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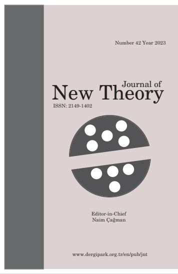

A Short Note on a Mus-Cheeger-Gromoll Type Metric

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Abstract — In this paper, we first show that the complete lift U^c to TM of a vector field U on M is an infinitesimal fiber-preserving conformal transformation if and only if U is an infinitesimal homothetic transformation of (M, g). Here, (M, g) is a Riemannian manifold and TM is its tangent bundle with a Mus-Cheeger-Gromoll type metric $\tilde{g}$. Secondly, we search for some conditions under which $\left(\overset{h}{\nabla}, \tilde{g}\right)$ is a Codazzi pair on TM when (∇, g) is a Codazzi pair on M where $\overset{h}{\nabla}$ is the horizontal lift of a linear connection ∇ on M. We finally discuss the need for further research.

Keywords *Codazzi pair, infinitesimal fiber-preserving conformal transformation, infinitesimal homothetic transformation, Mus-Cheeger-Gromoll type metric, tangent bundle*

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1. Introduction

The Sasaki metric [1] and the Cheeger-Gromoll metric [2] are the well-known metrics on the tangent bundles of Riemannian manifolds. Moreover, many metrics on tangent bundles have been introduced by deforming these two metrics. The rescaled Sasaki metric [3], the twisted Sasaki metric [4], the Mus-Sasaki metric [5], the rescaled Cheeger-Gromoll metric [6], the generalized Cheeger-Gromoll metric [7], and the Cheeger-Gromoll type metric [8] are examples of these deformations. Moreover, Latti and Djaa [9] introduced a new deformation of the Cheeger-Gromoll metric $\tilde{g}$, called the Mus-Cheeger-Gromoll metric. They computed the Levi-Civita connection and studied the curvature properties of a tangent bundle with respect to this metric. This paper will deal with a special case of this metric.

A classical problem on a Riemannian manifold M is to find infinitesimal conformal transformations (conformal vector fields) on M. The vector field U on M is an infinitesimal conformal transformation if and only if there is a function ρ on M satisfying $L_U g = 2\rho g$ where L_U is the Lie derivative with respect to U. If ρ is a nonzero constant (resp. zero), then U is referred to as an infinitesimal homothetic transformation (resp. Killing vector field). Infinitesimal conformal transformations are studied on tangent bundles by many authors [10–14].

Statistical manifolds were studied first by Amari [15] in view of information geometry, and Lauritzen gave applications in [16]. These manifolds have a crucial role in statistics as the statistical model often fashions a geometrical manifold. The geometry of statistical structures on tangent bundles is

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an actual topic. These structures were examined with respect to various Riemannian metrics such as the Sasaki metric [17], the Cheeger-Gromoll metric, and a $g-$natural metric which consists of three classic lifts of the metric g [18], the twisted Sasaki metric, and the gradient Sasaki metric [19].

In this paper, we prove that the complete lift U^c to TM of a vector field U on M is an infinitesimal fiber-preserving conformal transformation (IFPCT) on TM if and only if U is an infinitesimal homothetic transformation (IHT) of (M, g). We also investigate conditions under which $\left(\overset{h}{\nabla}, \tilde{g}\right)$ is a Codazzi pair on TM when (∇, g) is a Codazzi pair on M, where $\overset{h}{\nabla}$ is the horizontal lift of a linear connection ∇ on M.

