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Research Article

Approximate spectral cosynthesis in the harmonically weighted Dirichlet spaces

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Abstract

For a finite positive Borel measure μ on the unit circle, let $\mathcal{D}(\mu)$ be the associated harmonically weighted Dirichlet space. A shift invariant subspace $\mathcal{M}$ recognizes strong approximate spectral cosynthesis if there exists a sequence of shift invariant subspaces $\mathcal{M}_k$, with finite codimension, such that the orthogonal projections onto $\mathcal{M}_k$ converge in the strong operator topology to the orthogonal projection onto $\mathcal{M}$. If μ is a finite sum of atoms, then we show that shift invariant subspaces of $\mathcal{D}(\mu)$ admit strong approximate spectral cosynthesis.

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1. Introduction

Let H^2 be the usual Hardy space and $dA(re^{it}) = \frac{1}{\pi} r dr dt$ be the normalized area measure on the unit disc $\mathbb{D}$. Let μ be a finite positive Borel measure on the unit circle $\mathbb{T}$. The harmonically weighted Dirichlet space $\mathcal{D}(\mu)$ is the set of all functions $f \in H^2$ such that

$$\mathcal{D}_\mu(f) = \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where P_μ is the Poisson integral of the measure μ:

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{it}-z|^2} d\mu(t).$$

It is well known that the space $\mathcal{D}(\mu)$ is a Hilbert space with respect to the norm $||.||_\mu$ given by $||f||_\mu^2 = ||f||_{H^2}^2 + \mathcal{D}_\mu(f)$. Moreover, these spaces are reproducing kernel Hilbert spaces, that is, for each $z \in \mathbb{D}$, there exists a function $k_z^\mu \in \mathcal{D}(\mu)$, called the reproducing kernel, such that for every $f \in \mathcal{D}(\mu)$, $f(z) = \langle f, k_z^\mu \rangle_\mu$. If μ is the normalized Lebesgue measure m, then $\mathcal{D}(m)$ is the classical Dirichlet space $\mathcal{D}$, and if $\mu \equiv 0$, then we define $\mathcal{D}(0) = H^2$.

The shift operator S on $\mathcal{D}(\mu)$, that is multiplication by z, is a bounded linear operator. A (closed) subspace $\mathcal{M}$ of $\mathcal{D}(\mu)$ is called invariant if S maps $\mathcal{M}$ into itself. The collection of all invariant subspaces is denoted by $\mathrm{Lat}(S, \mathcal{D}(\mu))$. For $f \in \mathcal{D}(\mu)$ the invariant subspace generated by f, denoted by $[f]_{\mathcal{D}(\mu)}$, is $[f]_{\mathcal{D}(\mu)} = \mathrm{clos}_{\mathcal{D}(\mu)}\{pf : p \text{ a polynomial}\}$. This is the

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smallest closed invariant subspace containing f. If $[f]_{\mathcal{D}(\mu)} = \mathcal{D}(\mu)$, then f is called cyclic for $\mathcal{D}(\mu)$.

The precise knowledge of invariant subspaces and cyclic functions for the Hardy space H^2 is known by Beurling‘s theorem [1]. In fact, if $\mathcal{M} \in \mathrm{Lat}(S, H^2)$ is nontrivial, then $\mathcal{M} = \theta H^2$, where θ is an inner function, that is, θ is a bounded holomorphic function on the unit disc such that $|\theta(e^{it})| = 1$ a.e. on $\mathbb{T}$; and outer functions are cyclic. However, we do not have complete characterization of $\mathrm{Lat}(S, \mathcal{D}(\mu))$, and do not know which functions are cyclic in $\mathcal{D}(\mu)$. These problems are still open. For the study of invariant subspaces and cyclic functions, see e.g. [3, 7–10, 17–22].

Approximate spectral synthesis of a subspace, which was suggested by Nikolskii, is a process of reconstructing it not only by the root vectors that are contained in it but also the limit process of a sequence of subspaces (see [16, 24]). To be more precise, let X be a Banach space of analytic functions on the unit disk $\mathbb{D}$. Suppose X is invariant with respect to the shift operator S. A subspace $\mathcal{M}$ of X, which is invariant with respect to the shift operator S, is said to admit strong approximate spectral cosynthesis if there exists a sequence $\mathcal{M}_n$ of invariant subspaces such that $\dim(X/\mathcal{M}_n) < \infty$, $\mathcal{M} = \underline{\lim}\mathcal{M}_n$ and $\mathcal{M}^{\perp} = \underline{\lim}\mathcal{M}_n^{\perp}$, where $\underline{\lim}\mathcal{M}_n := \{x \in X : \exists\, x_n \in \mathcal{M}_n \text{ with } x_n \to x\}$ (see Definition 4 of [24]). If X is a Hilbert space, then we can restate the above definition in light of Lemma 2 of [24] (also see Lemma 4.2 of [14]) as follows: an invariant subspace $\mathcal{M} \subseteq X$ is said to admit strong approximate spectral cosynthesis if there exists a sequence $\mathcal{M}_n$ of invariant subspaces, with $\dim(X/\mathcal{M}_n) < \infty$, such that the orthogonal projections P_n onto $\mathcal{M}_n$ converge in the strong operator topology (SOT) to the orthogonal projection P onto $\mathcal{M}$.

