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Operators Applied to Lifts with Respect to the Diagonal Lifts of Affinor Fields Along a Cross-Section on $T_q^p(M)$

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

Abstract

In this paper firstly, we study the operators associated with the diagonal lift and applied to vertical and horizontal lifts. Secondly, we get the conditions of almost holomorphic vector fields with respect to the diagonal lift.

Keywords: *Cross-section, Tachibana operators, Vishnevskii operators, Diagonal lift, Horizontal lift, Vertical lift.*
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1. Introduction

Let M_n be n-dimensional differentiable manifold of class C^∞, $T_q^p(M_n)$ its tensor bundle of type (p,q), and π the natural projection $T_q^p(M_n) \to M_n$. Let $x^j, j = 1, ..., n$ be local coordinates in neighborhood U of a point x of M_n.Then a tensor t of type (p,q) at $x \in M_n$ which is an element of $T_q^p(M_n)$ is expressible in the form

$$(x^j, t_{j_1\ldots j_q}^{i_1\ldots i_p}) = (x^j, x^{\bar{j}}), x^{\bar{j}} = t_{j_1\ldots j_q}^{i_1\ldots i_p}, \bar{j} = n+1, ..., n+n^{p+q},$$

whose $t_{j_1\ldots j_q}^{i_1\ldots i_p}$ are components of t with respect to the natural frame ∂_j. We may consider $(x^j, x^{\bar{j}})$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T_q^p(M_n)$. To a transformation of local coordinates of $M_n : x^{j'} = x^{j'}(x^j)$, there corresponds in $T_q^p(M_n)$ the coordinates transformation

$$\begin{aligned} x^{j'} &= x^{j'}(x^j) \\ x^{j'} &= t_{j_1'\ldots j_q'}^{i_1'\ldots i_p'} = A_{i_1}^{i_1'} \ldots A_{i_p}^{i_p'} A_{j_1'}^{j_1} \ldots A_{j_q'}^{j_q}, t_{j_1\ldots j_q}^{i_1\ldots i_p} = A_{(i)}^{(i')} A_{(j')}^{(j)} x^{\bar{j}}, \end{aligned} \tag{1.1}$$

where

$$A_{(i)}^{(i')} A_{(j')}^{(j)} = A_{i_1}^{i_1'} \ldots A_{i_p}^{i_p'} A_{j_1'}^{j_1} \ldots A_{j_q'}^{j_q}, A_{i_1}^{i_1'} = \frac{\partial x^{i'}}{\partial x^i}, A_{j_1'}^{j_1} = \frac{\partial x^j}{\partial x^{j'}}.$$

Let $A \in \Im_q^p(M_n)$. Then there is a unique vector field $A^V \in \Im_0^1(T_q^p(M_n))$ such that for $\alpha \in T_q^p(M_n)$

$$(A(\imath\alpha))^V = \alpha(A) \circ \pi = (\alpha(A))^V,$$

where $(\alpha(A))^V$ is the vertical lift of the function $\alpha(A) \in F(M_n)$. We call A^V the vertical lift of $A \in T_q^p(M_n)$ to $T_q^p(M_n)$ (see [4, 5]). The vertical lift A^V has components of the form

$$A^V = \begin{pmatrix} (A^j)^V \\ (A^{\bar{j}})^V \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1\ldots j_q}^{j_1\ldots j_p} \end{pmatrix} \tag{1.2}$$

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with respect to the coordinates $\left(x^j, x^{\overline{j}}\right)$ in $T_q^p(M_n)$.

