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Research Article

Directs estimates and a Voronovskaja-type formula for Mihesan operators

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ABSTRACT. We present an estimate for the rate of convergence of Mihesan operators in polynomial weighted spaces. A Voronovskaja-type theorem is included.

Keywords: Mihesan operators, rate of convergence, polynomial weighted spaces, Voronovskaja-type theorems.

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1. INTRODUCTION

For $|t| < 1$, let us consider the expansion

$$\frac{e^{at}}{(1-t)^y} = \sum_{k=0}^{\infty} p_{k,a}(y)\frac{t^k}{k!}, \qquad p_{k,a}(y) = \sum_{i=0}^{k}\binom{k}{i}(y)_i a^{k-i}. \tag{1.1}$$

Recall that, for $i \in \mathbb{N}$, $(x)_i = x(x+1)\cdots(x+i-1)$, while $(x)_0 = 1$. If we take $y = n$ and $t = x/(x+1)$, then

$$e^{ax/(x+1)}(1+x)^n = \sum_{k=0}^{\infty}\frac{p_{k,a}(n)}{k!}\left(\frac{x}{1+x}\right)^k.$$

For $a \geq 0$, Mihesan [9] defined

$$B_n^a(f,x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n}\right), \quad \text{where} \quad W_{n,k}^a(x) = e^{-ax/(x+1)}\frac{p_{k,a}(n)x^k}{k!(1+x)^{n+k}}.$$

We also write $B_n^a(f(t),x)$ instead of $B_n^a(f,x)$. Notice that, for $a = 0$, $B_n^a(f)$ is just a Baskakov operator.

In this paper, we present a Voronovsakaja type result for the operators B_n^a in a weighted space $C_\varrho[0,\infty)$ defined as follows: for the weight $\varrho(x) = 1/(1+x)^q$ ($q \geq 0$ a fixed real),

$$C_\varrho[0,\infty) = \Big\{ f \in C[0,\infty) \,:\, \|f\|_\varrho < \infty \Big\},$$

where $\|f\|_\varrho = \sup_{x\geq 0}|\varrho(x)f(x)|$. In order to present a simple proof, here we only consider the case $q \geq 3/2$.

It is known (see [9]) that, if $f \in C[0,\infty)$ and there exist positive constants A and B such that $|f(x)| \leq Be^{Ax}$, then $B_n^a(f)$ is well defined. Hence, $B_n^a(f)$ is defined for all $f \in C_\varrho[0,\infty)$.

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In [11] and [13] some pointwise asymptotic expansions were given for Mihesan operators. But we remark that the results are not correct. For instance, in Theorem 4.1 of [11] and in Theorem 2.2 in [13] a term $g'(x)B_n^a(t-x,x)$ should be added.

The paper is organized as follows. In Section 2 we recall some known results. Section 3 is very technical. We need estimates for the moments of the operators $M_{n,k}^a(x)$ for $1 \le k \le 6$. Finally, in Section 5 we present the Voronovskaja type theorem. We remark that some different quantitative Voronovskaja theorems were given in [6] and [7]. An inverse result will appear in another paper.

In what follows C and C_i will denote absolute constants. They may be different on each occurrence.

2. Known results

It is known that (see [9])

$$B_n^a(1,x) = 1, \qquad B_n^a(t,x) = x + \frac{ax}{n(1+x)}, \tag{2.2}$$

and

$$B_n^a((t-x)^2,x) = \frac{\varphi^2(x)}{n} + \frac{1}{n^2}\frac{ax}{(1+x)}\frac{(a+1)x+1}{(1+x)} = \frac{\varphi^2(x)}{n}\Big(1+\frac{a}{n(1+x)^2}+\frac{a^2x}{n(1+x)^3}\Big). \tag{2.3}$$

It was verified in [10] that

$$\varphi^2(x)\Big(\frac{d}{dx}W_{n,k}^a(x)\Big) = \Big(k - nx - \frac{ax}{x+1}\Big)W_{n,k}^a(x). \tag{2.4}$$

Theorem 2.1. (see [4]) *If $a \ge 0$ and $q \ge 0$ are real numbers, there exists a constant $M_q(a)$ such that*

$$M_a(q) := \sup_{n>1}\sup_{x\ge 0}\frac{B_n^a((1+t)^q,x)}{(1+x)^q} < \infty.$$

Proposition 2.1. (see [4]) *If $a > 0$, $r \in [0,1]$, there exists a constant C such that for each integer $n > 1$ and each $x \ge 0$,*

$$B_n^a\Big(\frac{1}{(1+t)^r},x\Big) \le \Big(\frac{n}{n-1}\Big)^r\frac{1}{(1+x)^r}.$$

Remark 2.1. It is known a similar result for $a = 0$. We can use the arguments in [4] to verify that, if $a \ge 0$ and $r \in [0,2]$, there exists a constant $C = C(a,r)$ such that, for $n > 2$,

$$B_n^a\Big(\frac{1}{(1+t)^r},x\Big) \le \frac{C}{(1+x)^r} \tag{2.5}$$

(see also [5]).

