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*Research Article*

# Identities of Generalized Pell and Pell-Lucas Sequences

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**ABSTRACT:** This paper presents sums of generalized Pell and Pell-Lucas sequences. Tulay Yagmur introduced these sequences in 2019. We have used their Generating function, Binet's formula and Induction method to derive the identities. We establish some connection formulae of involving them. Also, we present its two cross two matrix representation.

**Keywords:** Generalized Pell sequence, Generalized Pell-Lucas sequence, Binet's formula and Generating function.

## 1. INTRODUCTION

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences.

The sequence of Fibonacci numbers [6], $F_n$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \; n \geq 2 \; with \; F_0 = 0, \; F_1 = 1 \tag{1}$$

The sequence of Lucas numbers [6], $L_n$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \; n \geq 2 \; with \; L_0 = 2, \; L_1 = 1 \tag{2}$$

The sequence of Pell numbers [7], $P_n$ is defined by

$$P_n = 2P_{n-1} + P_{n-2}, \; n \geq 2 \; with \; P_0 = 0, \; P_1 = 1 \tag{3}$$

The sequence of Pell-Lucas numbers [7], $Q_n$ is defined by

$$Q_n = 2Q_{n-1} + Q_{n-2}, \; n \geq 2 \; with \; Q_0 = 2, \; Q_1 = 2 \tag{4}$$

Goksal Bilgici [1], defined new generalizations of Fibonacci and Lucas sequences for any real nonzero numbers a and b,

$$f_k = 2af_{k-1} + (b - a^2) f_{k-2}, \; k \geq 2 \; with \; f_0 = 0, \; f_1 = 1 \tag{5}$$

$$l_k = 2al_{k-1} + (b - a^2) l_{k-2}, \; k \geq 2 \; with \; l_0 = 2, \; l_1 = 2a \tag{6}$$

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Tulay Yagmur [9], defined generalizations of Pell and Pell-Lucas sequences
$p_k = 2ap_{k-1} + (b-a^2)p_{k-2}$ , $k \geq 2$ *with* $p_0 = 0$, $p_1 = 1$
$q_k = 2aq_{k-1} + (b-a^2)q_{k-2}$ , $k \geq 2$ *with* $q_0 = 2$, $q_1 = 2a$

The main objective of this study is to give some Explicit sums of generalized Pell and Pell-Lucas sequences. Moreover, we introduce the special sums of the generalized Pell and Pell-Lucas sequences and prove them using Binet's formula.

## 2. PRELIMINARIES

In this section, we review basic definitions and introduce relevant facts.

For $n \geq 2$ , The generalized Pell sequence [9], is defined by

$$p_k = 2ap_{k-1} + (b-a^2)p_{k-2} \text{ , } k \geq 2 \text{ with } p_0 = 0,\ p_1 = 1 \tag{7}$$

First few generalized Pell numbers are

$$\{p_n\} = \{0, 1, 2a, 3a^2 + b, 4a^3 + 4ab, 5a^4 + 10a^2b + b^2, ...\}$$

For $n \geq 2$ , The generalized Pell-Lucas sequence [9], is defined by

$$q_k = 2aq_{k-1} + (b-a^2)q_{k-2} \text{ , } k \geq 2 \text{ with } q_0 = 2,\ q_1 = 2a \tag{8}$$

First few generalized Jacobsthal-Lucas numbers are

$$\{q_n\} = \{2, 2a, 2a^2 + 2b, 2a^3 + 6ab, 2a^4 + 12a^2b + 2b^2, 2a^5 + 20a^3b + 10ab^2, ...\}$$

Any nonzero real numbers are in (7) and (8).

If $a = 1$ & $b = 2$, then we obtained classical Pell sequence and Pell-Lucas sequences,
If $a = \frac{1}{2}$ & $b = \frac{9}{4}$, then we obtained classical Jacobsthal and Jacobsthal-Lucas sequences,
If $a = \frac{1}{2}$ & $b = \frac{5}{4}$, then we obtained classical Fibonacci and Lucas sequences,
If $a = \frac{3}{2}$ & $b = \frac{1}{4}$, then we obtained classical Mersenne and Fermat sequences.