If $\mu = \sum_{j=1}^{n} c_j \delta_{\zeta_j}$ with $c_j > 0$ and $\zeta_j \in \mathbb{T}$, where δ_{ζ_j} is the Dirac measure at ζ_j, D. Guillot gave the complete characterization of the $\mathrm{Lat}(S, \mathcal{D}(\mu))$ in [12]. In this case, using D. Guillot’s characterization, our aim in this paper is to study the approximate spectral cosynthesis problem for the harmonically weighted Dirichlet space $\mathcal{D}(\mu)$. We have the following theorem.

Theorem 1.1. *Let* $\mu = \sum\limits_{j=1}^{n} c_j \delta_{\zeta_j}$ *with* $c_j > 0$ *and* $\zeta_j \in \mathbb{T}$, *and* $\mathcal{D}(\mu)$ *be the associated harmonically weighted Dirichlet space. Suppose that* $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$ *is nontrivial. Then* $\mathcal{M}$ *admits strong approximate spectral cosynthesis.*

For the Hardy space H^2 the corresponding problem has a solution as a consequence of Beurling’s theorem and Caratheodory-Schur theorem (see [25]). For certain weighted Bergman spaces, spectral synthesis problem was considered by S.M. Shimorin in [24]. He showed that invariant subspaces of index 1 admit strong approximate spectral cosynthesis. In [25], the author has a similar result for the weighted Dirichlet spaces and a partial result for the classical Dirichlet space.

As an application, in the final remark we will explain that Theorem 1.1 answers a certain case of a question posed by J. B. Conway, and D. Hadwin (see [5]).

The plan will be as follows. In the next section, we will give a background on the harmonically weighted Dirichlet spaces $\mathcal{D}(\mu)$. In the last section we will prove Theorem 1.1 as Theorem 3.3.

2. The harmonically weighted Dirichlet spaces

Given an analytic function f on the open unit disc $\mathbb{D}$ and $\zeta \in \mathbb{T}$, we denote by $f(\zeta)$ the radial limit $\lim_{r \to 1^-} f(r\zeta)$, whenever it exists. It is well known that if $f \in H^2$, then the radial limit $f(\zeta)$ exists almost everywhere on the unit circle $\mathbb{T}$.

Given a finite positive Borel measure μ on the unit circle $\mathbb{T}$, the harmonically weighted Dirichlet space $\mathcal{D}(\mu)$ can alternatively be defined as the set of all analytic functions $f \in H^2$

such that

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\zeta(f) d\mu(\zeta) < \infty,$$

where $\mathcal{D}_\zeta(f)$ is the local Dirichlet integral of f at $\zeta \in \mathbb{T}$ given by

$$\mathcal{D}_\zeta(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\zeta)|^2}{|e^{it} - \zeta|^2} \frac{dt}{2\pi}.$$

Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$ and $f \in \mathcal{D}(\mu)$. Denote by $\mathcal{Z}_{\mathbb{T}}(f)$ the set of points in $\mathbb{T}$ where the radial limit of f is zero, that is,

$$\mathcal{Z}_{\mathbb{T}}(f) = \{\zeta \in \mathbb{T} : \lim_{r \to 1^-} f(r\zeta) = 0\},$$

and

$$\mathcal{Z}_{\mathbb{T}}(\mathcal{M}) = \bigcap_{f \in \mathcal{M}} \mathcal{Z}_{\mathbb{T}}(f).$$

Another notion that plays an important role is capacity. To define it we consider the harmonic Dirichlet space $\mathcal{D}^h(\mu)$, associated with μ,

$$\mathcal{D}^h(\mu) := \{f \in L^2(\mathbb{T}) : \mathcal{D}_\mu(f) < \infty\}.$$

The space $\mathcal{D}^h(\mu)$ is a Hilbert space and the norm on it is $||f||_\mu^2 := ||f||_{L^2}^2 + \mathcal{D}_\mu(f)$. For any subset $E \subset \mathbb{T}$, the capacity c_μ of E is defined by

$$c_\mu(E) := \inf\{||f||_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ a.e. on a neighborhood of } E\}.$$

For every Borel subset $F \subset \mathbb{T}$, we have $c_\mu(F) = \sup\{c_\mu(K) : K \subset F \text{ is compact}\}$ (see Corollary 3.3 of [12]). If $\mu = m$, the Lebesgue measure on $\mathbb{T}$, it is well known that the c_μ capacity and logarithmic capacity are equivalent. See [11] for more details. A property holds c_μ-quasi-everywhere, denoted by $c_\mu - q.e.$, if it holds everywhere except on a set of zero c_μ-capacity.

