaterniyonları, Lucas kuaterniyonları, Genelleştirilmiş 3-parametreli kuaterniyonlar.

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1. Introduction

Fibonacci and Lucas numbers may be the most popular sequences among integer sequences. For $n > 1$, Fibonacci numbers satisfy the second order recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Lucas numbers also satisfy the same recurrence relation, namely, for $n > 1$

$$L_n = L_{n-1} + L_{n-2}$$

but the initial conditions are $L_0 = 2$ and $L_1 = 1$. Although there are many interesting relation between these two sequences, the most important one is the following identity

$$L_n = F_{n-1} + F_{n+1}.$$

Generating functions fort he Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

and

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2}$$

respectively. Binet's formulas for these are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^b$$

respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are roots of the equation $x^2 - x - 1 = 0$. The root α is the golden ratio and it is a well-known real number among metallic ratios. We can refer to (Koshy, 2001) for more information about Fibonacci and Lucas numbers.

Sir R. Hamilton introduced quaternions as an extension of complex numbers. A quaternion q is shown $q = a + bi + cj + dk$ where a, b, c, d are reals and i, j, k satisfy the following conditions

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The set of all quaternions is

$$\mathcal{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$

Due to the lack of commutativity, Hamilton quaternions form a skew field. Following Hamilton, a number of different quaternion algebras such as split-quaternions, semi-quaternions, split-semi quaternions, ¼-quaternions and commutative quaterions were studied.

Recently, Şentürk and Ünal (2022) introduced 3-parameter generalized quaternions. The set of 3-parameter generalized quaternions is

$$\mathbb{K} = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

where the versors $1, i, j$ and k satisfy the following multiplication rules

Table 1. Multiplication rules of versors

	1	i	j	k
1	1	i	j	k
i	i	$-\lambda_1\lambda_2$	$\lambda_1 k$	$-\lambda_2 j$
j	j	$-\lambda_1 k$	$-\lambda_1\lambda_3$	$\lambda_3 i$
k	k	$\lambda_2 j$	$-\lambda_3 i$	$-\lambda_2\lambda_3$

It shold be noted that λ_1, λ_2 and λ_3 are arbitrary real numbers.

Fibonacci quaternions were introduced by Horadam (1963). He also used two generalizations of Fibonacci numbers to generalize Fibonacci quaternions. Similar to Horadam's study, Iyer (1969) defined Lucas quaternions. Halıcı (2012) gave Binet's formulas for the Fibonacci and Lucas quaternions and this study was a milestone in this theory, as Binet's formulas allowed to obtain identities between terms of Fibonacci and Lucas quaternion sequences. Following Halıcı, a number of study investigating Fibonacci and Lucas quaternions or their generalizations have been done. Fibonacci split quaternions (Akyiğit, Kösal & Tosun, 2013) and dual Fibonacci quaternions (Nurkan & Güven, 2015) are examples of using classical Fibonacci and Lucas numbers over a quaternion algebra. Another type of studies in literature used generalizations of Fibonacci and Lucas numbers over any quaternion algebra (Akyiğit, Kösal & Tosun, 2014; Aydın, 2021; Bilgici, Tokeşer & Ünal 2017; Flaut & Savin, 2015; Halıcı & Karataş, 2017; Polatlı, Kızılateş & Kesim, 2016; Tan, Yılmaz & Şahin, 2016; Yüce & Aydın, 2016).

2. Definitions, Generating Functions and Binet's Formulas

Definitions of Fibonacci and Lucas generalized 3-parameter quaternions are given in the following.

Definition 2.1. For any non-negative integer n, nth Fibonacci generalized 3-parameter quaternion is

$$\mathcal{F}_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k$$

and nth Lucas generalized 3-parameter quaternion is

$$\mathcal{L}_n = L_n + L_{n+1} i + L_{n+2} j + L_{n+3} k$$

where F_n and L_n are the classical Fibonacci and Lucas numbers.

Corollary 2.2. For any non-negative integer n, Fibonacci and Lucas generalized 3-parameter quaternion satisfy the following recurrence relations

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$$

and

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}$$

respectively.

By using the identities $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$ for the classical Fibonacci and Lucas numbers, we obtain the following relations

$$\mathcal{F}_{-n} = (-1)^{n+1}(F_n - F_{n+1}i + F_{n+2}j - F_{n+3}k)$$

and

$$\mathcal{L}_{-n} = (-1)^{n}(L_n - L_{n+1}i + L_{n+2}j - L_{n+3}k).$$

Binet's formulas for the Fibonacci and Lucas generalized 3-parameter quaternions are in the next theorem.