Lemma 2.1. (see [8, Prop. 3.3]) *Assume $r \ge 0$, $m,p \in \mathbb{R}$, and $m-r+1>0$. Then for $x > 0$ and $t \ge 0$, one has*

$$\Big|\int_x^t \frac{(t-s)^m}{s^r}(1+u)^p ds\Big| \le \frac{|t-x|^{m+1}}{(m-r+1)x^r}\Big((1+x)^p + (1+t)^p\Big).$$

3. Estimates for the moments

The moment of order $j \in \mathbb{N}_0$ of the operator B_n^a is defined by

$$M_{n,j}^a(x) = B_n^a((t-x)^j, x).$$

In this work, we need estimates of $M_{n,j}^a(x)$ for $0 \le j \le 6$. We remark that in Lemma 2.3 of [13] some computations were given for $M_{n,3}^a(x)$ and $M_{n,4}^a(x)$, but they are complicated. Here, we follow a different approach. In Lemma 1 of [2], it is asserted that $M_{n,j}^a(x) = O(n^{-[(j+1)/2]})$, where $[\alpha]$ denotes the integer part of α, but the estimate is not correct (see (2.3)). A similar assertion was given in [1], where the authors defined the Mihesan operators in the form

$$B_n^a(f,x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\Big(\frac{k}{n+1}\Big).$$

Notice that

$$M_{n,0}^a(x) = 1 \qquad \text{and} \qquad M_{n,1}^a(x) = \frac{ax}{n(1+x)}. \tag{3.6}$$

We can use an iterative process to obtain representation for other moments of the operators.

Lemma 3.2. *If $a \ge 0$, $j, n \in \mathbb{N}$, $n > 1$, and $x \ge 0$, then*

$$M_{n,j+1}^a(x) = \frac{\varphi^2(x)}{n}\Big(jM_{n,j-1}^a(x) + \frac{a}{(1+x)^2} M_{n,j}^a(x) + \frac{d}{dx} M_{n,j}^a(x)\Big).$$

Proof. Taking into account (2.4), one has

$$\begin{aligned}
M_{n,j+1}^a(x) &= B_n^a((t-x)(t-x)^j, x)\\
&= \sum_{k=0}^{\infty}\Big(\frac{k}{n}-x\Big)\Big(\frac{k}{n}-x\Big)^j W_{n,k}^a(x)\\
&= \frac{1}{n}\frac{ax}{x+1} M_{n,j}^a(x) + \frac{1}{n}\sum_{k=0}^{\infty}\Big(k - nx - \frac{ax}{x+1}\Big)\Big(\frac{k}{n}-x\Big)^j W_{n,k}^a(x)\\
&= \frac{1}{n}\frac{ax}{x+1} M_{n,j}^a(x) + \frac{\varphi^2(x)}{n}\sum_{k=0}^{\infty}\Big(\frac{k}{n}-x\Big)^j \frac{d}{dx} W_{n,k}^a(x)\\
&= \frac{1}{n}\frac{ax}{x+1} M_{n,j}^a(x) + \frac{j\varphi^2(x)}{n} M_{n,j-1}^a(x) + \frac{\varphi^2(x)}{n}\frac{d}{dx} M_{n,j}^a(x)\\
&= \frac{\varphi^2(x)}{n}\Big(\frac{a}{(1+x)^2} M_{n,j}^a(x) + jM_{n,j-1}^a(x) + \frac{d}{dx} M_{n,j}^a(x)\Big).
\end{aligned}$$

□

Since in Lemma 3.2 the derivative appears, in order to estimate $M_{n,6}^a(x)$ we should study other derivatives of the previous moments.

Lemma 3.3. *Assume $a \ge 0$. There exists a constant C such that, if $n > 1$ and $x \ge 0$, then*

$$M_{n,2}^a(x) \le C\frac{\varphi^2(x)}{n}, \qquad \Big|\frac{d}{dx} M_{n,2}^a(x)\Big| \le C\frac{1+x}{n}, \qquad \Big|\frac{d^2}{dx^2} M_{n,2}^a(x)\Big| \le \frac{C}{n},$$

$$\Big|\frac{d^3}{dx^3} M_{n,2}^a(x)\Big| \le \frac{C}{n^2(1+x)^4} \qquad \text{and} \qquad \Big|\frac{d^4}{dx^4} M_{n,2}^a(x)\Big| \le \frac{C}{n^2(1+x)^5}.$$

Proof. It follows from (2.3) that

$$\frac{d}{dx}M_{n,2}^a(x) = \frac{1}{n}\Big(1+2x+\frac{a}{n(1+x)^2}+\frac{2a^2x}{n(1+x)^3}\Big).$$

Hence

$$0 \le \frac{d}{dx}M_{n,2}^a(x) \le \frac{1+2x+a(1+2ax)}{n} \le \frac{C(1+x)}{n}.$$