Let ∇ be a symmetric affine connection on M_n.We define the horizontal lift $\nabla^H = \widetilde{\nabla}_V \in \Im_0^1\left(T_q^p(M_n)\right)$ of $V \in \Im_0^1(M_n)$ to $T_q^p(M_n)$ by [5]

$$(V(\imath\alpha))^H = \imath(\nabla_V \alpha), \alpha \in T_q^p(M_n).$$

The horizontal lift V^H of $V \in \Im_0^1(M_n)$ to $T_q^p(M_n)$ has components

$$V^H = \left(\begin{array}{c} V^j \\ V^s \left(\sum\limits_{\mu=1}^{q} \Gamma_{sj}^m, t_{j_1 \ldots m \ldots j_q}^{i_1 \ldots i_p} - \sum\limits_{\lambda=1}^{p} \Gamma_{sm}^{i\lambda} t_{j_1 \ldots j_q}^{i_1 \ldots m \ldots i_q} \right) \end{array} \right) \tag{1.3}$$

with respect to the coordinates $\left(x^j, x^{\overline{j}}\right)$ in $T_q^p(M_n)$[1, 4], where Γ_{ij}^k are local components of ∇ in M_n.

Suppose that there is given a tensor field $\xi \in T_q^p(M_n)$.Then the correspondence $x \to \xi_x, \xi_x$ being the value of ξ at $x \in M_n$,determines a mapping $\sigma_\xi : M_n \to T_q^p(M_n)$, such that $\pi \circ \sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T_q^p(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1 \ldots k_q}^{l_1 \ldots l_p}\left(x^k\right)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$\left\{ \begin{array}{c} x^k = x^k \\ x^{\overline{k}} = \xi_{k_1 \ldots k_q}^{l_1 \ldots l_p}\left(x^k\right) \end{array} \right. \tag{1.4}$$

with respect to the coordinates $\left(x^k, x^{\overline{k}}\right)$ in $T_q^p(M_n)$. Differentiating (1.4) by x^j, we see that n tangent vector fields B_j to $\sigma_\xi(M_n)$ have components

$$\left(B_j^K\right) = \left(\frac{\partial x^K}{\partial x^j}\right) = \left(\begin{array}{c} \delta_j^k \\ \partial_j \xi_{k_1 \ldots k_q}^{l_1 \ldots l_p} \end{array} \right) \tag{1.5}$$

with respect to the natural frame $\left\{\partial_k, \partial_{\overline{k}}\right\}$ in $T_q^p(M_n)$.

On the other hand, the fibre is locally expressed by [4]

$$\left\{ \begin{array}{c} x^k = const, \\ t_{k_1 \ldots k_q}^{l_1 \ldots l_p} = t_{k_1 \ldots k_q}^{l_1 \ldots l_p}, \end{array} \right. \tag{1.6}$$

$t_{k_1 \ldots k_q}^{l_1 \ldots l_p}$ being considered as parameters.

Let $A, B \in \Im_q^p(M_n), V, W \in \Im_0^1(M_n)$ and $\varphi \in \Im_1^1(M_n)$. Let R denotes the curvature tensor field of the connection ∇. Then (see [1, 4])

$$\left\{ \begin{array}{c} \left[A^V, B^V\right] = 0 \\ \left[V^H, A^V\right] = (\nabla_V A)^V \\ \left[V^H, \widetilde{\gamma}\varphi - \gamma\varphi\right] = \widetilde{\gamma}\left(L_V \varphi + (\nabla V)\varphi - \varphi(\nabla V)\right) - \gamma\left(L_V \varphi + (\nabla V)\varphi - \varphi(\nabla V)\right) \\ \left[V^H, W^H\right] = [V, W]^H + (\widetilde{\gamma} - \gamma) R(V, W) \end{array} \right. \tag{1.7}$$

where $\widetilde{\gamma}\varphi - \gamma\varphi$ is a vector field in $T_q^p(M_n)$ defined by [4],

$$\widetilde{\gamma}\varphi - \gamma\varphi = \left(\begin{array}{c} 0 \\ \sum_{\mu=1}^{q} t_{j_1 \ldots m \ldots j_q}^{i_1 \ldots i_p} \varphi_{j_p}^m - \sum_{\lambda=1}^{p} t_{j_1 \ldots j_q}^{i_1 \ldots m \ldots i_p} \varphi_m^{i\lambda} \end{array} \right) \tag{1.8}$$

