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Complex Interval Matrix and Its Some Properties

Halise Levent [1,*], Yilmaz Yilmaz [1]

[1] *Department of Mathematics, Faculty of Science and Art, Inonu University, 44280, Malatya, Turkey.*

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Abstract. In this paper, we present the notion of complex interval matrix. Further, we discuss the algebraic structure of the set of all $(m \times n)$ complex interval matrices by using tools of quasilinear functional analysis. Finally, we put a norm on the space of the complex interval matrices and we calculate the norm of a complex interval matrices.

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1. Introduction

It is well known that interval analysis is a tool to gauge a mathematical problem for ranges of values of its parameters. For this reason there has been increasing interest in interval analysis, [1, 2, 4, 8, 10]. An interval x is the compact-convex subset of real numbers and x is denoted by $x = [\underline{x}, \overline{x}]$, where $\underline{x}$ and $\overline{x}$ are the left and right endpoints of x, respectively [10]. Further, if $\underline{x} = \overline{x}$ then we say that x is a degenerate interval and it can be shown by $\{x\}$ or $[x, x]$. The set of all real intervals is denoted by $\mathbb{I}_{\mathbb{R}}$.

In many application areas, the concept of matrix has a major role such as mathematics, physics, social sciences and statics. Many real world problems may contain uncertainties due to environmental factors. Therefore, traditional classical matrices are useless and we need interval matrices. With this motivation, many authors have done a lot of research on interval matrices such as Ganesan [6], Hansen [7] and Rohn [11], etc.

In our previous studies, we introduced the concept of complex interval in order to make a healthier examination in the areas where interval analysis was used. Hence, in [9] we defined the notion of complex interval as

$$u = \left[\underline{u_r}, \overline{u_r}\right] + i\left[\underline{u_s}, \overline{u_s}\right],$$

where $\left[\underline{u_r}, \overline{u_r}\right]$ and $\left[\underline{u_s}, \overline{u_s}\right]$ are real intervals and $i = \sqrt{-1}$ is the complex unit. $\left[\underline{u_r}, \overline{u_r}\right]$ and $\left[\underline{u_s}, \overline{u_s}\right]$ are called real and imaginary part of u, respectively. Further, $\left[\underline{u_r}, \overline{u_r}\right]$ and $\left[\underline{u_s}, \overline{u_s}\right]$ are called real and imaginary part of x, respectively. Unfortunately, both $\mathbb{I}_{\mathbb{R}}$ and $\mathbb{I}_{\mathbb{C}}$ have an algebraic structure which is not linear space which is called as a "quasilinear space" by Aseev in 1986 [3]. This work presents an approach for analysis of set-valued functions.

In this paper, we present the notion of complex interval matrix by using interval analysis and quasilinear functional analysis. We show that the set of all complex interval matrices is a quasilinear space. Further, we can calculate the norm of a complex interval matrix thanks to the norm that we put on the quasilinear space of all complex interval matrices.

*Corresponding Author
Email addresses: halisebanazili44@gmail.com (H. Levent), yyilmaz44@gmail.com (Y. Yılmaz)

2. Quasilinear Spaces of Complex Intervals

The aim of this section is to present basic notations and results. Let us start the definition of quasilinear space in [3]:

A set X is called a quasilinear space on field $\mathbb{K}$ if a partial order relation ”$\leq$”, an algebraic sum operation, and an operation of multiplication by real or complex numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{K}$:

$$x \leq x,$$
$$x \leq z \text{ if } x \leq y \text{ and } y \leq z,$$
$$x = y \text{ if } x \leq y \text{ and } y \leq x,$$
$$x + y = y + x,$$
$$x + (y + z) = (x + y) + z,$$
$$\text{there exists an element (zero) } \theta \in X \text{ such that } x + \theta = x,$$
$$\alpha(\beta x) = (\alpha\beta)x,$$
$$\alpha(x + y) = \alpha x + \alpha y,$$
$$1x = x,$$
$$0x = \theta,$$
$$(\alpha + \beta)x \leq \alpha x + \beta x,$$
$$x + z \leq y + v \text{ if } x \leq y \text{ and } z \leq v,$$
$$\alpha x \leq \alpha y \text{ if } x \leq y.$$