The estimates for the other derivatives follow from the identities

$$\frac{d^2}{dx^2}M_{n,2}^a(x) = \frac{1}{n}\Big(2-\frac{2a}{n(1+x)^3}+\frac{2a^2(1-2x)}{n(1+x)^4}\Big),$$
$$\frac{d^3}{dx^3}M_{n,2}^a(x) = \frac{1}{n^2}\Big(\frac{6a}{(1+x)^4}-\frac{4a^2(1-x)}{(1+x)^5}\Big),$$
$$\frac{d^4}{dx^4}M_{n,2}^a(x) = \frac{1}{n^2}\Big(\frac{4!\,a}{(1+x)^5}+\frac{4a^2(6-4x)}{(1+x)^6}\Big).$$

□

Assume $nx \ge 1$. Notice that $1+x \le nx+x = (n+1)x$. Hence $(1+x)^2 \le (n+1)\varphi^2(x)$ and

$$\frac{1+x}{n} \le \frac{\sqrt{n+1}\varphi(x)}{n} \le \sqrt{2}\frac{\varphi(x)}{\sqrt{n}}.$$

Lemma 3.4. *Suppose that $a \ge 0$. There exists a constant C such that, for $n > 1$ and $x \ge 0$, one has*

$$|M_{n,3}^a(x)| \le C\frac{(1+x)\varphi^2(x)}{n^2}, \qquad \Big|\frac{d}{dx}M_{n,3}^a(x)\Big| \le \frac{C\varphi^2(x)}{n^2},$$

$$\Big|\frac{d^2}{dx^2}M_{n,3}^a(x)\Big| \le \frac{C(1+x)}{n^2} \quad \textit{and} \quad \Big|\frac{d^3}{dx^3}M_{n,3}^a(x)\Big| \le \frac{C}{n^2}.$$

Proof. From Lemma 3.2, one has

$$M_{n,3}^a(x) = \frac{\varphi^2(x)}{n}\Big\{2M_{n,1}^a(x)+\frac{a}{(1+x)^2}M_{n,2}^a(x)+\frac{d}{dx}M_{n,2}^a(x)\Big\}$$
$$= \frac{\varphi^2(x)}{n}\Big\{\frac{2ax}{n(1+x)}+\frac{a}{(1+x)^2}M_{n,2}^a(x)+\frac{d}{dx}M_{n,2}^a(x)\Big\}.$$

Taking into account (2.3) and Lemma 3.3, we obtain

$$|M_{n,3}^a(x)| \le C_1\frac{\varphi^2(x)}{n}\Big\{\frac{x}{n(1+x)}+\frac{x}{n(1+x)}+\frac{1+x}{n}\Big\} \le C_2\frac{(1+x)\varphi^2(x)}{n^2}.$$

Moreover, for $x \ge 0$,

$$n\frac{d}{dx}M_{n,3}^a(x) = \frac{d}{dx}\Big\{\frac{2ax^2}{n}+\frac{ax}{(1+x)}M_{n,2}^a(x)+\varphi^2(x)\frac{d}{dx}M_{n,2}^a(x)\Big\}$$
$$= \frac{4ax}{n}+\frac{a}{(1+x)^2}M_{n,2}^a(x)+\frac{ax}{(1+x)}\frac{d}{dx}M_{n,2}^a(x)+\varphi^2(x)\frac{d^2}{dx^2}M_{n,2}^a(x).$$

Hence,

$$\Big|\frac{d}{dx}M_{n,3}^a(x)\Big| \le \frac{C_1\varphi^2(x)}{n}\Big(\frac{4}{n^2}+\frac{1}{n(1+x)^2}+\frac{a}{(1+x)}\frac{1}{n}+\frac{1}{n}\Big) \le \frac{C_2}{n^2}.$$

On the other hand

$$n\frac{d^2}{dx^2}M^a_{n,3}(x) = \frac{4a}{n} - \frac{2a}{(1+x)^3}M^a_{n,2}(x) + \frac{2a}{(1+x)^2}\frac{d}{dx}M^a_{n,2}(x) + \Big(\frac{ax}{(1+x)} + (1+2x)\Big)\frac{d^2}{dx^2}M^a_{n,2}(x) + \varphi^2(x)\frac{d^3}{dx^3}M^a_{n,2}(x).$$

Therefore,

$$\Big|\frac{d^2}{dx^2}M^a_{n,3}(x)\Big| \le \frac{C_1}{n}\Big(\frac{1}{n} + \frac{\varphi^2(x)}{n(1+x)^3} + \frac{1+x}{n(1+x)^2} + \frac{1+x}{n} + \frac{\varphi^2(x)}{n^2(1+x)^4}\Big) \le \frac{C_2(1+x)}{n^2}.$$