Theorem 2.3. For any integer n, nth Fibonacci generalized 3-parameter quaternion is

$$\mathcal{F}_n = \frac{\tilde{\alpha}\alpha^n - \tilde{\beta}\beta^n}{\alpha - \beta}$$

and nth Lucas generalized 3-parameter quaternion is

$$\mathcal{L}_n = \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n$$

where $\tilde{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\tilde{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$.

Proof. From the Binet's formula for the Fibonacci numbers, we have

$$\begin{aligned}\mathcal{F}_n &= F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k \\ &= \frac{1}{\alpha-\beta}[(\alpha^n - \beta^n) + (\alpha^{n+1} - \beta^{n+1})i + (\alpha^{n+2} - \beta^{n+2})j \\ &\qquad + (\alpha^{n+3} - \beta^{n+3})k] \\ &= \frac{1}{\alpha-\beta}[(1 + \alpha i + \alpha^2 j + \alpha^3 k)\alpha^n \\ &\qquad - (1 + \beta i + \beta^2 j + \beta^3 k)\beta^n].\end{aligned}$$

The last equation gives Binet's formula for the Fibonacci generalized 3-parameter quaternions. Binet's formula for the Lucas generalized 3-parameter quaternions can obtained in a similar way. ■

We need the following two relations for later use.

Lemma 2.4. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be as given in Theorem 2.3. Thus, we have

$$\tilde{\alpha}\,\tilde{\beta} = M + \sqrt{5}\,N$$

and

$$\tilde{\beta}\tilde{\alpha} = M - \sqrt{5}\,N$$

where

$$M = \mathcal{F}_1 + \lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_2\lambda_3 + j + k$$

and

$$N = -\lambda_3 i - \lambda_2 j + \lambda_1 k.$$

Proof. From the definitions of $\tilde{\alpha}$ and $\tilde{\beta}$, proof is straightforward. ■

Corollary 2.5. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be as given in Theorem 2.3, we have

$$\tilde{\alpha}\,\tilde{\beta} + \tilde{\beta}\tilde{\alpha} = M.$$

Generating functions for the Fibonacci and Lucas generalized 3-parameter quaternions are in the next theorem.

Theorem 2.6. Generating functions for the sequences $\{\mathcal{F}_n\}_{n=0}^{\infty}$ and $\{\mathcal{L}_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} \mathcal{F}_n x^n = \frac{i + j + 2k + (1 + j + k)x}{1 - x - x^2}$$

and

$$\sum_{n=0}^{\infty} \mathcal{L}_n x^n = \frac{2 + i + 3j + 4k + (-1 + 2i + j + 3k)x}{1 - x - x^2}$$

respectively.

Proof. Let $\mathcal{F}(x)$ be the sequences $\sum_{n=0}^{\infty} \mathcal{F}_n x^n$. Thus, we have

$$\mathcal{F}(x) = \mathcal{F}_0 + \mathcal{F}_1 x + \sum_{n=2}^{\infty} \mathcal{F}_n x^n.$$

If we multiply the last equation by $-x$ and $-x^2$, we obtain

$$-x\mathcal{F}(x) = -\mathcal{F}_0 x - \sum_{n=2}^{\infty} \mathcal{F}_{n-1} x^n$$

and

$$-x^2\mathcal{F}(x) = -\sum_{n=2}^{\infty} \mathcal{F}_{n-2} x^n.$$

Summing the last three equations and Corollary 2.2 give

$$(1 - x - x^2)\mathcal{F}(x) = \mathcal{F}_0 + (\mathcal{F}_1 - \mathcal{F}_0)x.$$

Thus we obtain the first identity in theorem. The second identity can be obtained similarly. ■

3. Results

In this section, we give generalizations of some well-known identities. We start with Vajda's identities given in the following theorem.

Theorem 3.1. For any integers r, s and t, following equations hold

$$\mathcal{F}_{r+s}\mathcal{F}_{r+t} - \mathcal{F}_r\mathcal{F}_{r+s+t} = (-1)^{r+1}F_s(-MF_t + NL_t)$$

and

$$\mathcal{L}_{r+s}\mathcal{L}_{r+t} - \mathcal{L}_r\mathcal{L}_{r+s+t} = (-1)^{r}5F_s(-MF_t + NL_t).$$

Proof.