The most popular examples are $\Omega(E)$ and $\Omega_C(E)$ which are defined as the sets of all non-empty closed bounded and non-empty convex closed bounded subsets of any normed linear space E, respectively. Both are a quasilinear space with the inclusion relation“$\subseteq$”, the algebraic sum operation

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}},$$

where the closure is taken on the norm topology of E.and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

Especially,. $\mathbb{I}_{\mathbb{R}}$ is a quasilinear space with the Minkowski sum and scalar multiplication operations are defined by

$$x + y = \left[\underline{x}, \overline{x}\right] + \left[\underline{y}, \overline{y}\right] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

and

$$\lambda x = \begin{cases} \left[\lambda\underline{x}, \lambda\overline{x}\right] & , \quad \lambda \geq 0 \\ \left[\lambda\overline{x}, \lambda\underline{x}\right] & , \quad \lambda < 0, \end{cases}$$

$x, y \in \mathbb{I}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$, respectively.

The Minkowski sum and scalar multiplication on $\mathbb{I}_{\mathbb{C}}$ are defined by

$$\begin{aligned} u + v &= \left[\underline{u_r}, \overline{u_r}\right] + i\left[\underline{u_s}, \overline{u_s}\right] + \left[\underline{v_r}, \overline{v_r}\right] + i\left[\underline{v_s}, \overline{v_s}\right] \\ &= \left[\underline{u_r} + \underline{v_r}, \overline{u_r} + \overline{v_r}\right] + i\left[\underline{u_s} + \underline{v_s}, \overline{u_s} + \overline{v_s}\right] \\ &= \left\{a + ib : a \in \left[\underline{u_r} + \underline{v_r}, \overline{u_r} + \overline{v_r}\right],\ b \in \left[\underline{u_s} + \underline{v_s}, \overline{u_s} + \overline{v_s}\right]\right\} \end{aligned}$$

and

$$\begin{aligned} \lambda u &= \lambda\left[\underline{u_r}, \overline{u_r}\right] + i\left(\lambda\left[\underline{v_s}, \overline{v_s}\right]\right) \\ &= \left\{\lambda a + i\lambda b : a \in \left[\underline{u_r}, \overline{u_r}\right],\ b \in \left[\underline{v_s}, \overline{v_s}\right]\right\} \end{aligned}$$

on $\mathbb{I}_{\mathbb{C}}$, where $i = \sqrt{-1}$ and $\lambda \in \mathbb{C}$. Further, the relation

$$u \leq v \text{ iff } \left[\underline{u_r}, \overline{u_r}\right] \subseteq \left[\underline{v_r}, \overline{v_r}\right] \text{ and } \left[\underline{u_s}, \overline{u_s}\right] \subseteq \left[\underline{v_s}, \overline{v_s}\right]$$

is a partial order relation on $\mathbb{I}_{\mathbb{C}}$. Thus, $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space, [9].

If X is a quasilinear space and $Y \subseteq X$, then Y is called a *subspace of* X whenever Y is a quasilinear space with the same partial order and the restriction to Y of the operations on X. Y is subspace of a quasilinear space X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{K}$, $\alpha x + \beta y \in Y$. Proof of this theorem is quite similar to its classical linear space analogue. Let Y be a subspace of a quasilinear space X and suppose each element x in Y has an inverse in Y. Then, the partial order on Y is determined by the equality. In this case Y is a linear subspace of X, [13].

An element x in a quasilinear space X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. Also, X_r stands for the set of all regular elements of X while X_s stands for the sets of all singular elements and zero in X. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular, symmetric* and *singular subspaces* of X, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of X is a linear space while the singular one is nonlinear at all. Further, $\mathbb{I}_{\mathbb{C}}$ is a closed subspace of $\Omega(\mathbb{C})$, [5].