For any positive integer *k*,

If $a = 1$ & $b = (1+k)$, then we obtained *k*-Pell and *k*-Pell-Lucas sequences,
If $a = \frac{k}{2}$ & $b = \left(\frac{4+k^2}{4}\right)$, then we obtained *k*-Fibonacci and *k*-Lucas sequences,
If $a = \frac{k}{2}$ & $b = \left(\frac{8+k^2}{4}\right)$, then we obtained *k*- Jacobsthal and *k*- Jacobsthal-Lucas sequences.

Generating function for generalized Pell and Pell-Lucas numbers are

$$\sum_{k=0}^{\infty} p_k x^k = \frac{x}{1-2ax-(b-a^2)x^2} \tag{9}$$

$$\sum_{k=0}^{\infty} q_k x^k = \frac{2-2ax}{1-2ax-(b-a^2)x^2} \tag{10}$$

The Binet's formula for generalized Pell and Pell-Lucas numbers are

$$p_k = \frac{\Re_1^k - \Re_2^k}{\Re_1 - \Re_2} \tag{11}$$

$$q_k = \Re_1^k + \Re_2^k \tag{12}$$

where $\Re_1$ and $\Re_2$ are the roots of the characteristic equation,

$$x^2 - 2ax - (b-a^2) = 0 \tag{13}$$

with $\Re_1 = a+\sqrt{b}$, $\Re_2 = a-\sqrt{b}$ ; $\Re_1+\Re_2 = 2a$ , $\Re_1-\Re_2 = 2\sqrt{b}$ ,$\Re_1\Re_2 = a^2-b$.

Also $p_{-k} = \frac{-1}{(a^2-b)^k} p_k$ *and* $q_{-k} = \frac{1}{(a^2-b)^k} q_k$ .

## 3. IDENTITIES OF THE GENERALIZED PELL AND PELL-LUCAS SEQUENCE

This section introduces and proves some interesting identities of generalized Pell and Pell-Lucas sequences.

### 3.1. Explicit Sums of generalized Pell and Pell-Lucas Sequence

This section studies the sums of generalized Pell and Pell-Lucas sequences. This enables us to give in a straightforward way several formulas for the sums of such numbers.

**Theorem 1:** Explicit sum formula for generalized Pell sequence

$$p_k = \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \binom{k-i-1}{i} (2a)^{k-2i-1} (b-a^2)^i \tag{14}$$

**Proof:** The proof is clear from the generating function of generalized Pell sequence.

**Theorem 2:** Explicit sum Formula generalized Pell-Lucas sequence

$$q_k = 2\sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k-i}{i} (2a)^{k-2i} (b-a^2)^i - \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \binom{k-i-1}{i} (2a)^{k-2i} (b-a^2)^i \tag{15}$$

**Proof:** The proof is clear from the generating function of generalized Pell-Lucas sequence.

**Lemma 3:** For every $s\ and\ t$, the following equality holds

$$p_{s(n+2)+t} = q_s\ p_{s(n+1)+t} - (a^2-b)^s\ p_{sn+t} \tag{16}$$

**Proof:** From the Binet's formula of generalized Pell and Pell-Lucas sequence,

$$\begin{aligned}
q_s\ p_{s(n+1)+t} &= \left(\Re_1^s + \Re_2^s\right)\left(\frac{\Re_1^{s(n+1)+t} - \Re_2^{s(n+1)+t}}{\Re_1 - \Re_2}\right) \\
&= \frac{1}{\Re_1 - \Re_2}\left[\Re_1^{s(n+2)+t} + (a^2-b)^s \Re_1^{sn+t} - (a^2-b)^s \Re_2^{sn+t} - \Re_2^{s(n+2)+t}\right] \\
&= \frac{1}{\Re_1 - \Re_2}\left[\left\{\Re_1^{s(n+2)+t} - \Re_2^{s(n+2)+t}\right\} + (a^2-b)^s\left(\Re_1^{sn+t} - \Re_2^{sn+t}\right)\right] \\
&= p_{s(n+2)+t} + (a^2-b)^s\ p_{sn+t}
\end{aligned}$$

then, equality becomes,

$$p_{s(n+2)+t} = q_s\ p_{s(n+1)+t} - (a^2-b)^s\ p_{sn+t}$$

**Theorem 4:** For fixed integers $s,\ t\ with\ 0 \le t \le s-1$, the following equality holds

$$\sum_{i=0}^{n} p_{si+t} = \frac{p_{s(n+1)+t} - (a^2-b)^t\ p_{s-t} - p_t - (a^2-b)^s\ p_{sn+t}}{q_s - (a^2-b)^s - 1} \tag{17}$$