2. Preliminary

Let M be an $n-$dimensional $(n > 1)$ Riemannian manifold and ∇ be a linear connection on M. The tangent bundle TM of the manifold M is a $2n-$dimensional differentiable manifold, and it is defined by disjoint tangent spaces at distinct points on M. If $\{N, x^i\}$ is a local coordinate system in M, then $\{\pi^{-1}(N), x^i, x^{\bar{i}} = u^i, \bar{i} = n+1, ..., 2n\}$ is a local coordinate system in TM where π is the projection defined by $\pi : TM \to M$. We have a decomposition

$$TTM = VTM \oplus HTM$$

for the tangent bundle of TM where the vertical subspace VTM is spanned by $\left\{\frac{\partial}{\partial u^i} := \left(\frac{\partial}{\partial x^i}\right)^v\right\}$ and the horizontal subspace HTM is spanned by $\left\{\frac{\delta}{\delta x^i} := \left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - u^m \Gamma^j_{mi} \frac{\partial}{\partial u^j}\right\}$. Here, Γ^j_{mi} are the Christoffel symbols of ∇. The vertical, horizontal, and the complete lift of a vector field $U = U^i \frac{\partial}{\partial x^i}$ are defined by, respectively,

$$U^v = U^i \frac{\partial}{\partial u^i}, \quad U^h = U^i \frac{\partial}{\partial x^i} - u^s \Gamma^m_{si} U^i \frac{\partial}{\partial u^m}, \quad \text{and} \quad U^c = U^i \frac{\partial}{\partial x^i} + u^s \frac{\partial U^i}{\partial x^s} \frac{\partial}{\partial u^i} \tag{1}$$

where we used Einstein's summation. In the sequel, for brevity, we denote $\frac{\partial}{\partial x^i}$, $\frac{\delta}{\delta x^i}$, and $\frac{\partial}{\partial u^i}$ by ∂_i, δ_i, and $\partial_{\bar{i}}$, respectively.

If ∇ is a torsionless linear connection, then the Lie brackets of the vertical lift and the horizontal lift of vector fields fulfill the following relations:

$$\left[U^h, V^h\right] = [U, V]^h - (R(U, V)u)^v, \quad \left[U^h, V^v\right] = (\nabla_U V)^v, \quad \text{and} \quad [U^v, V^v] = 0 \tag{2}$$

where R is the curvature of ∇ [20].

The frame $\{E_\lambda\} = \{E_i, E_{\bar{i}}\}$ adapted to the torsionless linear connection ∇ is given by

$$E_i = \delta^m_i \partial_m - u^s \Gamma^m_{si} \partial_{\bar{m}} \quad \text{and} \quad E_{\bar{i}} = \delta^m_i \partial_{\bar{m}}$$

Moreover, $\left\{dx^h, \delta u^h = du^h + u^c \Gamma^h_{cd} dx^d\right\}$ is the dual frame of $\{E_\lambda\}$. We can rewrite Lie Brackets 2 according to the adapted frame as follows:

$$[E_i, E_j] = u^s R^k_{ijs} E_{\bar{k}}, \quad \left[E_i, E_{\bar{j}}\right] = \Gamma^k_{ij} E_{\bar{k}}, \quad \text{and} \quad \left[E_{\bar{i}}, E_{\bar{j}}\right] = 0$$

where R^k_{jis} are the components of R. Vector Fields 1 are expressed as, according to the adapted frame,

$$U^v = U^i E_{\bar{i}}, U^h = U^i E_i \quad \text{and} \quad U^c = U^i E_i + u^s \nabla_s U^i E_{\bar{i}} \tag{3}$$

We have the following Lie derivatives with respect to $\tilde{U} = v^k E_k + v^{\bar{k}} E_{\bar{k}}$ [13]

$$\begin{aligned} L_{\tilde{U}} E_k &= -\partial_k v^c E_c + \left\{ u^a v^b R^c_{kba} - v^{\bar{a}} \Gamma^c_{ak} - E_k(v^{\bar{c}}) \right\} E_{\bar{c}} \\ L_{\tilde{U}} E_{\bar{k}} &= \left\{ v^a \Gamma^c_{ak} - E_{\bar{k}}(v^{\bar{c}}) \right\} E_{\bar{c}} \\ L_{\tilde{U}} dx^k &= \partial_n v^k dx^n \\ L_{\tilde{U}} \delta u^k &= -\left\{ u^a v^b R^k_{nba} - v^{\bar{a}} \Gamma^k_{an} - E_n(v^{\bar{k}}) \right\} dx^n - \left\{ v^a \Gamma^k_{an} - E_{\bar{n}}(v^{\bar{k}}) \right\} \delta u^n \end{aligned}$$