Finally

$$n\frac{d^3}{dx^3}M^a_{n,3}(x) = \frac{6aM^a_{n,2}(x)}{(1+x)^4} - \frac{6a}{(1+x)^3}\frac{d}{dx}M^a_{n,2}(x)) + \varphi^2(x)\frac{d^4}{dx^4}M^a_{n,2}(x) + \Big(\frac{3a}{(1+x)^2} + 2\Big)\frac{d^2}{dx^2}M^a_{n,2}(x) + \Big(\frac{ax}{(1+x)} + 2(1+2x)\Big)\frac{d^3}{dx^3}M^a_{n,2}(x)$$

and we obtain

$$\Big|\frac{d^3}{dx^3}M^a_{n,3}(x)\Big| \le \frac{C_1}{n^2}\Big(\frac{\varphi^2(x)}{(1+x)^4} + \frac{(1+x)}{(1+x)^3} + 1 + \frac{1+x}{n(1+x)^4} + \frac{\varphi^2(x)}{n(1+x)^5}\Big) \le \frac{C_2}{n^2}.$$

□

Lemma 3.5. *Suppose that $a \ge 0$. There exists a constant C such that, for $n > 1$ and $x \ge 0$, one has*

$$0 \le M^a_{n,4}(x) \le C\frac{\varphi^4(x)}{n^2}, \quad \Big|\frac{d}{dx}M^a_{n,4}(x)\Big| \le \frac{C(1+x)\varphi^2(x)}{n^2} \quad \text{and} \quad \Big|\frac{d^2}{dx^2}M^a_{n,4}(x)\Big| \le \frac{C(1+x)^2}{n^2}.$$

Proof. First, we consider the representation

$$M^a_{n,4}(x) = \frac{\varphi^2(x)}{n}\Big(3M^a_{n,2}(x) + \frac{a}{(1+x)^2}M^a_{n,3}(x) + \frac{d}{dx}M^a_{n,3}(x)\Big).$$

It follows from Lemma 3.4 that

$$0 \le M^a_{n,4}(x) \le C_1\frac{\varphi^2(x)}{n}\Big(\frac{\varphi^2(x)}{n} + \frac{(1+x)\varphi^2(x)}{n^2(1+x)^2} + \frac{\varphi^2(x)}{n^2}\Big) \le C_2\frac{\varphi^4(x)}{n^2}.$$

For the first derivative, we obtain

$$\begin{aligned} n\frac{d}{dx}M^a_{n,4}(x) &= (1+2x)\Big(3M^a_{n,2}(x) + \frac{a}{(1+x)^2}M^a_{n,3}(x) + \frac{d}{dx}M^a_{n,3}(x)\Big) \\ &\quad + \varphi^2(x)\Big(3\frac{d}{dx}M^a_{n,2}(x) - \frac{2aM^a_{n,3}(x)}{(1+x)^3} + \frac{a}{(1+x)^2}\frac{d}{dx}M^a_{n,3}(x) + \frac{d^2}{dx^2}M^a_{n,3}(x)\Big) \\ &= 3(1+2x)M^a_{n,2}(x) + 3\varphi^2(x)\frac{d}{dx}M^a_{n,2}(x) + \frac{a}{(1+x)^2}M^a_{n,3}(x) \\ &\quad + \Big(1+2x + \frac{ax}{(1+x)}\Big)\frac{d}{dx}M^a_{n,3}(x) + \varphi^2(x)\frac{d^2}{dx^2}M^a_{n,3}(x). \end{aligned}$$

Hence

$$\Big|\frac{d}{dx}M^a_{n,4}(x)\Big| \le \frac{C_1}{n}\Big(\frac{(1+x)\varphi^2(x)}{n} + \frac{(1+x)\varphi^2(x)}{n^2} + \frac{(1+x)\varphi^2(x)}{n^2}\Big)$$
$$\le \frac{C_1(1+x)\varphi^2(x)}{n^2}.$$

For the second derivative, we consider the identity

$$\begin{aligned}
n\frac{d^2}{dx^2}M^a_{n,4}(x) &= 6M^a_{n,2}(x) + 6(1+2x)\frac{d}{dx}M^a_{n,2}(x) + 3\varphi^2(x)\frac{d^2}{dx^2}M^a_{n,2}(x)\\
&- \frac{2a}{(1+x)^3}M^a_{n,3}(x) + \Big(\frac{a}{(1+x)^2} + \Big(2 - \frac{a}{(1+x)^2}\Big)\frac{d}{dx}M^a_{n,3}(x)\\
&+ \Big((1+2x) - \frac{ax}{(1+x)} + (1+2x)\Big)\frac{d^2}{dx^2}M^a_{n,3}(x) + \varphi^2(x)\frac{d^3}{dx^3}M^a_{n,3}(x)\\
&= 6M^a_{n,2}(x) + 6(1+2x)\frac{d}{dx}M^a_{n,2}(x) + 3\varphi^2(x)\frac{d^2}{dx^2}M^a_{n,2}(x) - \frac{2a}{(1+x)^3}M^a_{n,3}(x)\\
&+ 2\frac{d}{dx}M^a_{n,3}(x) + \Big(2(1+2x) - \frac{ax}{(1+x)}\Big)\frac{d^2}{dx^2}M^a_{n,3}(x) + \varphi^2(x)\frac{d^3}{dx^3}M^a_{n,3}(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Big|\frac{d^2}{dx^2}M^a_{n,4}(x)\Big| &\le \frac{C_1}{n}\Big(\frac{\varphi^2(x)}{n} + \frac{(1+x)^2}{n} + \frac{\varphi^2(x)}{n}\\
&+ \frac{(1+x)\varphi^2(x)}{n^2(1+x)^3} + \frac{\varphi^2(x)}{n^2} + \frac{(1+x)^2}{n^2} + \frac{\varphi^2(x)}{n^2}\Big)\\
&= \frac{C_1(1+x)}{n^2}\Big(2x + (1+x) + \frac{1}{n} + \frac{2x}{n} + \frac{(1+x)}{n}\Big) \le \frac{C_2(1+x)^2}{n^2}.
\end{aligned}$$

