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Research Article

Screen generic lightlike submanifolds of semi-Riemannian product manifolds

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Abstract

In this paper, we introduce screen generic lightlike submanifolds of semi-Riemannian product manifolds. We investigate the integrability of various distributions and geometry of such submanifolds. Finally, we find a condition for minimal screen generic lightlike submanifolds. We also give examples.

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1. Introduction

In the generalization from Riemannian to semi-Riemannian manifolds, the induced metric may be degenerate (lightlike) therefore there is a natural existence of lightlike submanifolds and for which the local and global geometry is completely different than non-degenerate case. In lightlike case, the standard text book definitions do not make sense and one fails to use the theory of non-degenerate geometry in the usual way. The primary difference between the lightlike submanifolds and non-degenerate submanifolds is that in the first case, the normal vector bundle intersects with the tangent bundle. Thus, the study of lightlike submanifolds becomes more difficult and interesting from the study of non-degenerate submanifolds. It is well known that the intersection of the normal bundle and the tangent bundle of a submanifold of a semi-Riemannian manifold may be not trivial, it is more difficult and interesting to study the geometry of lightlike submanifolds than non-degenerate submanifolds. The geometry of lightlike submanifolds of a semi-Riemannian manifold was presented Duggal-Bejancu and Duggal-Şahin [4, 9] respectively.

Duggal and Bejancu [4] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds and Duggal and Şahin [7] introduced contact CR-lightlike submanifolds of indefinite Sasakian manifolds. Similar to CR-lightlike submanifolds, semi-invariant lightlike submanifolds of semi-Riemannian product manifolds were introduced by Atçeken and Kılıç in [1]. But CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases. Then, screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds were introduced in [5] and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds [7] were presented by Duggal and Şahin. Screen Cauchy-Riemann

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(SCR)-lightlike submanifolds, analogously, screen semi-invariant lightlike submanifolds, of semi-Riemannian product manifolds were introduced by Khursheed, Thakur and Advin [17] and Kılıç, Şahin and Keleş [18], respectively. But there is no inclusion relation between CR and SCR submanifolds, so Duggal and Şahin [6] presented a new class named GCR-lightlike submanifolds of indefinite Kaehler manifolds and generalized Cauchy-Riemann GCR-lightlike submanifolds of indefinite Sasakian manifolds [8] which is an umbrella for all these types of submanifolds. Kumar et al. studied GCR-lightlike submanifolds of a semi-Riemannian product manifold [19]. These types of submanifolds have been studied in various manifolds by many authors [12, 15, 20].

But CR-lightlike, screen CR-lightlike and generalized CR-lightlike do not contain real lightlike curves. For this reason, Şahin presented screen transversal lightlike submanifolds of indefinite Kaehler manifolds and show that such submanifolds contain lightlike real curves [23]. Screen transversal lightlike submanifolds of indefinite almost contact manifolds introduced in [24]. Screen transversal lightlike submanifolds of semi-Riemannian product manifolds were introduced in [25]. Further, such submanifolds have also studied in [10,11,14,21]. On the other hand, Doğan et al. [3] introduced a new class of lightlike submanifolds for indefinite Kaehler manifolds which particularly contain invariant lightlike, screen real lightlike and generic lightlike submanifolds and they called this submanifolds as screen generic lightlike submanifolds. After, Gupta introduced screen generic lightlike submanifolds of indefinite Sasakian manifolds [13].

In this paper, we introduce screen generic lightlike submanifolds of semi-Riemannian product manifolds. We investigate the integrability of various distributions and geometry of such submanifolds. Finally, we find a condition for minimal screen generic lightlike submanifolds. We also give examples.

2. Preliminaries

Let $\bar{M}$ be an $n-$dimensional differentiable manifold with a tensor field F of type $(1,1)$ on $\bar{M}$ such that

$$F^2 = I. \tag{2.1}$$

Then $\bar{M}$ is called an almost product manifold with almost product structure F. If we put

$$\pi = \frac{1}{2}(I+F),\ \sigma = \frac{1}{2}(I-F)$$

then we have

$$\pi + \sigma = I,\ \pi^2 = \pi,\ \sigma^2 = \sigma,\ \pi\sigma = \sigma\pi = 0,\ F = \pi - \sigma.$$

Thus π and σ define two complementary distributions and F has the eigenvalue of $+1$ or -1.

If an almost product manifold $\bar{M}$ admits a semi-Riemannian metric $\bar{g}$ such that

$$\bar{g}(FX, FY) = \bar{g}(X, Y) \tag{2.2}$$

for any vector fields X, Y on $\bar{M}$, then $\bar{M}$ is called a semi-Riemannian almost product manifold.

From (2.1) and (2.2), we have

$$\bar{g}(FX, Y) = \bar{g}(X, FY). \tag{2.3}$$

If, for any vector fields X, Y on $\bar{M}$,

$$\bar{\nabla} F = 0,\ \text{that is, } \bar{\nabla}_X FY = 0, \tag{2.4}$$

then $\bar{M}$ is called a semi-Riemannian product manifold, where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ [22].