1.1. Diagonal lifts along a cross-section

Let $\varphi \in \Im_1^1(M_n)$. We define a tensor field $\varphi^D \in \Im_1^1\left(T_q^p(M_n)\right)$ along the cross-section $\sigma_\xi(M_n)$ by [4]

$$\left\{ \begin{array}{c} \varphi^D\left(V^H\right) = (\varphi(V))^H, \forall V \in \Im_0^1(M_n) \\ \varphi^D\left(A^V\right) = -(\varphi(A))^V, \forall A \in \Im_q^p(M_n), \end{array} \right. \tag{1.9}$$

where $\varphi(A) = C(\varphi \otimes A) \in \Im_q^p(M_n)$ and call φ^D the diagonal lift of $\varphi \in \Im_1^1(M_n)$ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$. Then , from (1.9) we have

$$\begin{cases} (i)\left(\widetilde{\varphi}_L^K\right)^D\left(\widetilde{V}^L\right)^H = \left((\widetilde{\varphi}(V))^K\right)^H, \\ (ii)\left(\varphi_L^K\right)^D\left(\widetilde{A}^L\right)^V = -\left((\widetilde{\varphi}(A))^K\right)^V, \end{cases} \tag{1.10}$$

where $(\widetilde{\varphi}(A))^V = \begin{pmatrix} 0 \\ \left((\widetilde{\varphi}(A))^{\bar{k}}\right)^V \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_m^{l_1} A_{k_1 \dots k_q}^{m l_2 \dots l_p} \end{pmatrix}$

Let $\varphi \in \Im_1^1(M_n)$. We define $\varphi^H \in \Im_1^1\left(T_q^p(M_n)\right)$ along $\sigma_\xi(M_n)$ by [4]

$$\begin{cases} \varphi^H\left(V^H\right) = (\varphi(V))^H, \forall V \in \Im_0^1(M_n) \\ \varphi^H\left(A^V\right) = (\varphi(A))^V, \forall A \in \Im_q^p(M_n), \end{cases} \tag{1.11}$$

where $\varphi(A) = C(\varphi \otimes A) \in \Im_q^p(M_n)$ [7].

Theorem 1.1. *[4] If $\varphi, \phi \epsilon \Im_1^1(M_n)$, then with respect to symmetric affine connection ∇ in M_n, we have*

$$\varphi^D \phi^D + \phi^D \varphi^D = (\varphi\phi + \phi\varphi)^H, \tag{1.12}$$

$$\varphi^D \phi^H + \phi^D \varphi^H = \varphi^H \phi^D + \phi^H \varphi^D = (\varphi\phi + \phi\varphi)^D \tag{1.13}$$

Putting $\varphi = \phi$ in (1.12), we obtain

$$\varphi^D \varphi^D = (\varphi\varphi)^H, \quad \left(\varphi^D\right)^2 = \left(\varphi^2\right)^H. \tag{1.14}$$

Since $(id_{M_n})^H = id_{\Im_q^p(M_n)}$, using (1.14), we have

Theorem 1.2. *[4]If φ is almost complex structure in M_n, then the diagonal lift φ^D of φ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$ is an almost complex structure in $T_q^p(M_n)$.*