A real-valued function $\|.\|$ on the quasilinear space X is called a *norm* if the following conditions hold:

$$\|x\| > 0 \text{ if } x \neq 0,$$
$$\|x + y\| \leq \|x\| + \|y\|,$$
$$\|\alpha x\| = |\alpha| \, \|x\|,$$
$$\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|,$$

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that}$$
$$x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon \text{ then } x \preceq y,$$

where x, y, x_ε are arbitrary element in X and α is any scalar. A quasilinear space X with a norm defined on it, is called *normed quasilinear space*, [3].

For a normed linear space E, a norm on $\Omega(E)$ is defined by

$$\|A\|_\Omega = \sup_{a \in E} \|a\|_E \; .$$

Hence, $\Omega_C(E)$ and $\Omega(E)$ are normed quasilinear spaces. A norm on $\mathbb{I}_{\mathbb{R}}$ is defined by

$$\|x\| = \left\| \left[\underline{x}, \overline{x} \right] \right\| = \sup_{t \in [\underline{x}, \overline{x}]} |t| \, .$$

Moreover, $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space with the norm

$$\begin{aligned} \|X\|_{\mathbb{I}_{\mathbb{C}}} &= \sup \{|z| : z \in X\} \\ &= \sup\{|a + ib| : a \in \left[\underline{x_r}, \overline{x_r}\right], b \in \left[\underline{x_s}, \overline{x_s}\right]\}, \end{aligned}$$

for $X = \left[\underline{x_r}, \overline{x_r}\right] + i \left[\underline{x_s}, \overline{x_s}\right]$, [12].

3. Main Results

A complex interval matrix $\hat{A}$ is a matrix whose elements are complex intervals. A $(m \times n)$ complex interval matrix $\hat{A}$ is written as

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} & . & . & . & \hat{A}_{1n} \\ \hat{A}_{21} & \hat{A}_{22} & . & . & . & \hat{A}_{2n} \\ . & . & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ \hat{A}_{m1} & \hat{A}_{m2} & . & . & . & \hat{A}_{mn} \end{pmatrix}_{m \times n} = (\hat{A}_{ij})_{m \times n}$$

such that

$$\hat{A}_{ij} = [\underline{A}^r_{ij}, \overline{A}^r_{ij}] + i[\underline{A}^s_{ij}, \overline{A}^s_{ij}],$$

where $[\underline{A}^r_{ij}, \overline{A}^r_{ij}]$ and $[\underline{A}^s_{ij}, \overline{A}^s_{ij}]$ are real and imaginary part of the component $\hat{A}_{ij}$, respectively. The set of all $(m \times n)$ complex interval matrix is denoted by $\mathbb{I}^{m \times n}_{\mathbb{C}}$. For example,

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix} = \begin{pmatrix} [1, 2] + i[0, 1] & [-1, 3] \\ \{2\} + i[-1, 1] & [2, 3] + i\{5\} \end{pmatrix}$$

is a (2×2) complex interval matrix.

If $\hat{A} = (\hat{A}_{ij})_{m\times n}$ is complex interval matrix with $\hat{A}_{ij} = [\underline{A}^r_{ij}, \overline{A}^r_{ij}] + i[\underline{A}^s_{ij}, \overline{A}^s_{ij}]$ and $B = (b_{ij})_{m\times n}$ is a complex matrix with $b_{ij} = b^r_{ij} + ib^s_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$ such that $b^r_{ij} \in [\underline{A}^r_{ij}, \overline{A}^r_{ij}]$ and $b^s_{ij} \in [\underline{A}^s_{ij}, \overline{A}^s_{ij}]$, then we say that $B \in \hat{A}$.

We define arithmetic operations a complex interval matrices as follows:

(i):

$$\hat{A} + \hat{B} = (\hat{A}_{ij})_{m\times n} + (\hat{B}_{ij})_{m\times n} = (\hat{A}_{ij} + \hat{B}_{ij})_{m\times n}, \tag{3.1}$$

(ii):

$$\lambda\hat{A} = \lambda(\hat{A}_{ij})_{m\times n} = (\lambda\hat{A}_{ij})_{m\times n}, \lambda \in \mathbb{R}, \tag{3.2}$$

(iii):

$$\hat{A}.\hat{B} = (\hat{A}_{ik})_{m\times n}.(\hat{B}_{kj})_{n\times p} = (\hat{C}_{ij})_{m\times p}$$

such that

$$\hat{C}_{ij} = \sum_{k=1}^{n} \hat{A}_{ik}\hat{B}_{kj},$$

where $\hat{A}_{ik}\hat{B}_{kj}$ is the product of two complex intervals.