□

Corollary 3.1. *Suppose that $a \ge 0$. There exists a constant C such that, if $n > 1$ and $x \ge 0$, then*

$$B^a_{n,3}(|t-x|^3, x) \le C\frac{\varphi^3(x)}{n^{3/2}}.$$

Proof. It follows from the inequalities

$$B^a_{n,3}(|t-x|^3, x) \le \Big(M^a_{n,2}(x)M^a_{n,4}(x)\Big)^{1/2} \le C_1\frac{\varphi(x)}{\sqrt{n}}\frac{\varphi^2(x)}{n}.$$

□

Lemma 3.6. *Suppose that $a \ge 0$. There exists a constant C such that, if $n > 1$ and $x \ge 0$, then*

$$|M^a_{n,5}(x)| \le C\frac{(1+x)\varphi^4(x)}{n^3} \qquad \text{and} \qquad \Big|\frac{d}{dx}M^a_{n,5}(x)\Big| \le C\frac{(1+x)^2\varphi^2(x)}{n^3}.$$

Proof. For the fifth moment, one has

$$M^a_{n,5}(x) = \frac{\varphi^2(x)}{n}\Big(4M^a_{n,3}(x) + \frac{a}{(1+x)^2}M^a_{n,4}(x) + \frac{d}{dx}M^a_{n,4}(x)\Big).$$

It is clear that

$$|M^a_{n,5}(x)| \le C_1 \frac{\varphi^2(x)}{n}\Big(\frac{(1+x)\varphi^2(x)}{n^2} + \frac{\varphi^4(x)}{n^2(1+x)^2} + \frac{(1+x)\varphi^2(x)}{n^2}\Big)$$
$$\le C_2 \frac{(1+x)\varphi^4(x)}{n^3}.$$

For the first derivative one has

$$\frac{d}{dx}M^a_{n,5}(x) = \frac{(1+2x)}{n}\Big(4M^a_{n,3}(x) + \frac{a}{(1+x)^2}M^a_{n,4}(x) + \frac{d}{dx}M^a_{n,4}(x)\Big)$$
$$+ \frac{\varphi^2(x)}{n}\Big(4\frac{d}{dx}M^a_{n,3}(x) - \frac{2aM^a_{n,4}(x)}{(1+x)^3} + \frac{a}{(1+x)^2}\frac{d}{dx}M^a_{n,4}(x) + \frac{d^2}{dx^2}M^a_{n,4}(x)\Big).$$

Therefore

$$\Big|\frac{d}{dx}M^a_{n,5}(x)\Big| \le \frac{C_1}{n}\Big(\frac{(1+x)^2\varphi^2(x)}{n^2} + \frac{\varphi^4(x)}{n^2(1+x)} + \frac{(1+x)^2\varphi^2(x)}{n^2}$$
$$+ \frac{\varphi^4(x)}{n^2} + \frac{\varphi^6(x)}{n^2(1+x)^3} + \frac{C1+x)\varphi^4(x)}{n^2(1+x)^2} + \frac{(1+x)^2\varphi^2(x)}{n^2}\Big)$$
$$\le C_2\frac{(1+x)^2\varphi^2(x)}{n^3}.$$

□

Lemma 3.7. *Suppose that $a \ge 0$. There exists a constant C such that, if $n > 1$ and $x \ge 0$, then*

$$0 \le M^a_{n,6}(x) \le C\frac{(1+x)\varphi^4(x)}{n^3}\Big(x + \frac{1}{n}\Big).$$

Proof. As in Lemma 3.5, we obtain

$$M^a_{n,6}(x) = \frac{\varphi^2(x)}{n}\Big(5M^a_{n,4}(x) + \frac{a}{(1+x)^2}M^a_{n,5}(x) + \frac{d}{dx}M^a_{n,5}(x)\Big).$$

Taking into account Lemmas 3.5 and 3.6, we obtain

$$0 \le M^a_{n,6}(x) \le C_1\frac{\varphi^2(x)}{n}\Big(\frac{\varphi^4(x)}{n^2} + \frac{(1+x)\varphi^4(x)}{n^3(1+x)^2} + \frac{(1+x)^2\varphi^2(x)}{n^3}\Big)$$
$$\le C_1\frac{(1+x)\varphi^4(x)}{n^3}\Big(x + \frac{\varphi^2(x)}{n^3(1+x)^2} + \frac{(1+x)}{n}\Big) \le C_2\frac{(1+x)\varphi^4(x)}{n^3}\Big(x + \frac{1}{n}\Big).$$