1.2. Sasakian Metrics on $T_q^p(M_n)$

For each $P \in M_n$ the extension of the scalar product g (denoted also by g) is defined on the tensor space $\pi^{-1}(p) = T_q^p(P)$ by

$$g(A, B) = g_{i_1 t_1} \dots g_{i_p t_p} g^{j_1 l_1} \dots g^{j_q l_q} A_{j_1 \dots j_q}^{i_1 \dots i_p} B_{l_1 \dots l_q}^{t_1 \dots t_p}$$

for all $A, B \in T_q^p(P)$. A Sasakian metric ${}^S g$(or a diagonal lift of g) is defined on $T_q^p(M_n)$ by the three equations [11, 13]

$${}^S g\left(A^V, B^V\right) = (g(A, B))^V, A, B \in \Im_q^p(M_n), \tag{1.15}$$

$${}^S g\left(A^V, Y^H\right) = 0, \tag{1.16}$$

$${}^S g\left(X^H, Y^H\right) = (g(X, Y))^V, X, Y \in \Im_0^1(M_n). \tag{1.17}$$

These equations are easily seen to determine ${}^S g$ on $T_q^p(M_n)$ with respect to which the horizontal and vertical distributions are complementary and orthogonal.

We define the horizontal lift ∇^H of the Levi-Civita connection in M_n to $T_q^p(M_n)$ by the conditions

$$\begin{cases} \left(\nabla_{A^V} B^V\right) = 0, \left(\nabla_{A^V} Y^H\right) = 0, \\ \left(\nabla_{X^H} B^V\right) = (\nabla_X B)^V, \left(\nabla_{X^H} Y^H\right) = (\nabla_X Y)^H \end{cases} \tag{1.18}$$

for any $X, Y \in \Im_0^1(M_n)$ and $A, B \in \Im_q^p(M_n)$.

Let R denote the curvature tensor field of the Levi-Civita connection ∇. Then [5, 8]

$$\begin{aligned} \left[A^V, B^V\right] &= 0, \\ \left[X^H, A^V\right] &= (\nabla_X A)^V, \\ \left[X^H, Y^H\right] &= [X, Y]^H + (\widetilde{\gamma} - \gamma) R(X, Y), \end{aligned} \tag{1.19}$$

where $\widetilde{\gamma} - \gamma : \Im_1^1(M_n) \to \Im_0^1\left(T_q^p(M_n)\right)$ is the operator defined by

$$(\widetilde{\gamma} - \gamma)\varphi = \begin{pmatrix} 0 \\ \sum_{\mu=1}^q t_{j_1 \dots m \dots j_q}^{i_1 \dots i_p} \varphi_{j_p}^m - \sum_{\lambda=1}^p t_{j_1 \dots j_q}^{i_1 \dots m \dots i_p} \varphi_m^{i\lambda} \end{pmatrix} \tag{1.20}$$

for any $\varphi \in \Im_1^1(M_n)$ with respect to the adapted frame ,φ_m^i being local components of φ in M_n.

2. Main Results

2.1. The Tachibana operators applied to vertical and horizontal lifts with respect to almost complex structure φ^D along $\sigma_\xi(M_n)$.

Definition 2.1. Let $\varphi \in \Im_1^1(M_n)$, and $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M_n)$ be a tensor algebra over R. A map $\phi_\varphi |_{r+s\rangle 0}$: $\overset{*}{\Im}(M_n) \to \Im(M_n)$ is called a Tachibana operator or ϕ_φ operator on M_n if

$a)$ ϕ_φ is linear with respect to constant coefficient,
$b)$ $\phi_\varphi : \overset{*}{\Im}(M_n) \to \Im_{s+1}^r(M_n)$ for all r and s,
$c)$ $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in \overset{*}{\Im}(M_n)$,
$d)$ $\phi_{\varphi X} Y = -(L_Y \varphi)X$ for all $X, Y \in \Im_0^1(M_n)$ where L_Y is the Lie derivation with respect to Y,
$e)$

$$(\phi_{\varphi X}\eta)Y = (d(\imath_Y\eta))(\varphi X) - (d(\imath_Y(\eta o\varphi)))X + \eta((L_Y\varphi)X) = \phi X(\imath_Y\eta) - X(\imath_{\varphi Y}\eta) + \eta((L_Y\varphi)X) \tag{2.1}$$

for all $\eta \in \Im_1^0(M_n)$ and $X, Y \in \Im_0^1(M_n)$, where $\imath_Y\eta = \eta(Y) = \eta \overset{C}{\otimes} Y$, $\overset{*}{\Im}{}_s^r(M_n)$ the module of all pure tensor fields of type (r, s) on M_n according to the affinor field φ [2, 3, 9, 12](see [10] for applied to pure tensor field).