Consider a m-dimensional submanifold (M, g) of a $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the induced metric g on M is degenerate and rank of the radical distribution $Rad(TM)$ of TM is r, $1 \le r \le m$, then (M, g) is called a lightlike submanifold of $(\bar{M}, \bar{g})$. While the normal bundle $TM^{\perp}$ of the tangent bundle TM is defined as

$$TM^{\perp} = \cup_{x \in M}\{u \in T_x\bar{M} \mid \bar{g}(u, v) = 0, \ \forall v \in T_xM\}, \tag{2.5}$$

the radical distribution $Rad(TM)$ of TM is defined as

$$Rad(TM) = \cup_{x \in M}\{\xi \in T_xM \mid g(u, \xi) = 0, \ \forall u \in T_x M, \ \xi \neq 0\}. \tag{2.6}$$

It is clear that $Rad(TM) = TM \cap TM^{\perp}$. On the other hand we know that both TM and $TM^{\perp}$ are degenerate vector subbundles. So, there exist complementary non-degenerate *screen distribution* $S(TM)$ and *co-screen distribution (or screen transversal bundle)* $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively. Then we can write the following decompositions:

$$TM = S(TM) \perp Rad(TM), \tag{2.7}$$

$$TM^{\perp} = S(TM^{\perp}) \perp Rad(TM). \tag{2.8}$$

Similarly, $S(TM)$ has an orthogonal complementary bundle $S(TM)^{\perp}$ in $T\bar{M}$ such that

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}, \tag{2.9}$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$.

Theorem 2.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM) \mid_U)$ consists of smooth sections $\{N_1, ..., N_r\}$ of $S(TM^{\perp})^{\perp} \mid_U$ such that*

$$\bar{g}(\xi_i, N_j) = \delta_{ij}, \ \bar{g}(N_i, N_j) = 0, \ i, j = 1, .., r,$$

where $\{\xi_1, ..., \xi_r\}$ is a basis of $\Gamma(Rad(TM))$ [4].

This result implies that there exists a complementary (but not orthogonal) vector bundle $tr(TM)$ to TM in $T\bar{M}|_M$, which called transversal vector bundle, such that the following decompositions hold:

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{2.10}$$

and

$$S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM). \tag{2.11}$$

Using the above equations we can write

$$T\tilde{M} \mid_M = TM \oplus tr\,(TM) = \{Rad(TM) \oplus ltr\,(TM)\} \perp S\,(TM) \perp S(TM^{\perp}). \tag{2.12}$$

There exist four cases for a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}) :$

Case 1: M is called r - lightlike if $r < min\{m, n\}$.
Case 2: M is called co - isotropic if $r = n < m$, *i.e.*, $S(TM^{\perp}) = \{0\}$.
Case 3: M is called isotropic if $r = m < n$, *i.e.*, $S(TM) = \{0\}$.
Case 4: M is called totally lightlike if $r = m = n$, *i.e.*, $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations of M are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.13}$$

and

$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ are belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_U is a linear endomorphism of $\Gamma(TM)$.
According to (2.12), considering the projection morphisms L and S of $tr\,(TM)$ on $ltr\,(TM)$ and $S(TM^\perp)$, respectively, (2.13) and (2.14) become

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y), && (2.15)\\ \tilde{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X,N), && (2.16)\\ \tilde{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X,W), && (2.17)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S(TM^\perp))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^l_X N, D^l(X,W) \in \Gamma(ltr\,(TM))$ and $\nabla^s_X W, D^s(X,N) \in \Gamma(S(TM^\perp))$. Then, by using (2.15)-(2.17) and taking into account that $\tilde{\nabla}$ is a metric connection we obtain

$$\begin{aligned} g(h^s(X,Y),W) + g(Y, D^l(X,W)) &= g(A_W X, Y), && (2.18)\\ g(D^s(X,N),W) &= g(A_W X, N), && (2.19)\\ g(h^l(X,Y),\xi) + g(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) &= 0. && (2.20)\end{aligned}$$

Let $\bar{P}$ be a projection of TM on $S(TM)$. Thus, using (2.7) we can obtain

$$\begin{aligned}\nabla_X \bar{P}Y &= \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y)\xi, && (2.21)\\ \nabla_X \xi &= -A^*_\xi X - \nabla^{*t}_X \xi, && (2.22)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.
Considering above equations, we derive

$$\begin{aligned}\bar{g}(h^l(X,\bar{P}Y),\xi) &= g(A^*_\xi X, \bar{P}Y), && (2.23)\\ \bar{g}(h^*(X,\bar{P}Y),N) &= g(A_N X, \bar{P}Y), && (2.24)\\ \bar{g}(h^l(X,\xi),\xi) &= 0, \quad A^*_\xi \xi = 0. && (2.25)\end{aligned}$$

We know that the induced connection ∇ on M, generally is not metric connection. If we consider that $\bar{\nabla}$ is a metric connection and use (2.15), we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y), \tag{2.26}$$

i.e., ∇ is not a metric connection. However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$.