□

4. The rate of convergence

Set

$$K(f,t)_\varrho = \inf_{g\in D(\varrho)}\Big(\|f-g\|_\varrho + t\Big(\|\varphi^2 g''\|_\varrho + a\|\varphi g'\|_\varrho\Big)\Big). \tag{4.7}$$

Theorem 4.2. *Assume $a, q \ge 0$ are real numbers, and $\varrho(x) = 1/(1+x)^q$. There exists a constant $C = C(a,q)$, such that for $n > 2$ and $f \in C_\varrho[0,\infty)$, one has*

$$\|B^a_n(f) - f\|_\varrho \le CK\Big(f, \frac{1}{n}\Big)_\varrho,$$

here $K(f,t)_\varrho$ is defined in (4.7).

Proof. If $x > 0$ and $g \in D(\varrho)$, we will use the representation

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(s)(t-s)ds.$$

It follows from Proposition 2.1 that

$$\begin{aligned}\Big|\int_x^t g''(s)(t-s)ds\Big| &\le \|\varphi^2 g''\|_\varrho \Big|\int_x^t \frac{(t-s)}{\varphi^2(s)\varrho(s)}ds\Big| \\ &= \|\varphi^2 g''\|_\varrho \Big|\int_x^t \frac{(t-s)}{s}(1+s)^{q-1}ds\Big| \\ &\le \|\varphi^2 g''\|_\varrho \frac{(t-x)^2}{x}((1+x)^{q-1}+(1+t)^{q-1}).\end{aligned}$$

Taking into account (2.3), Hölder inequality, Theorem 2.1 (see also (2.5)), and Lemma 3.5, we obtain

$$\begin{aligned}&\Big|B_n^a\Big(\int_x^t g''(s)(t-s)ds, x\Big)\Big| \\ \le&\frac{\|\varphi^2 g''\|_\varrho}{x}\Big((1+x)^{q-1}B_n^a((t-x)^2,x) + B_n^a((t-x)^2(1+t)^{q-1},x)\Big) \\ \le&\frac{\|\varphi^2 g''\|_\varrho}{x}\Big(C_1(1+x)^{q-1}\frac{\varphi^2(x)}{n} + (B_n^a((1+t)^{2q-2},x))^{1/2}\sqrt{M_{n,4}^a(x)}\Big) \\ \le&C_2\frac{\|\varphi^2 g''\|_\varrho}{x}\Big(\frac{x(1+x)^q}{n} + (1+x)^{q-1}\frac{\varphi^2(x)}{n}\Big) \le C_3\frac{\|\varphi^2 g''\|_\varrho}{n\varrho(x)}.\end{aligned}$$

Since

$$B_n^a(g,x) - g(x) = g'(x)B_n^a(t-x,x) + B_n^a\Big(\int_x^t g''(s)(t-s)ds, x\Big),$$

from the previous estimate and (3.6), one has

$$\begin{aligned}\varrho(x)\Big|B_n^a(g,x)-g(x)\Big| &\le \Big|\varrho(x)g'(x)B_n^a(t-x,x)\Big| + \Big|B_n^a\Big(\int_x^t g''(s)(t-s)ds,x\Big)\Big| \\ &\le \varrho(x)\Big|g'(x)B_n^a(t-x,x)\Big| + B_n^a\Big(\Big|\int_x^t g''(s)(t-s)ds\Big|,x\Big) \\ &\le \varrho(x)|g'(x)|\frac{ax}{n(1+x)} + C_3\frac{\|\varphi^2 g''\|_\varrho}{n} \\ &\le \frac{a\sqrt{x}}{n(1+x)^{3/2}}\|\varphi g'\|_\varrho + C_3\frac{\|\varphi^2 g''\|_\varrho}{n} \le \frac{C_4}{n}\Big(\|\varphi^2 g''\|_\varrho + a\|\varphi g'\|_\varrho\Big).\end{aligned}$$

Therefore, for any $g \in D(\varrho)$,

$$\begin{aligned}\|B_n^a(f)-f\|_\varrho &\le \|f-g\|_\varrho + \|B_n^a(f-g)\|_\varrho + \|B_n^a(g)-g\|_\varrho \\ &\le C\Big\{\|f-g\|_\varrho + \frac{1}{n}\Big(\|\varphi^2 g''\|_\varrho + a\|\varphi g'\|_\varrho\Big)\Big\}.\end{aligned}$$

□

Remark 4.2. *For $g \in D(\varrho)$, the previous proof also yields the inequality*

$$\|B_n^a(f)-f\|_\varrho \le C\Big\{\|f-g\|_\varrho + \frac{1}{n}\Big(\|\varphi^2 g''\|_\varrho + a\|g'\|_\varrho\Big)\Big\}.$$

But, the estimate with $\|\varphi g'\|_\varrho$ is more convenient to study the inverse result.