Theorem 2.1. *For L_X the operator Lie derivation with respect to X, $\varphi^D \in \Im_1^1\left(T_q^p(M_n)\right)$ the diagonal lift of $\varphi \in \Im_1^1(M_n)$ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$ defined by (1.9) is an almost complex structure in $T_q^p(M_n)$, ϕ_{φ^D} the Tachibana operator on M_n, we get the following formulas*

$$\begin{aligned} i)\ \phi_{\varphi^D V^H} W^H &= \left(L_{\varphi(V)}W - \varphi L_V W\right)^H + (\widetilde{\gamma} - \gamma)\left(R(\varphi(V), W) - \varphi R(V, W)\right), \\ ii)\ \phi_{\varphi^D A^V} V^H &= (\nabla_V \varphi(A))^V - (\varphi \nabla_V A)^V = ((\nabla_V \varphi) A)^V, \\ iii)\ \phi_{\varphi^D V^H} A^V &= \left(\psi_{\varphi(V)} A\right)^V, \\ iv)\ \phi_{\varphi^D A^V} B^V &= 0, \end{aligned}$$

where R is the curvature tensor of ∇, $A, B \in \Im_q^p(M_n)$, $V, W \in \Im_0^1(M_n)$ and $\varphi \in \Im_1^1(M_n)$.

Proof. $i)$

$$\begin{aligned} \phi_{\varphi^D V^H} W^H &= -\left(L_{W^H}\varphi^D\right)V^H = -L_{W^H}\varphi^D V^H + \varphi^D L_{W^H} V^H \\ &= -\left[W^H, (\varphi(V))^H\right] + \varphi^D\left[W^H, V^H\right] \\ &= \left[(\varphi(V))^H, W^H\right] - \varphi^D\left(\left[V^H, W^H\right]\right) \\ &= \left[(\varphi(V)), W\right]^H + (\widetilde{\gamma} - \gamma) R(\varphi(V), W) \\ &\quad - \varphi^D\left([V, W]^H + (\widetilde{\gamma} - \gamma) R(V, W)\right) \\ &= \left[(\varphi(V)), W\right]^H + (\widetilde{\gamma} - \gamma) R(\varphi(V), W) \\ &\quad - (\varphi[V, W])^H - (\widetilde{\gamma} - \gamma)\varphi R(V, W) \\ &= \left(L_{\varphi(V)}W - \varphi L_V W\right)^H \\ &\quad + (\widetilde{\gamma} - \gamma)\left(R(\varphi(V), W) - \varphi R(V, W)\right) \end{aligned}$$

$ii)$

$$\begin{aligned} \phi_{\varphi^D A^V} V^H &= -\left(L_{V^H}\varphi^D\right)A^V = -L_{V^H}\varphi^D A^V + \varphi^D L_{V^H} A^V \\ &= -L_{V^H} - (\varphi(A))^V + \varphi^D(\nabla_V A)^V \\ &= (\nabla_V \varphi(A))^V - (\varphi \nabla_V A)^V = ((\nabla_V \varphi) A)^V \end{aligned}$$

$iii)$

$$\begin{aligned}\phi_{\varphi^D V^H} A^V &= -(L_{A^V}\varphi^D)V^H = -L_{A^V}\varphi^D V^H + \varphi^D L_{A^V} V^H \\ &= -\left[A^V, (\varphi(V))^H\right] + \varphi^D\left[A^V, V^H\right] \\ &= \left[(\varphi(V))^H, A^V\right] - \varphi^D\left[V^H, A^V\right] \\ &= \left(\nabla_{\varphi(V)}A\right)^V - \varphi^D\left(\nabla_V A\right)^V \\ &= \left(\nabla_{\varphi(V)}A\right)^V + \left(\varphi\left(\nabla_V A\right)\right)^V) \\ &= \left(\nabla_{\varphi(V)}A - \varphi\left(\nabla_V A\right)\right)^V \\ &= \left(\psi_{\varphi(V)}A\right)^V\end{aligned}$$