**Proof:** From the Binet's formula of generalized Pell sequence,

$$\begin{aligned}
\sum_{i=0}^{n} p_{si+t} &= \sum_{i=0}^{n} \frac{\Re_1^{si+t} - \Re_2^{si+t}}{\Re_1 - \Re_2} \\
&= \frac{1}{\Re_1 - \Re_2}\left[\sum_{i=0}^{n} \Re_1^{si+t} - \sum_{i=0}^{n} \Re_2^{si+t}\right] \\
&= \frac{1}{\Re_1 - \Re_2}\left[\frac{\Re_1^{sn+t+s} - \Re_1^t}{\Re_1^s - 1} - \frac{\Re_2^{sn+t+s} - \Re_2^t}{\Re_2^s - 1}\right] \\
&= \frac{1}{(a^2-b)^s - q_s + 1}\left[(a^2-b)^s\ p_{sn+t} - p_{s(n+1)+t} + p_t + (a^2-b)^t\ p_{s-t}\right] \\
&= \frac{p_{s(n+1)+t} - (a^2-b)^t\ p_{s-t} - p_t - (a^2-b)^s\ p_{sn+t}}{q_s - (a^2-b)^s - 1}
\end{aligned}$$

This completes the proof.

**Corollary 5:** Sum of odd new generalized Pell sequence,

If $s=2m+1$ then Eq. (17) is

$$\sum_{i=0}^{n} p_{(2m+1)i+t} = \frac{p_{(2m+1)(n+1)+t} - (a^2-b)^t p_{2m+1-t} - p_t - (a^2-b)^{(2m+1)} p_{(2m+1)n+t}}{q_{(2m+1)} - (a^2-b)^{(2m+1)} - 1} \tag{18}$$

For example

(1) If $m=0$ then $\mathrm{s}=1$

$$\sum_{i=0}^{n} p_{i+t} = \frac{p_{n+t+1} - (a^2-b)^t p_{1-t} - p_t - (a^2-b) p_{n+t}}{2a-(a^2-b)-1} \tag{19}$$

(i) For $t=0$ : $\sum_{i=0}^{n} p_i = \frac{p_{n+1} - 1 - (a^2-b) p_n}{2a-(a^2-b)-1}$

(2) If $m=1$ then $\mathrm{s}=3$

$$\sum_{i=0}^{n} p_{3i+t} = \frac{p_{3n+t+3} - (a^2-b)^t p_{3-t} - p_t - (a^2-b)^3 p_{3n+t}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1} \tag{20}$$

(i) For $t=0$ : $\sum_{i=0}^{n} p_{3i} = \frac{p_{3n+3} - (3a^2+b) - (a^2-b)^3 p_{3n}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(ii) For $t=1$ : $\sum_{i=0}^{n} p_{3i+1} = \frac{p_{3n+4} - 2a(a^2+b) - 1 - (a^2-b)^3 p_{3n+1}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(iii) For $t=2$ : $\sum_{i=0}^{n} p_{3i+2} = \frac{p_{3n+5} - (a^2-b)^2 - 2a - (a^2-b)^3 p_{3n+2}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(3) If $m=2$ then $\mathrm{s}=5$

$$\sum_{i=0}^{n} p_{5i+t} = \frac{p_{5n+t+5} - (a^2-b)^t p_{5-t} - p_t - (a^2-b)^5 p_{5n+t}}{q_5 - (a^2-b)^5 - 1} \tag{21}$$

(i) For $t=0$ : $\sum_{i=0}^{n} p_{5i} = \frac{p_{5n+5} - p_5 - (a^2-b)^5 p_{5n}}{q_5 - (a^2-b)^5 - 1}$

(ii) For $t=1$ : $\sum_{i=0}^{n} p_{5i+1} = \frac{p_{5n+6} - (a^2-b) p_4 - 1 - (a^2-b)^5 p_{5n+1}}{q_5 - (a^2-b)^5 - 1}$

(iii) For $t=2$ : $\sum_{i=0}^{n} p_{5i+2} = \frac{p_{5n+7} - (a^2-b)^2 p_3 - 2a - (a^2-b)^5 p_{5n+2}}{q_5 - (a^2-b)^5 - 1}$