Theorem 2.2 ([4]). *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\tilde{M}$. Then the induced connection ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ .*

3. Screen generic lightlike submanifolds

Definition 3.1. Let M be a real r-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then we say that M is a screen generic lightlike submanifold if the following conditions are satisfied:

(A) $Rad(TM)$ is invariant respect to F, that is,

$$F(Rad(TM)) = Rad(TM). \tag{3.1}$$

(B) There exists a subbundle D_0 of $S(TM)$ such that

$$D_0 = F(S(TM)) \cap S(TM) \tag{3.2}$$

where D_0 is a non-degenerate distribution on M.

From definition of a screen generic lightlike submanifold, we obtain that there exists a complementary non-degenerate distribution $D^{'}$ to D_0 in $S(TM)$ such that,

$$S(TM) = D_0 \oplus D^{'},$$

where

$$F(D^{'}) \nsubseteq S(TM) \text{ and } F(D^{'}) \nsubseteq S(TM^{\perp}).$$

Let P_0, P_1 and Q be the projection morphisms on D_0, $Rad(TM)$ and $D^{'}$, respectively. Then we have, for any $X \in \Gamma(TM)$,

$$\begin{aligned} X &= P_0X + P_1X + QX \\ &= PX + QX, \end{aligned} \tag{3.3}$$

where $D = D_0 \perp Rad(TM)$, D is invariant and $PX \in \Gamma(D)$, $QX \in \Gamma(D^{'})$.
From (3.3) we get

$$FX = fX + \omega X \tag{3.4}$$

where fX and ωX are tangential and transversal parts of FX, respectively. Besides, it is clear that $F(D^{'}) \neq D^{'}$.
On the other hand, for a vector field $Y \in \Gamma(D^{'})$, we have

$$FY = fY + \omega Y$$

such that $fY \in \Gamma(D^{'})$ and $\omega Y \in \Gamma(S(TM^{\perp}))$.
Similarly, for $W \in \Gamma(tr(TM))$, we get following decomposition

$$FW = BW + CW \tag{3.5}$$

where BW is tangential part and CW is transversal part of FW, respectively.
We say that M is a proper screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$ if $D_0 \neq \{0\}$ and $D^{'} \neq \{0\}$.

Definition 3.2. Let M be a r-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. We say that M is a SCR-lightlike submanifold of $\bar{M}$, if the following conditions are satisfied:

(i) There exists real non-null distributions D and $D^{\perp}$ such that

$$S(TM) = D \oplus D^{\perp},\ F(D^{\perp}) \subset S(TM^{\perp}),\ D \cap D^{\perp} = \{0\},$$

where $D^{\perp}$ is orthogonal complement to D in $S(TM)$.

(ii) The distribution D and $RadTM$ are invariant with respect to F.

Corollary 3.3. *A SCR-lightlike submanifold is a screen generic lightlike submanifold such that distribution $D^{'}$ is totally anti-invariant, that is,*

$$S(TM^{\perp}) = \omega D^{'} \oplus \mu,$$

where μ is a non-degenerate invariant distribution.

Similar to Definition of generic lightlike submanifolds given by Jin-Lee [16], we have:

Definition 3.4. Let M be a r-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. If there exists a screen distribution $S(TM)$ of M, such that

$$F(S(TM^{\perp})) \subset S(TM)$$

then, M is called a generic r-lightlike submanifold.

Corollary 3.5. *A generic r-lightlike submanifold is a screen generic lightlike submanifold with $\mu = 0$.*

The tangential bundle TM of M have following decomposition:

$$TM = D \oplus D^{'}.$$

Proposition 3.6. *Any screen generic lightlike submanifold M of a semi-Riemannian product manifold $\bar{M}$ is an invariant lightlike submanifold if $D^{'} = \{0\}$.*

Example 3.7. Let $(\tilde{M} = \mathbb{R}_2^{10}, \tilde{g})$ be a 10-dimensional semi-Euclidean space with signature $(-,+,+,-,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ be the standard coordinate system of $\mathbb{R}_2^{10}$. If we define a mapping F by

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (x_1, x_3, x_2, x_6, x_5, x_4, x_8, x_7, x_9, x_{10}),$$

then $F^2 = I$ and F is an almost product structure on $\mathbb{R}_2^{10}$. Let M be a submanifold of $\tilde{M}$ defined by

$$\begin{aligned} x_1 &= u_1,\, x_2 = u_4 \cos\alpha,\, x_3 = u_5 \cos\alpha,\;\; x_4 = u_4,\;\; x_5 = u_1, \\ x_6 &= u_5,\, x_7 = 0, x_8 = u_4 \sin\alpha,\, x_9 = u_2 + u_3,\, x_{10} = u_2 - u_3. \end{aligned}$$