$iv)$

$$\begin{aligned}\phi_{\varphi^D A^V} B^V &= -\left(L_{B^V}\varphi^D\right)A^V = -L_{B^V}\varphi^D A^V + \varphi^D L_{B^V} A^V \\ &= -L_{B^V} - (\varphi(A))^V \\ &= 0\end{aligned}$$

□

2.2. The Vishnevskii Operators applied to vertical and horizontal lifts with respect to almost complex structure φ^D along $\sigma_\xi(M_n)$.

Definition 2.2. Suppose now that ∇ is a linear connection on M_n, and let $\varphi \in \Im_1^1(M_n)$. We can replace the condition $d)$ of defination 2.1 by

$$d')\ \ \psi_{\varphi X}Y = \nabla_{\varphi X}Y - \varphi\nabla_X Y \tag{2.2}$$

for any $X, Y \in \Im_0^1(M_n)$. Then we can consider a new operator by a Vishnevskii operator or $\psi_\varphi-$operator on M_n, we shall mean a map $\psi_\varphi : \overset{*}{\Im}(M_n) \to \Im(M_n)$, which satisfies conditions $a), b), c), e)$ of definition 2.1 and the condition (d') [2, 3, 8, 9].

Let $\omega \in \Im_1^0(M_n)$. Using Definition 2.2, we have

$$\begin{aligned}(\psi_\varphi\omega)(X,Y) &= (\psi_{\varphi X}\omega)Y \\ &= (\varphi X)(\iota_Y\omega) - X(\iota_{\varphi Y}\omega) - \omega\left(\nabla_{\varphi X}Y - \varphi\left(\nabla_X Y\right)\right) \\ &= \left(\nabla_{\varphi X}\omega - \nabla_X(\omega\circ\varphi)\right)Y\end{aligned} \tag{2.3}$$

for any $X, Y \in \Im_0^1(M_n)$,where $(\omega\circ\varphi)Y = \omega(\varphi Y)$. From (2.3) we see that $\psi_{\varphi X}\omega = \nabla_{\varphi X}\omega - \nabla_X(\omega\circ\varphi)$ is a $1-$form [9].

Theorem 2.2. *For the horizontal lift ∇^H of the Levi-Civita connection ∇ in M_n to $T_q^p(M_n)$, $\varphi^D \in \Im_1^1\left(T_q^p(M_n)\right)$ the diagonal lift of $\varphi \in \Im_1^1(M_n)$ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$ defined by (1.9) is an almost complex structure in $T_q^p(M_n)$, ψ_{φ^D} the Vishnevskii operator or $\psi_\varphi-$operator on M_n, we get the following formulas*

$$\begin{aligned} i)\ \psi_{\varphi^D V^H}A^V &= \left(\psi^A_{\varphi(V)}\right)^V, \\ ii)\ \psi_{\varphi^D V^H}W^H &= \left(\psi_{\varphi(V)}W\right)^H, \\ iii)\ \psi_{\varphi^D A^V}B^V &= 0, \\ iv)\ \phi_{\varphi^D A^V}B^V &= 0,\end{aligned}$$

where R is the curvature tensor of ∇, $A, B \in \Im_q^p(M_n)$, $V, W \in \Im_0^1(M_n)$ and $\varphi \in \Im_1^1(M_n)$.