The horizontal lift connection $\overset{h}{\nabla}$ of a linear connection ∇ is given by

$$\overset{h}{\nabla}_{U^h} V^h = (\nabla_U V)^h, \quad \overset{h}{\nabla}_{U^h} V^v = (\nabla_U V)^v, \quad \text{and} \quad \overset{h}{\nabla}_{U^v} V^h = \overset{h}{\nabla}_{U^v} V^v = 0$$

Remark that ∇ is a flat and torsionless linear connection if and only if $\overset{h}{\nabla}$ is a torsionless linear connection [20].

The Mus-Cheeger-Gromoll metric G_{mc} on TM is defined by

$$\begin{aligned} G_{mc}\left(U^h, V^h\right) &= g(U,V) \\ G_{mc}\left(U^h, V^v\right) &= 0 \\ G_{mc}\left(U^v, V^v\right) &= f(x)\omega(r^2)(g(U,V) + \alpha(r^2) g(U,u) g(V,u)) \end{aligned}$$

for every vector fields U and V on M where $f : M \to \mathbb{R}_+$ and $\omega, \alpha : \mathbb{R} \to \mathbb{R}_+$ are three functions and $r^2 = g(u,u)$ [9].

Particular cases of the metric G_{mc} are listed below:

i. If $f = 1$, $\omega = \frac{1}{1+r^2}$, and $\alpha = 1$, then G_{mc} is the Cheeger-Gromoll metric [2].

ii. If $f = 1$, $\omega = \left(\frac{1}{1+r^2}\right)^p$, $\alpha = cons.$, then G_{mc} is the generalized Cheeger-Gromoll metric [7].

iii. If $\omega = \left(\frac{1}{1+r^2}\right)^p$ and $\alpha = cons.$, then G_{mc} is the rescaled vertically generalized Cheeger-Gromoll metric [21].

In this paper, we consider a Mus-Cheeger-Gromoll type metric $\tilde{g}$ by assuming $\omega(r^2) = \frac{1}{1+r^2}$.

Definition 2.1. Let (M,g) be a Riemannian manifold and ∇ be a linear connection on M. The couple (g, ∇) is called a Codazzi pair if the following Codazzi equations are valid:

$$(\nabla_U g)(V,W) = (\nabla_V g)(W,U) = (\nabla_W g)(U,V)$$

for all vector fields U, V, and W on M. In this case, (M,g,∇) is referred to as a Codazzi manifold and ∇ is called a Codazzi connection. Moreover, if ∇ is torsionless, then (M,g,∇) is a statistical connection.

3. Main Results

Let $g = g_{ij}\ dx^i dx^j$ is the Riemannian metric g on M. Then, the local expression of the Mus-Cheeger-Gromoll type metric $\tilde{g}$ is

$$\tilde{g} = g_{ij} dx^i dx^j + h_{ij} \delta x^i \delta x^j$$

where $h_{ij} = \frac{f}{1+r^2}(g_{ij} + \alpha g_{im} g_{jn} u^m u^n)$. If $G_1 = g_{ij} dx^i dx^j$ and $G_2 = h_{ij} \delta x^i \delta x^j$, then

$$\tilde{g} = G_1 + G_2$$

Besides, recall that a vector field $\tilde{U}$ with components $\left(v^h, v^{\bar{h}}\right)$ on TM is a fibre preserving (FP) if and only if v^h has components $\left(x^h\right)$.

The following lemma states the Lie derivatives of G_1 and G_2.