Then TM is spanned by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_5,\, Z_2 = \partial x_9 + \partial x_{10},\, Z_3 = \partial x_9 - \partial x_{10}, \\ Z_4 &= \cos\alpha \partial x_2 + \partial x_4 + \sin\alpha \partial x_8,\;\; Z_5 = \cos\alpha \partial x_3 + \partial x_6. \end{aligned}$$

Hence M is a $1-$lightlike submanifold of $\mathbb{R}_2^{10}$ with $Rad(TM) = Span\{Z_1\}$, $D_0 = Span\{Z_2, Z_3\}$ and $D^{'} = Span\{Z_4, Z_5\}$. It is easy to see that $FZ_1 = Z_1$, $FZ_2 = Z_2$ and $FZ_3 = Z_3$. By direct calculations, we get the lightlike transversal bundle spanned by

$$N = \frac{1}{2}(-\partial x_1 + \partial x_5)$$

and the screen transversal bundle spanned by

$$\begin{aligned} W_1 &= \sin\alpha \partial x_7,\, W_2 = \cos\alpha \partial x_2 + \partial x_4 - \sin\alpha \partial x_8, \\ W_3 &= \partial x_2 - \cos\alpha \partial x_4,\, W_4 = \partial x_3 - \cos\alpha \partial x_6, \end{aligned}$$

where $\mu = Span\{W_3, W_4\}$, $FW_3 = W_4$ and $FN = N$. Since

$$\begin{aligned} FZ_4 &= Z_5 + W_1, \\ FZ_5 &= \frac{Z_4 + W_2}{2}, \end{aligned}$$

then M is a screen generic lightlike submanifold.

Theorem 3.8. *There exist no coisotropic, isotropic or totally lightlike proper screen generic lightlike submanifold M of a semi-Riemannian product manifold $\bar{M}$. Any screen generic isotropic, coisotropic or totally lightlike submanifold M is an invariant submanifold.*

Proof. Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. If M is isotropic, then $S(TM) = \{0\}$ which implies that $D_0 = \{0\}$ and $D^{'} = \{0\}$. Hence we get $TM = Rad(TM) = F(Rad(TM))$, which is invariant respect to F.

If M is coisotropic, then $S(TM^{\perp}) = 0$ implies $\mu = 0$ and the real parts of $\omega(D^{'}) = \{0\}$. Thus, $TM = D_0 \oplus f(D^{'}) \oplus Rad(TM)$ and M is invariant.

Finally, if M is totally lightlike, then $S(TM) = \{0\}$ and $S(TM^{\perp}) = \{0\}$. Hence, $TM = Rad(TM)$, which implies M is invariant.

So, it is clear that there exist no coisotropic, isotropic or totally lightlike proper screen generic lightlike submanifolds and the proof is completed. □

Theorem 3.9. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. If ∇ is a metric connection, then $h^s(X, FY)$ has no components in $\omega D'$. Conversely, if*

$$\bar{g}(h^l(X, fU), FY) = \bar{g}(h^s(X, FY), \omega U) \tag{3.6}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$, then the induced connection ∇ is a metric connection.

Proof. Suppose that ∇ is a metric connection. From (2.1) and (2.4) we have

$$\bar{\nabla}_X Y = F(\bar{\nabla}_X FY) \tag{3.7}$$

for any $X \in \Gamma(TM)$, and $Y \in \Gamma(Rad(TM))$. Using (2.15) and (2.22) we obtain

$$\bar{\nabla}_X Y = F(-A^*_{FY} X + \nabla^{*t}_X FY + h^l(X, FY) + h^s(X, FY)). \tag{3.8}$$

Using (3.4) and (3.5) in (3.8) and taking the tangential parts of this equation, we get

$$\nabla_X Y = -f A^*_{FY} X + f\nabla^{*t}_X FY + Bh^s(X, FY). \tag{3.9}$$

From Theorem 2.2 we know that induced connection ∇ is a metric connection if and only if $Rad(TM)$ is a parallel distribution. Asume that $Rad(TM)$ is parallel, then $g(\nabla_X Y, U) = 0$. From the above equation, we derive

$$g(\nabla_X Y, U) = \bar{g}(h^s(X, FY), \omega U) \tag{3.10}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$. Thus $h^s(X, FY)$ has no components in $\omega D'$.

Conversely, we suppose that

$$\bar{g}(h^l(X, fU), FY) = \bar{g}(h^s(X, FY), \omega U) \tag{3.11}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$. On the other hand, from (2.1) and (2.4) we have

$$\bar{g}(\bar{\nabla}_X FY, FU) = \bar{g}(\bar{\nabla}_X Y, U). \tag{3.12}$$

Using (2.15), (2.22), (3.4) and (3.5), we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_X FY, FU) &= \bar{g}(-A^*_{FY} X + \nabla^{*t}_X FY + h^l(X, FY) + h^s(X, FY), FU) \\ &= -\bar{g}(A^*_{FY} X, fU) + \bar{g}(h^s(X, FY), wU) \end{aligned} \tag{3.13}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM)$. Using (3.11), (3.12) and (3.13), we get

$$g(\nabla_X Y, U) = 0$$

which shows that $\nabla_X Y \in \Gamma(Rad(TM))$. This completes the proof. □

Theorem 3.10. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D_0 is integrable if and only if the followings hold:*

$$g(\nabla^*_X FY - \nabla^*_Y FX, fZ) = g(B(h^s(X, FY) - h^s(Y, FX)), Z), \tag{3.14}$$

$$h^*(X, FY) = h^*(Y, FX), \tag{3.15}$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D')$. Also, D is integrable if and only if (3.14) holds.