5. A Voronovskaja-type theorem

Theorem 5.3. *Assume $a, q \geq 0$ and $\varrho(x) = 1/(1+x)^q$. There exists a constant C such that, if $n > 1$, $g \in C^3[0,\infty)$ and $g, \varphi^2 g'', \varphi^3 g''' \in C_\varrho[0,\infty)$, then*

$$\left\| B_n^a(g) - g - \frac{ae_1}{n(1+e_1)} g' - \frac{\varphi^2 g''}{2n} \right\|_\varrho \leq C\Big(\frac{\|\varphi^2 g''\|_\varrho + \|\varphi^2 g'''\|_\varrho}{n^2} + \frac{\|\varphi^3 g'''\|_\varrho}{n^{3/2}} \Big),$$

where $e_1(x) = x$.

Proof. For $g \in C^3[0,\infty)$, we use the Taylor expansion

$$g(t) = g(x) + g'(x)(t-x) + \frac{1}{2} g''(x)(t-x)^2 + \frac{1}{2} \int_x^t g'''(s)(t-s)^2 ds,$$

to obtain the representation

$$B_n^a(g,x) - Q_n^a(g,x) = \frac{1}{2} B_n^a\Big(\int_x^t g'''(s)(t-s)^2 ds, x \Big),$$

with

$$\begin{aligned} Q_n^a(g,x) &= g(x) + g'(x) B_n^a(t-x,x) + \frac{1}{2} g''(x) B_n^a((t-x)^2, x) \\ &= g(x) + g'(x) M_{n,1}^a(x) + \frac{1}{2} g''(x) M_{n,2}^a(x). \end{aligned}$$

We should estimate

$$B_n^a\Big(\Big| \int_x^t g'''(s)(t-s)^2 ds \Big|, x \Big) = \sum_{k=0}^{\infty} \Big| \int_x^{k/n} g'''(s) \Big(\frac{k}{n} - s \Big)^2 ds \Big| W_{n,k}^a(x).$$

(A) Suppose $0 \leq x < 1/n$. In this case, we consider the inequality

$$\begin{aligned} &\sum_{k=0}^{\infty} \Big| \int_x^{k/n} g'''(s) \Big(\frac{k}{n} - s \Big)^2 ds \Big| W_{n,k}^a(x) \\ \leq &\|\varphi^2 g'''\|_\varrho \sum_{k=0}^{\infty} \Big| \int_x^{k/n} \Big(\frac{k}{n} - s \Big)^2 \frac{ds}{\varphi^2(s)\varrho(s)} \Big| W_{n,k}^a(x). \end{aligned}$$

For $k = 0$, one has

$$\begin{aligned} W_{n,0}^a(x) \int_0^x \frac{s^2}{\varphi^2(s)\varrho(s)} ds = W_{n,0}^a(x) \int_0^x s(1+s)^{q-1} ds &\leq \frac{x^2(1+x)^{q-1}}{(1+x)^n} \\ &\leq \frac{(1+x)^q}{n^2}. \end{aligned}$$

On the other hand, if $q \geq 1$, then

$$\begin{aligned}
\sum_{k=1}^{\infty} W_{n,k}^{a}(x) \int_{x}^{k/n} \frac{(k/n-s)^2}{\varphi^2(s)\varrho(s)} ds &\leq \frac{1}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^{a}(x) \int_{x}^{k/n} \frac{(k/n-s)^2}{\sqrt{s}(1+s)^{1-q}} ds \\
&\leq \frac{1}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^{a}(x)(1+k/n)^{q-1}(k/n-x)^2 \int_{x}^{k/n} \frac{ds}{\sqrt{s}} \\
&= \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^{a}(x)(1+k/n)^{q-1}(k/n-x)^2(\sqrt{k/n}-\sqrt{x}) \\
&\leq \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} W_{n,k}^{a}(x)(1+k/n)^{q-1}(k/n-x)^{5/2} \\
&\leq \frac{2}{\sqrt{x}} \sqrt{M_{n,5}(x)} \Big(\sum_{k=0}^{\infty} (1+k/n)^{2q-2} W_{n,k}^{a}(x) \Big)^{1/2} \\
&\leq \frac{C_1\sqrt{1+x}}{\sqrt{x}} \frac{\varphi^2(x)}{n^{3/2}} (1+x)^{q-1} \leq C_2 \frac{\sqrt{x}}{n^{3/2}} (1+x)^q \leq \frac{C_1(1+x)^q}{n^2},
\end{aligned}$$

where we use Lemma 3.6 and Theorem 2.1.