Lemma 3.1. The Lie derivatives of G_1 and G_2 with respect to a FP vector field $\tilde{U}$ are

$$\begin{aligned}
L_{\tilde{U}}G_1 &= (L_U g_{ij})dx^i dx^j \\
L_{\tilde{U}}G_2 &= -2h_{mj}\{u^b v^c R^m_{icb} - v^{\bar{b}}\Gamma^m_{bi} - E_i(v^{\bar{m}})\}dx^i\delta u^j + \{L_U h_{ij} - 2h_{mj}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\overline{m}}) \\
&\quad + \tfrac{1}{1+r^2}v^{\overline{m}}u^s(-2g_{ms}h_{ij} + 2f\alpha' g_{is}g_{jt}g_{mn}u^t u^n + f\alpha(g_{js}g_{im} + g_{jm}g_{is})\}\delta u^i\delta u^j
\end{aligned}$$

where $L_U g_{ij}$ is the components of $L_U g$ and $\nabla_i v^m$ is the components of ∇_U.

Proof.
The proof is similar to the proof of Proposition 2.3 in [13]. □

The first main result of the paper is as follows:

Theorem 3.2. If TM is the tangent bundle of (M, g) equipped with the Mus-Cheeger-Gromoll type metric $\tilde{g}$, then the complete lift U^c of a vector field U is an IFPCT of $(TM, \tilde{g})$ if and only if U is an IHT of (M, g).

Proof.
If $\tilde{U}$ is an IFPCT of $(TM, \tilde{g})$, then there exists a smooth function Ω satisfying

$$L_{\tilde{U}}\tilde{g} = 2\Omega\tilde{g}$$

From Lemma 3.1,

$$\begin{aligned}
2\Omega g_{ij}dx^i dx^j + 2\Omega h_{ij}\delta x^i\delta x^j = (L_U g_{ij})dx^i dx^j &- 2h_{mj}\{u^b v^c R^m_{icb} - v^{\bar{b}}\Gamma^m_{bi} - E_i(v^{\bar{m}})\}dx^i\delta u^j \\
&+ \{L_U h_{ij} - 2h_{jm}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\overline{m}}) \\
&+ \tfrac{1}{1+r^2}v^{\overline{m}}u^s(-2g_{ms}h_{ij} + 2f\alpha' g_{is}g_{jt}g_{mn}u^t u^n + f\alpha(g_{js}g_{im} + g_{jm}g_{is})\}\delta u^i\delta u^j
\end{aligned}$$

It follows that

$$L_U g_{ij} = 2\Omega g_{ij} \tag{4}$$

$$u^b v^c R^m_{icb} - v^{\bar{b}}\Gamma^m_{bi} - E_i(v^{\bar{m}}) = 0$$

$$L_U h_{ij} - 2h_{jm}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\overline{m}}) + \frac{1}{1+r^2}v^{\overline{m}}u^s(-2g_{ms}h_{ij} + 2f\alpha' g_{is}g_{jt}g_{mn}u^t u^n + f\alpha(g_{js}g_{im} + g_{jm}g_{is})) = 2\Omega h_{ij}$$

From Equation 3, we can write the complete lift a vector field $U = v^k\partial_k$ as $U^c = v^k E_k + u^s\nabla_s v^k E_{\overline{k}}$. Thus,

$$L_U g_{ij} = 2\Omega g_{ij}$$

and

$$u^b(v^c R^m_{icb} - \nabla_i\nabla_b v^m) = 0 \tag{5}$$

Therefore, Equation 5 gives

$$\nabla_i\nabla_b v_j = v^c R_{icbj}$$

Using algebraic properties of the Riemannian curvature tensor,

$$\nabla_i\nabla_b v_j + \nabla_i\nabla_j v_b = 0 \tag{6}$$

Since $L_U g_{ij} = \nabla_i v_j + \nabla_j v_i$, from Equation 4,

$$\nabla_i v_j + \nabla_j v_i = 2\Omega g_{ij}$$

Taking the covariant derivative on both hand sides of the above equation,

$$\nabla_k(\nabla_i v_j) + \nabla_k(\nabla_j v_i) = 2(\nabla_k\Omega)g_{ij} \tag{7}$$