Proof. From definition of screen generic lightlike submanifold, D_0 is integrable iff for any $X, Y \in \Gamma(D_0)$, $[X, Y] \in \Gamma(D_0)$, that is,

$$g([X, Y], Z) = \bar{g}([X, Y], N) = 0$$

$Z \in \Gamma(D')$ and $N \in \Gamma(ltr(TM))$.

Using that $\bar{\nabla}$ is a metric connection and (2.2), (2.3), (2.4), (2.15), (2.21), (3.4) and (3.5), we derive

$$g([X, Y], Z) = \bar{g}(\nabla^*_X FY - \nabla^*_Y FX, fZ) - g(B(h^s(X, FY) - h^s(Y, FX)), Z)$$

$$g([X, Y], N) = \bar{g}(h^*(X, FY) - h^*(Y, FX), FN)$$

which hold (3.14) and (3.15). This completes proof. □

Theorem 3.11. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then, the distribution $D^{'}$ is integrable iff*

$$\nabla_Z fW - \nabla_W fZ - A_{\omega W}Z + A_{\omega Z}W \in \Gamma(D^{'}) \tag{3.16}$$

for any $Z, W \in \Gamma(D^{'})$.

Proof. From the definition of screen generic lightlike submanifold, $D^{'}$ is integrable iff

$$g([Z, W], X) = \bar{g}([Z, W], N) = 0$$

for any $Z, W \in \Gamma(D^{'})$, $X \in \Gamma(D_0)$ and $N \in \Gamma(ltr(TM))$. Considering (2.2), (2.15), (2.17) and (3.4), we get

$$g([Z, W], X) = g(\nabla_Z fW - \nabla_W fZ - A_{\omega W}Z + A_{\omega Z}W, FX).$$

From last equation it is easy to see that

$$\nabla_Z fW - \nabla_W fZ - A_{\omega W}Z + A_{\omega Z}W \tag{3.17}$$

has no components on $\Gamma(D_0)$. Similarly, from (2.2), (2.15), (2.17) and (3.4) we have

$$g([Z, W], N) = \bar{g}(\nabla_Z fW - \nabla_W fZ - A_{\omega W}Z + A_{\omega Z}W, FN).$$

Thus,

$$\nabla_Z fW - \nabla_W fZ - A_{\omega W}Z + A_{\omega Z}W \tag{3.18}$$

has no components on $\Gamma(Rad(TM))$. From (3.17) and (3.18), it is clear that $D^{'}$ is integrable iff $\nabla_Z fW - \nabla_W fZ - A_{\omega W}Z + A_{\omega Z}W \in \Gamma(D^{'})$. This completes the proof. □

Theorem 3.12. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the distribution D is parallel iff*

$$\nabla^*_X fZ - A_{\omega Z}X \text{ has no components on } \Gamma(D_0), \tag{3.19}$$

$$h^l(X, fZ) = -D^l(X, \omega Z), \tag{3.20}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'})$.

Proof. Since D is invariant, if $Y \in \Gamma(D)$, then $FY \in \Gamma(D)$. From the definition of screen generic lightlike submanifold, D is parallel iff

$$g(\nabla_X FY, Z) = 0$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{'})$. Considering that $\bar{\nabla}$ is a metric connection and (2.3), (2.4), (2.15), (2.17) and (3.4), we get

$$g(\nabla_X FY, Z) = -\bar{g}(\nabla_X fZ + h^l(X, fZ) - A_{\omega Z}X + D^l(X, \omega Z), Y).$$

This completes the proof. □

Theorem 3.13. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then $D^{'}$ is parallel if and only if*

$$\nabla_Z fW - A_{\omega W}Z \in \Gamma(D^{'}) \tag{3.21}$$

for any $Z, W \in \Gamma(D^{'})$.