(iv) For $t=3$ : $\sum_{i=0}^{n} p_{5i+3} = \frac{p_{5n+8} - (a^2-b)^3 2a - (3a^2+b) - (a^2-b)^5 p_{5n+3}}{q_5 - (a^2-b)^5 - 1}$

(v) For $t=4$ : $\sum_{i=0}^{n} p_{5i+4} = \frac{p_{5n+9} - (a^2-b)^4 - 4a(a^2+b) - (a^2-b)^5 p_{5n+4}}{q_5 - (a^2-b)^5 - 1}$

(vi) For $t=5$ : $\sum_{i=0}^{n} p_{5i+5} = \frac{p_{5n+10} - p_5 - (a^2-b)^5 p_{5n+5}}{q_5 - (a^2-b)^5 - 1}$

**Corollary 6:** Sum of even generalized Pell sequence, if $s = 2m$ then Eq. (17) is

$$\sum_{i=0}^{n} p_{2mi+t} = \frac{p_{2m(n+1)+t} - (a^2-b)^t p_{2m-t} - p_t - (a^2-b)^{2m} p_{2mn+t}}{q_{2m} - (a^2-b)^{2m} - 1} \tag{22}$$

For example

(1) If $m = 1$ then $s = 2$

$$\sum_{i=0}^{n} p_{2i+t} = \frac{p_{2n+2+t} - (a^2-b)^t p_{2-t} - p_t - (a^2-b)^2 p_{2n+t}}{q_2 - (a^2-b)^2 - 1} \tag{23}$$

(i) For $t = 0$ : $\sum_{i=0}^{n} p_{2i} = \frac{p_{2n+2} - 2a - (a^2-b)^2 p_{2n}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(ii) For $t = 1$ : $\sum_{i=0}^{n} p_{2i+1} = \frac{p_{2n+3} - (a^2-b) - 1 - (a^2-b)^2 p_{2n+1}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(iii) For $t = 2$ : $\sum_{i=0}^{n} p_{2i+2} = \frac{p_{2n+4} - 2a - (a^2-b)^2 p_{2n+2}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(2) If $m = 2$ then $s = 4$

$$\sum_{i=0}^{n} p_{4i+t} = \frac{p_{4n+4+t} - (a^2-b)^t p_{4-t} - p_t - (a^2-b)^4 p_{4n+t}}{q_4 - (a^2-b)^4 - 1} \tag{24}$$

(i) For $t = 0$ : $\sum_{i=0}^{n} p_{4i} = \frac{p_{4n+4} - p_4 - (a^2-b)^4 p_{4n}}{q_4 - (a^2-b)^4 - 1}$

(ii) For $t = 1$ : $\sum_{i=0}^{n} p_{4i+1} = \frac{p_{4n+5} - (a^2-b) p_3 - 1 - (a^2-b)^4 p_{4n+1}}{q_4 - (a^2-b)^4 - 1}$

(iii) For $t = 2$ : $\sum_{i=0}^{n} p_{4i+2} = \frac{p_{4n+6} - (a^2-b)^2 2a - 2a - (a^2-b)^4 p_{4n+2}}{q_4 - (a^2-b)^4 - 1}$

(iv) For $t = 3$ : $\sum_{i=0}^{n} p_{4i+3} = \frac{p_{4n+7} - (a^2-b)^3 - p_3 - (a^2-b)^4 p_{4n+3}}{q_4 - (a^2-b)^4 - 1}$

(v) For $t = 4$ : $\sum_{i=0}^{n} p_{4i+4} = \frac{p_{4n+8} - p_4 - (a^2-b)^4 p_{4n+4}}{q_4 - (a^2-b)^4 - 1}$

(3) If $m = 3$ then $s = 6$

$$\sum_{i=0}^{n} p_{6i+t} = \frac{p_{6n+6+t} - (a^2-b)^t p_{6-t} - p_t - (a^2-b)^6 p_{6n+t}}{q_6 - (a^2-b)^6 - 1} \tag{25}$$

(i) For $t = 0$ : $\sum_{i=0}^{n} p_{6i} = \frac{p_{6n+6} - p_6 - (a^2-b)^6 p_{6n}}{q_6 - (a^2-b)^6 - 1}$

(ii) For $t = 1$ : $\sum_{i=0}^{n} p_{6i+1} = \frac{p_{6n+7} - (a^2-b) p_5 - 1 - (a^2-b)^6 p_{6n+1}}{q_6 - (a^2-b)^6 - 1}$