Let us give an example for (iii):

Suppose that

$$\hat{A} = \begin{pmatrix} [-1,0] + i[1,2] & \{3\} + i[-1,2] \\ [0,1] + i\{5\} & [-2,-1] + i[-1,1] \end{pmatrix}_{2\times 2}$$

and

$$\hat{B} = \begin{pmatrix} [1,2] + i[0,1] \\ \{-1\} + i[0,2] \end{pmatrix}_{2\times 1}.$$

Then,

$$\begin{aligned} \hat{A}_{11}\hat{B}_{11} &= ([-1,0] + i[1,2])([1,2] + i[0,1]) \\ &= [-1,0][1,2] - [1,2][0,1] + i([1,2][1,2] + [-1,0][1,2]) \\ &= [-2,0] - [0,2] + i([1,4] + [-2,0]) = [-4,0] + i[-1,4], \end{aligned}$$

$$\begin{aligned} \hat{A}_{12}\hat{B}_{21} &= (\{3\} + i[-1,2])(\{-1\} + i[0,2]) \\ &= \{3\}\{-1\} - [-1,2][0,2] + i(\{3\}[0,2] + [-1,2]\{-1\}) \\ &= \{-3\} - [-2,4] + i([0,6] + [-2,1]) = [-5,1] + i[-2,7], \end{aligned}$$

$$\begin{aligned} \hat{A}_{11}\hat{B}_{11} + \hat{A}_{12}\hat{B}_{21} &= ([-4,0] + i[-1,4]) + ([-5,1] + i[-2,7]) \\ &= [-9,1] + i[-3,11], \end{aligned}$$

$$\begin{aligned} \hat{A}_{12}\hat{B}_{11} &= ([0,1] + i\{5\})([1,2] + i[0,1]) \\ &= [0,1][1,2] - \{5\}[0,1] + i(\{5\}[1,2] + [0,1][0,1]) \\ &= [0,2] - [0,5] + i([5,10] + [0,1]) = [-5,2] + i[5,11], \end{aligned}$$

$$\begin{aligned} \hat{A}_{22}\hat{B}_{21} &= ([-2,-1] + i[-1,1])(\{-1\} + i[0,2]) \\ &= [-2,-1]\{-1\} - [-1,1][0,2] + i([-2,-1][0,2] + \{-1\}[-1,1]) \\ &= [1,2] - [-2,2] + i([-4,-2] + [-1,1]) = [-1,4] + i[-5,-1], \end{aligned}$$

$$\begin{aligned} \hat{A}_{12}\hat{B}_{11} + \hat{A}_{22}\hat{B}_{21} &= ([-5,2] + i[5,11]) + ([-1,4] + i[-5,-1]) \\ &= [-6,6] + i[0,10]. \end{aligned}$$

Hence; we obtain that

$$\hat{A}.\hat{B} = \begin{pmatrix} [-9,1] + i[-3,11] \\ [-6,6] + i[0,10] \end{pmatrix}_{2\times 1}.$$

It is obvious that the product of two complex interval matrices is again a complex interval matrix.

The space $\mathbb{I}_{\mathbb{C}}^{m\times n}$ is not a linear space, since some complex interval matrices have not an additive inverse; e.g.; let

$$\hat{A}=\begin{pmatrix} [1,2]+i[-1,0] & [-1,1]+i\{2\} \\ \{3\}+i[0,2] & [-2,-1]+i[1,2] \end{pmatrix}. \tag{3.3}$$

Then,

$$\begin{aligned}\hat{A}+(-1)\hat{A} &= \begin{pmatrix} [1,2]+i[-1,0] & [-1,1]+i\{2\} \\ \{3\}+i[0,2] & [-2,-1]+i[1,2] \end{pmatrix}+\begin{pmatrix} [-2,-1]+i[0,1] & [-1,1]+i\{-2\} \\ \{-3\}+i[-2,0] & [1,2]+i[-2,-1] \end{pmatrix} \\ &= \begin{pmatrix} [-1,1]+i[-1,1] & [-2,2] \\ i[-2,2] & [-1,1]+i[-1,1] \end{pmatrix} \neq \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix},\end{aligned}$$

where $\theta=\begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix}\cong\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is unit element of the algebraic sum operation.