Binet formula for the Fibonacci generalized 3-parameter quaternions gives

$$\begin{aligned}&\mathcal{F}_{r+s}\mathcal{F}_{r+t} - \mathcal{F}_r\mathcal{F}_{r+s+t} \\ &= \frac{1}{(\alpha-\beta)^2}\left[\left(\tilde{\alpha}\alpha^{r+s} - \tilde{\beta}\beta^{r+s}\right)\left(\tilde{\alpha}\alpha^{r+t} - \tilde{\beta}\beta^{r+t}\right) \right.\\ &\qquad \left. - \left(\tilde{\alpha}\alpha^{r} - \tilde{\beta}\beta^{r}\right)\left(\tilde{\alpha}\alpha^{r+s+t} - \tilde{\beta}\beta^{r+s+t}\right)\right] \\ &= \frac{1}{(\alpha-\beta)^2}\left[\tilde{\alpha}\tilde{\beta}(\alpha^r\beta^{r+s+t} - \alpha^{r+s}\beta^{r+t}) \right.\\ &\qquad \left. + \tilde{\beta}\tilde{\alpha}(\alpha^{r+s+t}\beta^r - \alpha^{r+t}\beta^{r+s})\right] \\ &= \frac{(-1)^r}{(\alpha-\beta)^2}\left[\tilde{\alpha}\tilde{\beta}(\beta^{s+t} - \alpha^s\beta^t) \right.\\ &\qquad \left. + \tilde{\beta}\tilde{\alpha}(\alpha^{s+t}\beta^r - \alpha^t\beta^s)\right] \quad (\text{since } \alpha\beta = -1) \\ &= \frac{(-1)^{r+1}}{(\alpha-\beta)^2}\left[\tilde{\alpha}\tilde{\beta}\beta^t(\alpha^s - \beta^t) - \tilde{\beta}\tilde{\alpha}\alpha^t(\alpha^s - \beta^s)\right] \\ &= \frac{(-1)^{r+1}F_s}{\alpha-\beta}\left[\tilde{\alpha}\tilde{\beta}\beta^t - \tilde{\beta}\tilde{\alpha}\alpha^t\right]\end{aligned}$$

$$= \frac{(-1)^{r+1} F_s}{\alpha - \beta} \left[\tilde{\alpha} \tilde{\beta} \beta^t - \tilde{\beta} \tilde{\alpha} \alpha^t \right]$$

$$= \frac{(-1)^{r+1} F_s}{\alpha - \beta} \left[\beta^t (M + \sqrt{5} N) - \alpha^t (M - \sqrt{5} N) \right].$$

Final equation with Binet formulas for the Fibonacci and Lucas numbers gives the first equation in the theorem. The second equation can be proved in a similar way. ■

If we take $t \to -s$ in Vajda's identity, we obtain Catalan's identities for the Fibonacci and Lucas generalized 3-parameter quaternions given in the following corollary.

Corollary 3.2. For any integers r and s, following equations hold

$$\mathcal{F}_{r+s} \mathcal{F}_{r-s} - \mathcal{F}_r^2 = (-1)^{r+s+1} (M F_s^2 + N F_{2s})$$

and

$$\mathcal{L}_{r+s} \mathcal{L}_{r-s} - \mathcal{L}_r^2 = (-1)^{r+s} 5 (M F_s^2 + N F_{2s}).$$

If we take $s \to 1$ in Catalan's identity, we have Cassini's identities for the Fibonacci and Lucas generalized 3-parameter quaternions given in the following corollary.

Corollary 3.3. For any integer r, following equations hold

$$\mathcal{F}_{r+1} \mathcal{F}_{r-1} - \mathcal{F}_r^2 = (-1)^r (M + N)$$

and

$$\mathcal{L}_{r+1} \mathcal{L}_{r-1} - \mathcal{L}_r^2 = (-1)^{r+1} 5 (M + N).$$

Another well-identity is d'Ocagne's identity and it is in the next theorem.

Theorem 3.4. For any integers r and s, following equations hold

$$\mathcal{F}_r \mathcal{F}_{s+1} - \mathcal{F}_{r+1} \mathcal{F}_s = (-1)^s [M F_{r-s} + N L_{r-s}]$$

and

$$\mathcal{L}_r \mathcal{L}_{s+1} - \mathcal{L}_{r+1} \mathcal{L}_s = (-1)^{s+1} 5 [M F_{r-s} + N L_{r-s}].$$