Proof. We assume that $D^{'}$ is a parallel distribution. Then, for any $Z, W \in \Gamma(D^{'})$, $\nabla_Z W \in \Gamma(D^{'})$. In the other words, for any $X \in \Gamma(D_0)$ and $N \in \Gamma(ltr(TM))$,

$$g(\nabla_Z W, X) = \bar{g}(\nabla_Z W, N) = 0.$$

Using (2.2), (2.15), (2.17) and (3.4), we derive

$$g(\nabla_Z W, X) = \bar{g}(\nabla_Z fW - A_{\omega W} Z, FX)$$

and then

$$\nabla_Z fW - A_{\omega W} Z \tag{3.22}$$

has no components on $\Gamma(D_0)$. Similary, we obtain

$$\bar{g}(\nabla_Z W, N) = \bar{g}(\nabla_Z fW - A_{\omega W} Z, FN)$$

and from this equation, it is clear that

$$\nabla_Z fW - A_{\omega W} Z \tag{3.23}$$

has no components on $\Gamma(Rad(TM))$. Therefore, using (3.22) and (3.23), we have that $D^{'}$ is a parallel iff $\nabla_Z fW - A_{\omega W} Z \in \Gamma(D^{'})$. This completes the proof. □

Definition 3.14. We say that M is a D-geodesic screen generic lightlike submanifold if its second fundamental form h satisfies

$$h(X, Y) = 0, \tag{3.24}$$

for any $X, Y \in \Gamma(D)$. It is easy to see that M is a D-geodesic screen generic lightlike submanifold if

$$h^l(X, Y) = h^s(X, Y) = 0, \tag{3.25}$$

for any $X, Y \in \Gamma(D)$. On the other hand, if h satisfies

$$h(X, Y) = 0, \tag{3.26}$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{'})$, then M is called a mixed geodesic screen generic lightlike submanifold.

Proposition 3.15. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the distribution D is a totally geodesic foliation in $\bar{M}$ iff M is D-geodesic and D is parallel respect to ∇ on M.*

Proof. Assume that D defines a totally geodesic foliation in $\bar{M}$, that is, $\bar{\nabla}_X Y \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. Then, we have

$$\bar{g}(\bar{\nabla}_X Y, \xi) = \bar{g}(\bar{\nabla}_X Y, W) = \bar{g}(\bar{\nabla}_X Y, Z) = 0$$

for any $\xi \in \Gamma(Rad(TM))$, $Z \in \Gamma(D^{'})$ and $W \in \Gamma(S(TM^{\perp}))$. From (2.15), we derive

$$\begin{aligned} \bar{g}(\bar{\nabla}_X Y, \xi) &= \bar{g}(h^l(X, Y), \xi), \\ \bar{g}(\bar{\nabla}_X Y, W) &= \bar{g}(h^s(X, Y), W). \end{aligned}$$

Then it is clear that for any $X, Y \in \Gamma(D)$, $h^l(X, Y) = h^s(X, Y) = 0$. In other words, M is D-geodesic and D is parallel respect to ∇ on M.
Conversely, we suppose that M is D-geodesic and D is parallel respect to ∇ on M. Since $h^l(X, Y) = h^s(X, Y) = 0$ for any $X, Y \in \Gamma(D)$, then $\bar{\nabla}_X Y \in \Gamma(TM)$. On the other hand, since D is parallel on M, considering (2.15), we have $\bar{\nabla}_X Y \in \Gamma(D)$. This completes the proof. □

Theorem 3.16. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then, M is mixed geodesic iff the following conditions hold:*

(i) $D^l(X, \omega Z) = -h^l(X, fZ)$,
(ii) $g(A_{\omega Z} X - \nabla_X fZ, BW) = \bar{g}(h^s(X, fZ) + \nabla^s_X \omega Z, CW)$,

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'})$ and $W \in \Gamma(S(TM^{\perp}))$.

Proof. If M is mixed geodesic, then from (3.26), $\bar{g}(h^l(X,Z),\xi) = 0$ and $\bar{g}(h^s(X,Z),W) = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'})$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Thus from (2.15), we derive

$$\bar{g}(\bar{\nabla}_X Z, \xi) = 0.$$

Since distribution $Rad(TM)$ is invariant, we can replace $F\xi$ with ξ. Then we get

$$\bar{g}(\bar{\nabla}_X Z, F\xi) = 0.$$

Considering (2.3), (2.15), (2.17) and (3.4) in the last equation, we obtain

$$\bar{g}(h^l(X, fZ) + D^l(X, \omega Z), \xi) = 0. \tag{3.27}$$

Similarly, from (2.2) it is easy to derive

$$\bar{g}(\bar{\nabla}_X FZ, FW) = 0$$

and using (2.15), (2.17), (3.4) and (3.5) in the last equation, we have

$$g(\nabla_X fZ - A_{\omega Z}X, BW) + \bar{g}(h^s(X, fZ) + \nabla^s_X\ \omega Z, CW) = 0. \tag{3.28}$$

Thus the proof follows from (3.27) and (3.28). □

Proposition 3.17. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is mixed geodesic iff*

(i) $D^l(X, \omega Z) = -h^l(X, fZ)$,
(ii) $\omega Q(A_{\omega Z}X - \nabla_X fZ) = C(h^s(X, fZ) + \nabla^s_X \omega Z)$,

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'})$.