Now, let us the relation "$\ll$" such that

$$\hat{A}=(\hat{A}_{ij})_{m\times n}\ll\hat{B}=(\hat{B}_{ij})_{m\times n} \text{ iff } \hat{A}_{ij}\leq\hat{B}_{ij} \text{ for each } 1\leq i\leq m, 1\leq j\leq n,$$

where the relation "$\leq$" is the partial order relation on $\mathbb{I}_{\mathbb{C}}$. For example, let

$$\hat{A}=\begin{pmatrix} [0,1]+i[-1,2] & \{3\}+i[2,3] \\ [1,2]+i\{-2\} & [-1,3]+i[0,7] \end{pmatrix}$$

and

$$\hat{B}=\begin{pmatrix} [-1,1]+i[-2,3] & [1,4]+i[-1,5] \\ [-2,3]+i\{-2\} & [-2,4]+i[0,9] \end{pmatrix}.$$

It can easily be seen that $\hat{A}_{11}\ll\hat{B}_{11}$, $\hat{A}_{12}\ll\hat{B}_{12}$, $\hat{A}_{21}\ll\hat{B}_{21}$, $\hat{A}_{22}\ll\hat{B}_{22}$. Therefore, $\hat{A}\ll\hat{B}$.

Theorem 3.1. *$\mathbb{I}_{\mathbb{C}}^{m\times n}$ is a quasilinear space with the operations given by (3.1) and (3.2) and the partial order relation "$\ll$".*

Proof. Verification of the axioms (1-10) is easy. For $\hat{A}=(\hat{A}_{ij})_{m\times n}\in\mathbb{I}_{\mathbb{C}}^{m\times n}$ and $\alpha,\beta\in\mathbb{R}$ we write

$$(\alpha+\beta)\hat{A}=((\alpha+\beta)\hat{A}_{ij})\ll(\alpha\hat{A}_{ij}+\beta\hat{A}_{ij})=(\alpha\hat{A}_{ij})+(\beta\hat{A}_{ij})=\alpha\hat{A}+\beta\hat{A}$$

since $\hat{A}_{ij}\in\mathbb{I}_{\mathbb{C}}$ for each $1\leq i\leq m, 1\leq j\leq n$ and $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space. Further, if $\hat{A}\ll\hat{B}$ and $\hat{C}\ll\hat{D}$ then $\hat{A}_{ij}\leq\hat{B}_{ij}$ and $\hat{C}_{ij}\leq\hat{D}_{ij}$ for each $1\leq i\leq m, 1\leq j\leq n$. Since $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space, we say that $\hat{A}_{ij}\leq\hat{C}_{ij}$ and $\hat{B}_{ij}\leq\hat{D}_{ij}$ for each $1\leq i\leq m, 1\leq j\leq n$. This implies that $\hat{A}\ll\hat{C}$ and $\hat{B}\ll\hat{D}$. Now, suppose that $\hat{A}\ll\hat{B}$. Then, $\hat{A}_{ij}\leq\hat{B}_{ij}$ for each $1\leq i\leq m, 1\leq j\leq n$. Further, we write that $\alpha\hat{A}_{ij}\leq\alpha\hat{B}_{ij}$ for each $1\leq i\leq m, 1\leq j\leq n$ and for any $\alpha\in\mathbb{R}$. Therefore, $\alpha\hat{A}\ll\alpha\hat{B}$. □

Now, we will put a norm on the quasilinear space $\mathbb{I}_{\mathbb{C}}^{m\times n}$.