If $E \subseteq \mathbb{T}$ is a Borel set, define $\mathcal{D}_E(\mu)$ as follows;

$$\mathcal{D}_E(\mu) := \{f \in \mathcal{D}(\mu) : f = 0 \; c_\mu - q.e. \text{ on } E\}.$$

Note that it was shown in [12] (see Proposition 5.2) that $\mathcal{D}_E(\mu)$ is closed in $\mathcal{D}(\mu)$. Clearly it is invariant, so $\mathcal{D}_E(\mu) \in \mathrm{Lat}(S, \mathcal{D}(\mu))$.

Richter and Sundberg in [20] gave the following characterization of invariant subspaces of $\mathcal{D}(\mu)$.

Theorem 2.1. (see [20] Theorem 5.3) Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$ and let θ be the greatest common inner divisor of functions in $\mathcal{M}$. Then, there is an outer function $f \in \mathcal{D}(\mu)$ such that

$$\mathcal{M} = [\theta f]_{\mathcal{D}(\mu)} = \theta[f]_{\mathcal{D}(\mu)} \cap \mathcal{D} = [f]_{\mathcal{D}(\mu)} \cap \theta H^2.$$

In fact, f can be chosen so that f and θf are multipliers of $\mathcal{D}(\mu)$.

When the associated measure is a finite sum of atoms, using the above result of Richter and Sundberg, D. Guillot showed that

$$[f]_{\mathcal{D}(\mu)} = \{f \in \mathcal{D}(\mu) : f = 0 \; c_\mu - q.e. \text{ on } \mathcal{Z}_{\mathbb{T}}(f)\},$$

where $f \in \mathcal{D}(\mu)$ is an outer function (see Theorem 5.9 of [12]). Using these results, D. Guillot's characterization of invariant subspaces can be given in the following theorem.

Theorem 2.2. (See [12]) Let $\mu = \sum_{j=1}^{n} c_j \delta_{\zeta_j}$ with $c_j > 0$ and $\zeta_j \in \mathbb{T}$. Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$ and let θ be the greatest common inner divisor of functions in $\mathcal{M}$. Then

$$\mathcal{M} = \theta H^2 \cap \mathcal{D}_E(\mu),$$

where $E = \{\zeta \in supp\mu : c_\mu(\{\zeta\}) > 0 \text{ and } \zeta \in \mathcal{Z}_{\mathbb{T}}(\mathcal{M})\}$.

In [6] El-Fallah, Elmadani, and Kellay extended D. Guillot's characterization of invariant subspaces to harmonically weighted Dirichlet spaces $\mathcal{D}(\mu)$ where the associated measure μ has countable support (see Theorem 2 of [6]). The following theorem is well known.

Theorem 2.3. (See [3]) Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on a region $\Omega \subseteq \mathbb{C}$ and let $\{f_n\} \subseteq \mathcal{H}$. Then the following are equivalent:

(1) $f_n \to f$ weakly
(2) $||f_n|| \leq M$ and $f_n(z) \to f(z)$ for all $z \in \Omega$
(3) $||f_n|| \leq M$ and $f_n \to f$ locally uniformly.

3. Proof of Theorem 1.1

For the rest of the paper we will take μ to be finite sum of atoms, i.e., $\mu = \sum\limits_{j=1}^{n} c_j \delta_{\zeta_j}$ with $c_j > 0$ and $\zeta_j \in \mathbb{T}$. We begin with the following propositions which will be used in the sequel.

Proposition 3.1. *Let B be a finite Blaschke product with the zero set $\{\alpha_i\}_{i=1}^{k}$, and $F \in \mathcal{D}(\mu)$ be an outer function such that $F(\zeta_j) = 0$ for $j = 1, ..., n$. Then $BF \in \mathcal{D}(\mu)$.*

Proof. By the Richter-Sundberg formula (see Theorem 3.1 of [19]) the local Dirichlet integral $\mathcal{D}_\zeta(BF)$ of BF is

$$\mathcal{D}_\zeta(BF) = \sum_{i=1}^{k} \frac{1 - |\alpha_i|^2}{|\zeta - \alpha_i|^2} |F(\zeta)|^2 + \mathcal{D}_\zeta(F).$$