If $q < 1$, the proof is simpler because, for $x < 1/n$,

$$\frac{1}{(1+k/n)^{1-q}} \leq \frac{1}{(1+x)^{1-q}}.$$

Therefore

$$\varrho(x)|B_n^a(g,x) - Q_n^a(g,x)| \leq \frac{C}{n^2} \|\varphi^2 g'''\|_{\varrho}.$$

(B) Assume that $nx \geq 1$. In this case, we consider the inequality

$$\begin{aligned}
&\sum_{k=0}^{\infty} \Big| \int_{x}^{k/n} g'''(s) \Big(\frac{k}{n} - s\Big)^2 ds \Big| W_{n,k}^{a}(x) \\
\leq &\|\varphi^3 g'''\|_{\varrho} \sum_{k=0}^{\infty} \Big| \int_{x}^{k/n} \Big(\frac{k}{n} - s\Big)^2 \frac{ds}{\varphi^3(s)\varrho(s)} \Big| W_{n,k}^{a}(x).
\end{aligned}$$

Since $1/n \leq x$, it follows from Lemma 3.7

$$M_{n,6}^{a}(x) \leq C \frac{(1+x)\varphi^4(x)}{n^3} \Big(x + \frac{1}{n}\Big) \leq 2C \frac{\varphi^6(x)}{n^3}.$$

Moreover

$$B_n^a(|t-x|^3, x) \leq \sqrt{M_{n,6}^{a}(x)} \leq 2C \frac{\varphi^3(x)}{n^{3/2}}.$$

Thus, we can use Lemma 2.1, Lemma 3.6 and Theorem 2.1 to obtain

$$\begin{aligned}
\varrho(x)\,|B^a_n(g,x)-Q^a_n(g,x)| &\leq \frac{1}{2}\varrho(x)\|\varphi^3 g'''\|_\varrho B^a_n\Big(\Big|\int_x^t \frac{(t-s)^2}{\varphi^3(s)\varrho(s)}ds\Big|,x\Big)\\
&= \frac{1}{2}\varrho(x)\|\varphi^3 g'''\|_\varrho B^a_n\Big(\Big|\int_x^t \frac{(t-s)^2(1+s)^{q-3/2}}{s^{3/2}}ds\Big|,x\Big)\\
&\leq \frac{1}{3}\varrho(x)\|\varphi^3 g'''\|_\varrho B^a_n\Big(|t-x|^3\Big(\frac{(1+x)^{q-3/2}}{x^{3/2}}+\frac{(1+t)^{q-3/2}}{x^{3/2}}\Big),x\Big)\\
&= \frac{1}{3}\|\varphi^3 g'''\|_\varrho\Big(\frac{B^a_n(|t-x|^3,x)}{\varphi^3(x)}+\frac{\varrho(x)}{x^{3/2}}\sqrt{M^a_{n,6}(x)}\sqrt{B^a_n((1+t)^{2q-3},x)}\\
&\leq C_3\|\varphi^3 g'''\|_\varrho\Big(\frac{1}{n^{3/2}}+\frac{\varrho(x)}{x^{3/2}}\frac{\varphi^3(x)}{n^{3/2}}(1+x)^{q-3/2}\Big)\leq C_4\frac{\|\varphi^3 g'''\|_\varrho}{n^{3/2}}.
\end{aligned}$$

Taking into account (2.2) and (2.3), we obtain

$$\begin{aligned}
&\Big\|B^a_n(g)-g-\frac{ae_1}{n(1+e_1)}g'-\frac{\varphi^2 g''}{2n}\Big\|_\varrho\\
\leq&\Big\|B^a_n(g)-g-g'M^a_{n,1}-\frac{1}{2}g''M^a_{n,2}\Big\|_\varrho\\
&+\frac{1}{2}\Big\|\frac{\varphi^2 g''}{n}-\frac{\varphi^2(x)}{n}\Big(1+\frac{a}{n(1+x)^2}+\frac{a^2x}{n(1+x)^3}\Big)g''\Big\|_\varrho\\
=&\Big\|B^a_n(g)-g-g'M^a_{n,1}-\frac{1}{2}g''M^a_{n,2}\Big\|_\varrho+\frac{1}{n^2}\Big\|\Big(\frac{a}{(1+x)^2}+\frac{a^2x}{(1+x)^3}\Big)\varphi^2 g''\Big\|_\varrho\\
\leq&C\Big(\frac{\|\varphi^2 g''\|_\varrho+\|\varphi^2 g'''\|_\varrho}{n^2}+\frac{\|\varphi^3 g'''\|_\varrho}{n^{3/2}}\Big).
\end{aligned}$$

□

Corollary 5.2. *Under the assumptions of Theorem 5.3, one has*

$$\lim_{n\to\infty}\Big\|n\Big(B^a_n(g)-g\Big)-\frac{ae_1}{(1+e_1)}g'-\frac{\varphi^2 g''}{2}\Big\|_\varrho=0,$$

where $e_1(x)=x$.

Remark 5.3. *In Theorem 3.2 of [13], a Voronovskaja-type theorem was given for functions* $f\in C^3[0,\infty)$ *such that* $f,f',f'',f'''\in C_\varrho[0,\infty)$*, but the authors only considered the case* $q\in\mathbb{N}_0$*. Moreover, they only obtained pointwise convergence.*

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