Equations 6 and 7 show that $\nabla_k \Omega = 0$. Hence, Ω is constant. The proof of converse is clear. $\square$

In this part of this section, we deal with Codazzi pairs on M and TM. Let (M, g) be an $n-$dimensional $(n > 1)$ Riemannian manifold and $\left(\overset{h}{\nabla}, \tilde{g}\right)$ be a Codazzi pair on TM. Taking into account Definition 2.1, by direct calculation,

$$\left(\overset{h}{\nabla}_{\delta_i}\widetilde{g}\right)(\delta_j, \partial_{\bar{k}}) = \left(\overset{h}{\nabla}_{\delta_j}\widetilde{g}\right)(\partial_{\bar{k}}, \delta_i) = \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\delta_i, \delta_j) = 0$$

and

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \delta_k) = \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\delta_k, \partial_{\bar{i}}) = \left(\overset{h}{\nabla}_{\delta_k}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0$$

Moreover,

$$\left(\overset{h}{\nabla}_{\delta_i}\widetilde{g}\right)(\delta_j, \delta_k) = \nabla_i g_{jk}$$

and

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = \partial_{\bar{i}}\widetilde{g}(\partial_{\bar{j}}, \partial_{\bar{k}}), \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{i}}) = \partial_{\bar{j}}\widetilde{g}(\partial_{\bar{k}}, \partial_{\bar{i}}), \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = \partial_{\bar{k}}\widetilde{g}(\partial_{\bar{i}}, \partial_{\bar{j}})$$

Furthermore,

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = f\left\{\tfrac{-2u^m}{(1+r^2)^2}(g_{im}g_{jk} + \alpha u^s u^t g_{im}g_{js}g_{kt}) + \tfrac{2\alpha'}{1+r^2}g_{in}g_{js}g_{kt}u^n u^s u^t + \tfrac{\alpha u^s}{1+r^2}(g_{ji}g_{ks} + g_{js}g_{ki})\right\} \tag{8}$$

$$\left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{i}}) = f\left\{\tfrac{-2u^m}{(1+r^2)^2}(g_{jm}g_{ki} + \alpha u^s u^t g_{jm}g_{ks}g_{it}) + \tfrac{2\alpha'}{1+r^2}g_{jn}g_{ks}g_{it}u^n u^s u^t + \tfrac{\alpha u^s}{1+r^2}(g_{kj}g_{is} + g_{ks}g_{ij})\right\} \tag{9}$$

$$\left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = f\left\{\tfrac{-2u^m}{(1+r^2)^2}(g_{km}g_{ij} + \alpha u^s u^t g_{km}g_{is}g_{jt}) + \tfrac{2\alpha'}{1+r^2}g_{kn}g_{is}g_{jt}u^n u^s u^t + \tfrac{\alpha u^s}{1+r^2}(g_{ik}g_{js} + g_{is}g_{jk})\right\} \tag{10}$$

Equations 8-10 yield two cases.

Case 1) If $\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}}, \partial_{\bar{i}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0$, then, from Equation 8,

$$\left(\overset{h}{\nabla}_{\partial_{\bar{i}}}\widetilde{g}\right)(\partial_{\bar{j}}, \partial_{\bar{k}}) = f\left\{\frac{-2u^m}{1+r^2}(g_{im}g_{jk} + \alpha u^s u^t g_{im}g_{js}g_{kt}) + 2\alpha' g_{in}g_{js}g_{kt}u^n u^s u^t + \alpha u^s(g_{ji}g_{ks} + g_{js}g_{ki})\right\} = 0$$