Proof. Considering (2.1), (2.4), (2.15), (2.17) and (3.4), we obtain for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'})$,

$$\begin{aligned} h(X,Z) &= F(\nabla_X fZ + h^l(X, fZ) + h^s(X, fZ) \\ &\quad -A_{\omega Z}X + \nabla^s_X \omega Z + D^l(X, \omega Z)) - \nabla_X Z. \end{aligned}$$

Using (3.3)-(3.5) and taking transversal part of this equation, we have

$$\begin{aligned} h(X,Z) &= \omega Q(A_{\omega Z}X - \nabla_X fZ) - C(h^l(X, fZ) + D^l(X, \omega Z)) \\ &\quad -C(h^s(X, fZ) + \nabla^s_X \omega Z). \end{aligned}$$

Hence $h(X,Z) = 0 \Leftrightarrow$ (i) and (ii) are satisfied. □

Proposition 3.18. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then, for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$, we have*

$$\nabla_X Z = -fA_{\omega Z}X + f\nabla_X fZ + Bh^s(X, fZ) - B\nabla^s_X\ \omega Z + Bh^s(X, Z).$$

Proof. Using (2.1), (2.15), (2.17), (3.4) and (3.5), we have

$$\begin{aligned} \bar{\nabla}_X Z &= f\nabla_X fZ - +Bh^s(X, fZ) - fA_{\omega Z}X - B\nabla^s_X \omega Z \\ &\quad +\omega\nabla_X fZ + Ch^s(X, fZ) + \omega A_{\omega Z}X - C\nabla^s_X\ \omega Z \\ &\quad +Ch^l(X, fZ) - CD^l(X, \omega Z) \end{aligned}$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$. If we take tangential parts of last equation, then we obtain

$$\nabla_X Z = -fA_{\omega Z}X + f\nabla_X fZ + Bh^s(X, fZ) - B\nabla^s_X\ \omega Z + Bh^s(X, Z).$$

Then proof follows from last equation. □

4. Minimal screen generic lightlike submanifolds

Definition 4.1. We say that a lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:

(i) $h^s = 0$ on $Rad(TM)$ and

(ii) $traceh = 0$, where trace is written with respect to g restricted to $S(TM)$.

In Case 2, condition (i) is trivial. It has been shown in [2] that the above definition is independent of $S(TM)$ and $S(TM^\perp)$, but it depends on $tr(TM)$.

Example 4.2. Let $(\tilde{M} = \mathbb{R}^9_2, \tilde{g})$ be a 9-dimensional semi-Euclidean space with signature $(-,-,+,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be the standard coordinate system of $\mathbb{R}^9_2$. If we define a mapping F by

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = (x_1, x_2, x_3 \cos\alpha + x_4 \sin\alpha, x_3 \sin\alpha - x_4 \cos\alpha, x_5 \cos\alpha + x_6 \sin\alpha, x_5 \sin\alpha - x_6 \cos\alpha, x_7 \cos\alpha + x_8 \sin\alpha, x_7 \sin\alpha - x_8 \cos\alpha, x_9)$$

then $F^2 = I$ and F is a product structure on $\mathbb{R}^9_2$. Let M be a submanifold of $\tilde{M}$ given by

$$\begin{aligned} x_1 &= u_1 \sinh\alpha + u_2 \cosh\alpha, x_2 = u_1, \\ x_3 &= \sin u_3 \sinh u_4, x_5 = \sin u_3 \cosh u_4, x_4 = 0, x_6 = 0, \\ x_7 &= \sqrt{2} \cos u_3 \cosh u_4, x_8 = 0, x_9 = u_1 \cosh\alpha + u_2 \sinh\alpha, \end{aligned}$$

where u_i, $1 \le i \le 4$, are real parameters. Thus $TM = Span\{Z_1, Z_2, Z_3, Z_4\}$, where

$$\begin{aligned} Z_1 &= \sinh\alpha \partial x_1 + \partial x_2 + \cosh\alpha \partial x_9, Z_2 = \cosh\alpha \partial x_1 + \sinh\alpha \partial x_9, \\ Z_3 &= \cos u_3 \sinh u_4 \partial x_3 + \cos u_3 \cosh u_4 \partial x_5 - \sqrt{2} \sin u_3 \cosh u_4 \partial x_7, \\ Z_4 &= \sin u_3 \cosh u_4 \partial x_3 + \sin u_3 \sinh u_4 \partial x_5 + \sqrt{2} \cos u_3 \sinh u_4 \partial x_7, \end{aligned}$$

where $Rad(TM) = Span\{Z_1\}$ and $D_0 = Span\{Z_2\}$. By direct calculation, we derive that $ltr(TM)$ is spanned by

$$N = \frac{1}{2}(\sinh\alpha \partial x_1 - \partial x_2 + \cosh\alpha \partial x_9).$$

Also, the screen transversal bundle is spanned by

$$\begin{aligned} W_1 &= \cos u_3 \sinh u_4 \partial x_4 + \cos u_3 \cosh u_4 \partial x_6 - \sqrt{2} \sin u_3 \cosh u_4 \partial x_8, \\ W_2 &= \sin u_3 \cosh u_4 \partial x_4 + \sin u_3 \sinh u_4 \partial x_6 + \sqrt{2} \cos u_3 \sinh u_4 \partial x_8, \\ W_3 &= -\sqrt{2} \sinh u_4 \cosh u_4 \partial x_4 + \sqrt{2}(\sin^2 u_3 + \sinh^2 u_4) \partial x_6 + \sin u_3 \cos u_3 \partial x_8, \\ W_4 &= -\sqrt{2} \sinh u_4 \cosh u_4 \partial x_3 + \sqrt{2}(\sin^2 u_3 + \sinh^2 u_4) \partial x_5 + \sin u_3 \cos u_3 \partial x_7. \end{aligned}$$