(iii) For $t=2$ : $\sum_{i=0}^{n} p_{6i+2} = \frac{p_{6n+8} - (a^2-b)^2 p_4 - 2a - (a^2-b)^6 p_{6n+2}}{q_6 - (a^2-b)^6 - 1}$

(iv) For $t=3$ : $\sum_{i=0}^{n} p_{6i+3} = \frac{p_{6n+9} - (a^2-b)^2 p_3 - p_3 - (a^2-b)^6 p_{6n+3}}{q_6 - (a^2-b)^6 - 1}$

**Theorem 7:** For fixed integers $s,\ t\ with\ 0 \le t \le s-1$, the following equality holds

$$\sum_{i=0}^{n} (-1)^i p_{si+t} = \frac{(-1)^n p_{s(n+1)+t} + (-1)^n (a^2-b)^s p_{sn+t} - (a^2-b)^t p_{s-t} + p_t}{q_s + (a^2-b)^s + 1} \tag{26}$$

For different values of $s$ and $t$:

(i) $\sum_{i=0}^{n} (-1)^i p_i = \frac{(-1)^n p_{n+1} + (-1)^n (a^2-b) p_n - 1}{2a + a^2 - b + 1}$

(ii) $\sum_{i=0}^{n} (-1)^i p_{2i} = \frac{(-1)^n p_{2n+2} + (-1)^n (a^2-b)^2 p_{2n} - 2a}{(2a^2+2b) + (a^2-b)^2 + 1}$

(iii) $\sum_{i=0}^{n} (-1)^i p_{2i+1} = \frac{(-1)^n p_{2n+3} + (-1)^n (a^2-b)^2 p_{2n+1} - (a^2-b) + 1}{(2a^2+2b) + (a^2-b)^2 + 1}$

(iv) $\sum_{i=0}^{n} (-1)^i p_{4i} = \frac{(-1)^n p_{4n+4} + (-1)^n (a^2-b)^4 p_{4n} - p_4}{q_4 + (a^2-b)^4 + 1}$

(v) $\sum_{i=0}^{n} (-1)^i p_{4i+1} = \frac{(-1)^n p_{4n+5} + (-1)^n (a^2-b)^4 p_{4n+1} - (a^2-b) p_3 + 1}{q_4 + (a^2-b)^4 + 1}$

(vi) $\sum_{i=0}^{n} (-1)^i p_{4i+2} = \frac{(-1)^n p_{4n+6} + (-1)^n (a^2-b)^4 p_{4n+2} - (a^2-b)^2 2a + 2a}{q_4 + (a^2-b)^4 + 1}$

(vii) $\sum_{i=0}^{n} (-1)^i p_{4i+3} = \frac{(-1)^n p_{4n+7} + (-1)^n (a^2-b)^4 p_{4n+3} - (a^2-b)^3 + p_3}{q_4 + (a^2-b)^4 + 1}$

### 3.2. Product of Generalized Pell and Pell-Lucas Sequences

In this section, we present identities involving the product of generalized Pell and Pell-Lucas numbers and related identities consisting of even and odd terms.

**Theorem 8:** If $p_k\ and\ q_k$ are generalized Pell and Pell-Lucas numbers, then holds for every $k\ and\ s$,

I. $p_{2k+s} q_{2k+1} = p_{4k+s+1} + (a^2-b)^{2k+1} p_{s-1}$ (27)

II. $p_{2k+s} q_{2k+2} = p_{4k+s+2} + (a^2-b)^{2k+2} p_{s-2}$ (28)

III. $p_{2k+s} q_{2k} = p_{4k+s} + (a^2-b)^{2k} p_s$ (29)

**Theorem 9:**

I. $p_{2k-s}q_{2k+1} = p_{4k-s+1} + (a^2-b)^{2k+1} p_{-s-1}$ (30)

II. $p_{2k-s}q_{2k-1} = p_{4k-s-1} + (a^2-b)^{2k-1} p_{1-s}$ (31)

III. $p_{2k-s}q_{2k} = p_{4k-s} + (a^2-b)^{2k} p_{-s}$ (32)

**Theorem 10:**

I. $p_{2k}q_{2k+s} = p_{4k+s} - (a^2-b)^{2k} p_s$ (33)

II. $4bp_{2k}p_{2k+s} = q_{4k+s} - (a^2-b)^{2k} q_s$ (34)

III. $q_{2k}q_{2k+s} = q_{4k+s} + (a^2-b)^{2k} q_s$ (35)

The proof is clear by the Binet's formula of generalized Pell and Pell-Lucas numbers.