Proof. From the Binet formula for the Fibonacci generalized 3-parameter quaternions, we have

$$\mathcal{F}_r \mathcal{F}_{s+1} - \mathcal{F}_{r+1} \mathcal{F}_s$$

$$= \frac{1}{(\alpha - \beta)^2} \left[(\tilde{\alpha} \alpha^r - \tilde{\beta} \beta^r)(\tilde{\alpha} \alpha^{s+1} - \tilde{\beta} \beta^{s+1}) - (\tilde{\alpha} \alpha^{r+1} - \tilde{\beta} \beta^{r+1})(\tilde{\alpha} \alpha^s - \tilde{\beta} \beta^s) \right]$$

$$= \frac{1}{(\alpha - \beta)^2} \left[-\tilde{\alpha} \tilde{\beta} (\alpha^r \beta^{s+1} - \alpha^{r+1} \beta^s) - \tilde{\beta} \tilde{\alpha} (\alpha^{s+1} \beta^r - \alpha^s \beta^{r+1}) \right]$$

$$= \frac{1}{(\alpha - \beta)^2} \left[\tilde{\alpha} \tilde{\beta} \alpha^r \beta^s (\alpha - \beta) - \tilde{\beta} \tilde{\alpha} \alpha^s \beta^r (\alpha - \beta) \right]$$

$$= \frac{1}{\alpha - \beta} \left(\tilde{\alpha} \tilde{\beta} \alpha^r \beta^s - \tilde{\beta} \tilde{\alpha} \alpha^s \beta^r \right)$$

$$= \frac{(-1)^s}{\alpha - \beta} \left(\tilde{\alpha} \tilde{\beta} \alpha^{r-s} - \tilde{\beta} \tilde{\alpha} \beta^{r-s} \right)$$

$$= \frac{(-1)^s}{\alpha - \beta} \left[(M + \sqrt{5} N) \alpha^{r-s} - (M - \sqrt{5} N) \beta^{r-s} \right].$$

Binet formula fort he Fibonacci and Lucas numbers gives the first identity in theorem. The second one can be proved similarly. ■

The other identities for the Fibonacci and Lucas generalized 3-parameter quaternions are given in the next theorem. We will not prove these identities because theirs proofs based on Binet formulas for the Fibonacci and Lucas generalized 3-parameter quaternions similar to Vajda's and Catalan's identities.

Theorem 3.5. For any integers r, s and t, we have

$\mathcal{L}_r = \mathcal{F}_{r-1} + \mathcal{F}_{r+1},$

$\mathcal{L}_{r+s} \mathcal{F}_{r+t} - \mathcal{L}_{r+t} \mathcal{F}_{r+s} = 2(-1)^{r+s} M F_{t-s},$

$\mathcal{F}_r \mathcal{L}_s - \mathcal{L}_s \mathcal{F}_r = 2(-1)^r N \mathcal{L}_{s-r},$

$\mathcal{F}_r \mathcal{L}_s - \mathcal{L}_r \mathcal{F}_s = 2(-1)^s (M F_{r-s} + N L_{r-s}),$

$\mathcal{F}_r \mathcal{F}_s - \mathcal{F}_s \mathcal{F}_r = 2(-1)^{s+1} N F_{r-s},$

$\mathcal{L}_r \mathcal{L}_s - \mathcal{L}_s \mathcal{L}_r = 10(-1)^s N F_{r-s},$

$\mathcal{F}_{r+s} F_{r+s} - \mathcal{F}_{r-s} F_{r-s} = \mathcal{F}_{2r} F_{2s},$

$\mathcal{L}_{r+s} \mathcal{L}_{r+s} - \mathcal{L}_{r-s} \mathcal{L}_{r-s} = 5 \mathcal{F}_{2r} F_{2s},$

$\mathcal{F}_{r+s} L_{r+s} - \mathcal{F}_{r-s} L_{r-s} = \mathcal{F}_{2r+2s} - \mathcal{F}_{2r-2s},$

$\mathcal{L}_{r+s} L_{r+s} - \mathcal{L}_{r-s} L_{r-s} = \mathcal{L}_{2r+2s} - \mathcal{L}_{2r-2s},$

$5 \mathcal{F}_r^2 - \mathcal{L}_r^2 = 4(-1)^{r+1} M,$

$\mathcal{F}_{r+s} + (-1)^s \mathcal{F}_{r-s} = \mathcal{F}_r L_s,$

$\mathcal{L}_{r+s} + (-1)^s \mathcal{L}_{r-s} = \mathcal{L}_r L_s,$

$\mathcal{F}_{2r} = F_{r+1} \mathcal{F}_r + F_r \mathcal{F}_{r-1}.$

4. Conclusions

There is an icreasing interest in quaternions whose coefficients are integer sequences, especially Fibonacci and Lucas sequences. Recently, Şentürk and Ünal (2022) introduced 3-parameter generalized quaternions. This study aims to investigate these quaternions whose coefficients are Fibonacci and Lucas numbers. In this context, generating functions and Binet formulas for Fibonacci and Lucas generalized 3-parameter quaternions are important for calculating their properties and obtaining some generalization of well-known identities.

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