Since $FW_1 \neq W_2$, then it is easy to see that $\mu = Span\{W_3, W_4\}$,

$$\begin{aligned} FZ_3 &= \cos\alpha Z_3 + \sin\alpha W_1\,, \\ FZ_4 &= \cos\alpha Z_4 + \sin\alpha W_2. \end{aligned}$$

Then $D' = Span\{Z_3, Z_4\}$ and M is a screen generic lightlike submanifold of $\mathbb{R}^9_2$.

On the other hand, by direct computations and using Gauss and Weingarten formulas, we obtain

$$\bar{\nabla}_{Z_i} Z_T = 0\,, \quad i = 1, 2\,, \quad 1 \le T \le 4$$

and

$$\begin{aligned} h(Z_1, Z_1) &= 0\,, \; h(Z_2, Z_2) = 0, \\ h^s(Z_3, Z_3) &= -\frac{\sqrt{2} \sin u_3 \cosh u_4}{(\sin^2 u_3 + 2\sinh^2 u_4)(1 + \sin^2 u_3 + 2\sinh^2 u_4)} W_4, \\ h^s(Z_4, Z_4) &= \frac{\sqrt{2} \sin u_3 \cosh u_4}{(\sin^2 u_3 + 2\sinh^2 u_4)(1 + \sin^2 u_3 + 2\sinh^2 u_4)} W_4, \end{aligned}$$

that is, $h^s = 0$ on $Rad(TM)$ and

$$trace \mid_{S(TM)} h = 0.$$

Then, it is clear that M is not totally geodesic and, but it is a minimal screen generic lightlike submanifold of $\bar{M} = \mathbb{R}^9_2$.

Theorem 4.3. *Let M be a screen generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is minimal iff*

$$traceA^*_{\xi_k} \mid_{S(TM)} = traceA_{W_T} \mid_{S(TM)} = 0$$

where $\dim(TM) = m$, $\dim(tr(TM)) = n$, $\dim(Rad(TM)) = r$ *and* $W_T \in \Gamma(S(TM^{\perp}))$.

Proof. We take an quasi orthonormal frame $\{\xi_1, ..., \xi_r, e_1, ..., e_{m-r}, W_1, ..., W_n, N_1, ..., N_r\}$ such that $\{e_1, ..., e_a\}$ are tangent to D_0 and $\{e_{a+1}, ..., e_{m-r}\}$ are tangent to D'. First from (cf. [2], page 140), we know that $h^l = 0$ on $RadTM$.

From definition of minimal submanifold, we know that

$$\begin{aligned} traceh \mid_{S(TM)} &= traceh \mid_{D_0} + traceh \mid_{D'} \\ &= \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{j=1}^{b} h(U_j, U_j) \\ &= 0 \end{aligned}$$

and $h^s \mid_{Rad(TM)} = 0$. If we choose an orthonormal basis of $S(TM)$ as $\{e_i\}_{i=1}^{m-r}$, then we get

$$\begin{aligned} traceh \mid_{S(TM)} &= \sum_{i=1}^{a} \varepsilon_i \left[h^l(e_i, e_i) + h^s(e_i, e_i)\right] + \sum_{j=a+1}^{m-r} \varepsilon_j \left[h^l(e_j, e_j) + h^s(e_j, e_j)\right] \\ &= \sum_{i=1}^{a} \varepsilon_i [\frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(e_i, e_i), \xi_k) N_k + \frac{1}{n-r} \sum_{T=1}^{n-r} \bar{g}(h^s(e_i, e_i), W_T) W_T] \\ &\quad + \sum_{J=1}^{b} \varepsilon_j [\frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(e_j, e_j), \xi_k) N_k + \frac{1}{n-r} \sum_{T=1}^{n-r} \bar{g}(h^s(e_j, e_j), W_T) W_T]. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \bar{g}(h^l(e_i, e_i), \xi_k) N_k &= g(A^*_{\xi_k} e_i, e_i) N_k \text{ and} \\ \bar{g}(h^s(e_i, e_i), W_T) W_T &= g(A_{W_T} e_i, e_i) W_T, \end{aligned}$$

we derive

$$traceh \mid_{S(TM)} = traceA^*_{\xi_k} \mid_{D_0 \oplus D'} + traceA_{W_T} \mid_{D_0 \oplus D'} .$$

Therefore we get

$$traceA^*_{\xi_k} \mid_{D_0 \oplus D'} = 0 \text{ and } traceA_{W_T} \mid_{D_0 \oplus D'} = 0,$$

which proves assertion. □

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