Taking the derivative in the above equation with respect to $\partial_{\bar{h}}$,

$$\begin{aligned} 0 = f\Bigg\{ & \tfrac{-2\delta_h^m(1+r^2)+4u^m u^n g_{nh}}{(1+r^2)^2}(g_{im}g_{jk} + \alpha u^s u^t g_{im}g_{js}g_{kt}) - \tfrac{2u^m g_{mi}u^t(2\alpha' g_{hn}g_{js}g_{kt}u^n u^s + \alpha g_{tk}g_{jh} + \alpha g_{jt}g_{hk})}{1+r^2} \\ & - \tfrac{4u^m\alpha' g_{hn}g_{js}g_{kt}g_{im}u^s u^t u^n}{1+r^2} + (2\alpha' g_{hn}u^n u^s + \alpha\delta_h^s)(g_{ji}g_{ks} + g_{js}g_{ki})\Bigg\} \end{aligned}$$

because

$$\begin{aligned} 0 &= f\Bigg\{\frac{-2}{1+r^2}(g_{jk}g_{ih} + \alpha u^s u^t g_{ih}g_{js}g_{kt}) + \frac{4u^m u^n g_{nh}}{(1+r^2)^2}(g_{jk}g_{im} + u^s u^t g_{im}g_{js}g_{kt}) \\ &\quad -\frac{2u^m g_{mi}u^t(2\alpha' g_{hn}g_{js}g_{kt}u^n u^s + \alpha g_{tk}g_{jh} + \alpha g_{jt}g_{hk})}{1+r^2} - \frac{4u^m\alpha' g_{hn}u^n g_{js}g_{kt}g_{im}u^s u^t}{1+r^2} \\ &\quad +2\alpha' g_{hn}u^n u^s(g_{ji}g_{ks} + g_{js}g_{ki})\alpha(g_{ji}g_{kh} + g_{jh}g_{ki})\Bigg\} \end{aligned} \tag{11}$$

Equation 11 is satisfied, for all $(x, u) \in TM$. For zero section, i.e., $u = 0$, Equation 11 becomes

$$-2g_{jk}g_{ih} = 0$$

This is a contradiction, when i, j, and k run from 1 to n.

Case 2) If $\left(\overset{h}{\nabla}_{\partial_{\bar{\imath}}}\widetilde{g}\right)(\partial_{\bar{j}},\partial_{\bar{k}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{j}}}\widetilde{g}\right)(\partial_{\bar{k}},\partial_{\bar{\imath}}) = \left(\overset{h}{\nabla}_{\partial_{\bar{k}}}\widetilde{g}\right)(\partial_{\bar{\imath}},\partial_{\bar{j}}) \neq 0$, then, from Equations 8-10,

$$-2g_{jk}g_{ih} = -2g_{ki}g_{jh}$$

Hence, $g_{jk}g_{ih} = g_{ik}g_{jh}$. Multiplying both side of this equation by g^{jh}, $g_{ki} = ng_{ki}$. Thus, $n = 1$. This is a contradiction. Consequently, we can express the following result.

Theorem 3.3. Let TM be the tangent bundle of an $n-$dimensional $(n > 1)$ Riemannian manifold (M, g) equipped with the metric $\tilde{g}$ and ∇ be a linear connection on M. If (∇, g) is a Codazzi pair on M, then $\left(\overset{h}{\nabla}, \widetilde{g}\right)$ is not a Codazzi pair on TM.

4. Conclusion

The Mus-Cheeger-Gromoll metric is a new metric on the tangent bundle of a Riemannian manifold. In this paper, we studied the infinitesimal fiber-preserving property of the complete lift of a vector field and investigated the Codazzi pairs using the horizontal lift of a linear connection. Our findings suggest that these techniques could be applied to more general metrics in tangent bundles, opening up new avenues of research in this area. In addition, we believe that further investigation of the Mus-Cheeger-Gromoll metric could yield even more insights into the nature of Riemannian manifolds and their properties.

Author Contributions

The author read and approved the last version of the article.

Conflicts of Interest

The author declares no conflict of interest.

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