Proof. $i)$

$$\begin{aligned} \psi_{\varphi^D V^H} A^V &= \nabla^H_{\varphi^D V^H} A^V - \varphi^D \nabla^H_{V^H} A^V, \\ &= \nabla^H_{(\varphi(V))^H} A^V - \varphi^D (\nabla_V A)^V, \\ &= \left(\nabla_{\varphi(V)} A\right)^V + (\varphi(\nabla_V A))^V, \\ &= \left(\nabla_{\varphi(V)} A - \varphi(\nabla_V A)\right)^V, \\ &= \left(\psi^A_{\varphi(V)}\right)^V. \end{aligned}$$

$ii)$

$$\begin{aligned} \psi_{\varphi^D V^H} W^H &= \nabla^H_{\varphi^D V^H} W^H - \varphi^D \left(\nabla^H_{V^H} W^H\right), \\ &= \nabla^H_{(\varphi(V))^H} W^H - \varphi^D (\nabla_V W)^H, \\ &= \left(\nabla_{\varphi(V)} W\right)^H - (\varphi(\nabla_V W))^H, \\ &= \left(\nabla_{\varphi(V)} W - \varphi(\nabla_V W)\right)^H, \\ &= \left(\psi_{\varphi(V)} W\right)^H. \end{aligned}$$

$iii)$

$$\begin{aligned} \psi_{\varphi^D A^V} B^V &= \nabla^H_{\varphi^D A^V} B^V - \varphi^D \nabla^H_{A^V} B^V, \\ &= -\nabla^H_{(\varphi(A))^V} B^V, \\ &= 0. \end{aligned}$$

$iv)$

$$\begin{aligned} \psi_{\varphi^D A^V} V^H &= \nabla^H_{\varphi^D A^V} V^H - \varphi^D \nabla^H_{A^V} V^H, \\ &= -\nabla^H_{(\varphi(A))^V} V^H, \\ &= 0. \end{aligned}$$

□

Theorem 2.3. *If V is an holomorphic vector field with respect to almost complex structure φ , the curvature tensor R of ∇ satisfies $R(V, \varphi(W)) = \varphi R(V, W)$ for any $V, W \in \Im^1_0(M_n)$ and $\nabla\varphi = 0$, then its horizontal lift X^H to the $T^p_q(M)$ is an almost holomorfic vector field with respect to the almost complex structure $\varphi^D \in \Im^1_1\left(T^p_q(M_n)\right)$ the diagonal lift of $\varphi \in \Im^1_1(M_n)$ to $T^p_q(M_n)$ along $\sigma_\xi(M_n)$.*

Proof. $i)$

$$\begin{aligned} \left(L_{V^H}\varphi^D\right) W^H &= L_{V^H}\varphi^D W^H - \varphi^D L_{V^H} W^H \\ &= L_{V^H} (\varphi(W))^H - \varphi^D \left([V, W]^H + (\widetilde{\gamma} - \gamma) R(V, W)\right) \\ &= [V, \varphi(W)]^H + (\widetilde{\gamma} - \gamma) R(V, \varphi(W)) - \varphi^D [V, W]^H \\ &\quad -\varphi^D ((\widetilde{\gamma} - \gamma) R(V, W)) \\ &= (L_V \varphi(W))^H - (\varphi L_V W)^H + (\widetilde{\gamma} - \gamma) R(V, \varphi(W)) \\ &\quad - (\widetilde{\gamma} - \gamma) \varphi R(V, W) \\ &= ((L_V \varphi) W)^H + (\widetilde{\gamma} - \gamma) (R(V, \varphi(W)) - \varphi R(V, W)) \end{aligned}$$

$ii)$

$$\begin{aligned} \left(L_{V^H}\varphi^D\right) A^V &= L_{V^H}\varphi^D A^V - \varphi^D L_{V^H} A^V \\ &= -L_{V^H} (\varphi(A))^V - \varphi^D (\nabla_V A)^V \\ &= -(\nabla_V \varphi(A))^V + (\varphi \nabla_V A)^V \\ &= -((\nabla_V \varphi) A)^V \end{aligned}$$

□

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Author's contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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