Theorem 3.2. *$\mathbb{I}_{\mathbb{C}}^{m\times n}$ is a normed quasilinear space with the norm*

$$\left\|\hat{A}\right\|=\max_i\sum_j\left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}},$$

where $\hat{A}=(\hat{A}_{ij})_{m\times n}$ and $1\leq i\leq m, 1\leq j\leq n$.

Proof. (1) Suppose that $\hat{A}\neq\theta$. Then we write that $\hat{A}_{ij}\neq\theta$ for at least the fixed i and j. This means that $\left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}}\neq 0$ and so $\left\|\hat{A}\right\|=\max_i\sum_j\left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}}\neq 0$.

(2)

$$\begin{aligned}\left\|\hat{A}+\hat{B}\right\| &= \max_i \sum_j \left\|\hat{A}_{ij}+\hat{B}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \\ &\leq \max_i \sum_j (\left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} + \left\|\hat{B}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}}) \\ &= \max_i \sum_j \left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} + \max_i \sum_j \left\|\hat{B}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \\ &= \left\|\hat{A}\right\| + \left\|\hat{B}\right\|.\end{aligned}$$

(3)

$$\begin{aligned}\left\|\alpha\hat{A}\right\| &= \max_i \sum_j \left\|\alpha\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \\ &= \max_i \sum_j |\alpha| \left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \\ &= |\alpha| \max_i \sum_j \left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \\ &= |\alpha| \left\|\hat{A}\right\|.\end{aligned}$$

(4) Assume that, $\hat{A} \ll \hat{B}$. Then, $\hat{A}_{ij} \leq \hat{B}_{ij}$ for each $1 \leq i \leq m, 1 \leq j \leq n$. By the fourth condition of norm on $\mathbb{I}_{\mathbb{C}}$ we have that $\left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \leq \left\|\hat{B}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}}$. This implies that

$$\sum_j \left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \leq \sum_j \left\|\hat{B}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}}.$$

Hence, we have that

$$\max_i \sum_j \left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \leq \max_i \sum_j \left\|\hat{B}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}}$$

and so

$$\left\|\hat{A}\right\| \leq \left\|\hat{B}\right\|.$$

(5) Let $\varepsilon > 0$ be arbitrary and suppose that there exists an element $\hat{A}^{\varepsilon} \in \mathbb{I}_{\mathbb{C}}^{m\times n}$ such that $\hat{A} \ll \hat{B} + \hat{A}^{\varepsilon}$ and $\left\|\hat{A}^{\varepsilon}\right\| \leq \varepsilon$. Then, we write that $\hat{A}_{ij} \leq \hat{B}_{ij} + \hat{A}^{\varepsilon}_{ij}$ and $\left\|\hat{A}^{\varepsilon}_{ij}\right\| \leq \varepsilon$ for each $1 \leq i \leq m, 1 \leq j \leq n$. By the fifth condition of norm on $\mathbb{I}_{\mathbb{C}}$ we say that $\hat{A}_{ij} \leq \hat{B}_{ij}$ for each $1 \leq i \leq m, 1 \leq j \leq n$. This means that $\hat{A} \ll \hat{B}$.

□

Example 3.3. Let us calculate the norm of the complex interval matrix $\hat{A}$ given in (3.3):

$$\begin{aligned}\left\|\hat{A}\right\| &= \max_{1\leq i\leq 2} \sum_{1\leq j\leq 2} \left\|\hat{A}_{ij}\right\|_{\mathbb{I}_{\mathbb{C}}} \\ &= \max\{\|[1,2]+i[-1,0]\|_{\mathbb{I}_{\mathbb{C}}} + \|[-1,1]+i\{2\}\|_{\mathbb{I}_{\mathbb{C}}}, \\ &\quad \|\{3\}+i[0,2]\|_{\mathbb{I}_{\mathbb{C}}} + \|[-2,-1]+i[1,2]\|_{\mathbb{I}_{\mathbb{C}}}\} \\ &= \max\{\sqrt{5}+\sqrt{5}, \sqrt{13}+2\sqrt{2}\} \\ &= \sqrt{13}+2\sqrt{2}.\end{aligned}$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Both authors have created and written the article.

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