### 3.2. Sum and difference of squares Generalized Pell and Pell-Lucas Sequences

In this section, the sum and difference of generalized Pell and Pell-Lucas numbers are treated in the following theorem.

**Theorem 11:** $4b\left(p_{n+1}^2 + p_{n-1}^2\right) = q_{2n+2} + q_{2n-2} - 2(a^2-b)^{n-1}\{(a^2-b)^2+1\}$ (36)

**Theorem 12:** $4b\left(p_{n+1}^2 - p_{n-1}^2\right) = q_{2n+2} - q_{2n-2} - 2(a^2-b)^{n-1}\{(a^2-b)^2-1\}$ (37)

The proof is clear by the Binet's formula of generalized Pell and Pell-Lucas numbers.

### 3.3. Matrix Representation of Generalized Pell and Pell-Lucas Sequences

In this section, we present two crosses and two matrices for generalized Pell and Pell-Lucas sequences are given by $\mathrm{A} = \begin{bmatrix} 2a & 1 \\ (b-a^2) & 0 \end{bmatrix}$.

**Theorem 13:** For $n \in \mathbb{N}$ , we have $\begin{bmatrix} p_{n+1} \\ (b-a^2)p_n \end{bmatrix} = \mathrm{A}\begin{bmatrix} p_n \\ (b-a^2)p_{n-1} \end{bmatrix}$ (38)

**Proof:** To prove the result we will use induction on n. (6.1) is valid for n = 1. Suppose (38) is good for n, we get

$$\begin{bmatrix} p_{n+2} \\ (b-a^2)p_{n+1} \end{bmatrix} = \begin{bmatrix} 2ap_{n+1} + (b-a^2)p_n \\ (b-a^2)p_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 1 \\ (b-a^2) & 0 \end{bmatrix}\begin{bmatrix} p_{n+1} \\ (b-a^2)p_n \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 1 \\ (b-a^2) & 0 \end{bmatrix}\begin{bmatrix} 2a & 1 \\ (b-a^2) & 0 \end{bmatrix}\begin{bmatrix} p_n \\ (b-a^2)p_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 1 \\ (b-a^2) & 0 \end{bmatrix} \begin{bmatrix} 2ap_n + (b-a^2)p_{n-1} \\ (b-a^2)p_n \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 1 \\ (b-a^2) & 0 \end{bmatrix} \begin{bmatrix} p_{n+1} \\ (b-a^2)p_n \end{bmatrix}$$

$$= \mathrm{A} \begin{bmatrix} p_{n+1} \\ (b-a^2)p_n \end{bmatrix}$$

**Theorem 14:** For $n \in \mathbb{N}$, we have $\begin{bmatrix} q_{n+1} \\ (b-a^2)q_n \end{bmatrix} = \mathrm{A} \begin{bmatrix} q_n \\ (b-a^2)q_{n-1} \end{bmatrix}$ (39)

**Theorem 15:** For $, n \in \mathbb{N}$ we have $\begin{bmatrix} p_{n+1} \\ (b-a^2)p_n \end{bmatrix} = \mathrm{A}^n \begin{bmatrix} p_1 \\ (b-a^2)p_0 \end{bmatrix}$ (40)

**Theorem 16:** For $n \in \mathbb{N}$, we have $\begin{bmatrix} q_{n+1} \\ (b-a^2)q_n \end{bmatrix} = \mathrm{A}^n \begin{bmatrix} q_1 \\ (b-a^2)q_0 \end{bmatrix}$ (41)

## 4. CONCLUSIONS

This study describes explicit sums of generalized Pell and Pell-Lucas sequences. This enables us to give in a straightforward way several formulas for the sums of such generalized numbers. We describe some generalized identities involving the product of generalized Pell and Pell-Lucas sequences. Also, we present identities related to their sum and difference of squares involving them and its two cross two matrices and find exciting properties such as the nth power of the matrix.

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### Declaration of Competing Interest

The author declares that they have no known competing financial interests or personal relationships that could influence the work reported in this paper.

### Author Contribution

Pell and Pell-Lucas sequences were introduced by Tulay Yagmur in 2019. Yashwant K Panwar contributed 100% at every stage of the article.

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