Hence

$$\begin{aligned}\mathcal{D}_\mu(BF) &= \int_{\mathbb{T}} \sum_{i=1}^{k} \frac{1 - |\alpha_i|^2}{|\zeta - \alpha_i|^2} |F(\zeta)|^2 \, d\mu(\zeta) + \mathcal{D}_\mu(F) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n} c_j \frac{1 - |\alpha_i|^2}{|\zeta_j - \alpha_i|^2} |F(\zeta_j)|^2 + \mathcal{D}_\mu(F) < \infty.\end{aligned}$$

□

Proposition 3.2. *For $1 \leq j \leq n$, let $F_j(z) = z - \zeta_j$ and $F(z) = \prod\limits_{j=1}^{n} F_j(z)$. Then for each $j \in \{1, ..., n\}$, F_j is an outer function with $F_j \in \mathcal{D}(\mu)$.*

Further, $F(z)$ is an outer function with $F \in \mathcal{D}(\mu)$.

Proof. It is well known that F_j and F are outer functions (see [15] page 27). Since polynomials are dense in $\mathcal{D}(\mu)$ (see Corollary 3.8 of [18]), obviously we have F_j and F are in $\mathcal{D}(\mu)$. □

Now we are ready to prove our main result.

Theorem 3.3. *Let $\mu = \sum\limits_{j=1}^{n} c_j \delta_{\zeta_j}$ with $c_j > 0$ and $\zeta_j \in \mathbb{T}$, and $\mathcal{D}(\mu)$ be the associated harmonically weighted Dirichlet space. Suppose that $\mathcal{M} \in Lat(S, \mathcal{D}(\mu))$ is nontrivial. Then $\mathcal{M}$ admits strong approximate spectral cosynthesis.*

Proof. If $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$ is nontrivial, then by Theorem 2.2

$$\mathcal{M} = \theta H^2 \cap \mathcal{D}_E(\mu),$$

where θ is the greatest common inner divisor of the inner parts of the nonzero functions in $\mathcal{M}$.

Without loss of generality let $E = \{\zeta_1, ..., \zeta_n\}$. Notice that, in this case since $F(z) = \prod_{j=1}^{n}(z - \zeta_j) \in \mathcal{D}(\mu)$ is an outer function by Proposition 3.2, we have $\mathcal{D}_E(\mu) = [F]_{\mathcal{D}(\mu)}$ by Theorem 5.9 of [12]. Hence $\mathcal{M} = \theta H^2 \cap \mathcal{D}_E(\mu) = [\theta F]_{\mathcal{D}(\mu)}$ by Theorem 2.1, where θF is extremal for $\mathcal{M}$, that is, $\theta F \in \mathcal{M} \ominus z\mathcal{M}$, $||\theta F||_\mu = 1$ (see Theorem 3.2 of [20]). Using the Caratheodory-Schur theorem (see, e.g.,Theorem 5.5.1 of [23]), we can get a sequence of finite Blaschke products B_k such that $B_k \to \theta$ locally uniformly in the unit disc. Denote the zeros of the Blaschke product B_k by $\{\alpha_{k_i}\}_{i=1}^{t_k}$. Then by Proposition 3.1, $B_k F \in \mathcal{D}(\mu)$. Now set

$$\begin{aligned}\mathcal{M}_k = [B_k F]_{\mathcal{D}(\mu)} =& B_k H^2 \cap \mathcal{D}_E(\mu) = \{f \in \mathcal{D}(\mu) : f \in B_k H^2 \text{ and } f|_E = 0\}\\ =& \{f \in \mathcal{D}(\mu) : f(\alpha_{k_i}) = 0,\ i = 1, ..., t_k, \text{ and } f|_E = 0\}.\end{aligned}$$

Let P_k and P be the orthogonal projections onto $\mathcal{M}_k$ and $\mathcal{M}$, respectively. To finish the proof, we need to show that $P_k \to P$ in SOT, and $\mathcal{M}_k$ has finite codimension.

If $B_k F(z) \to \theta F(z)$ for all $z \in \mathbb{D}$, then $P_k \to P$ in SOT follows from Corollary 4.5 of [14], since $B_k F$ is extremal function for $\mathcal{M}_k$ and θF is extremal function for $\mathcal{M}$. That $B_k \to \theta$ locally uniformly on $\mathbb{D}$ implies that $B_k(z) \to \theta(z)$ for all $z \in \mathbb{D}$. But since $F \in \mathcal{D}(\mu)$ this implies that $B_k F(z) \to \theta F(z)$ for all $z \in \mathbb{D}$.

It is left to show that $\dim \mathcal{M}_k^{\perp} < \infty$. Since the $c_\mu(\{\zeta_i\}) > 0$ for each $i = 1, ..., n$, the evaluation functional $f \mapsto f(\zeta_i)$ is bounded on $\mathcal{D}(\mu)$, so there exists a reproducing kernel $k^\mu_{\zeta_i} \in \mathcal{D}(\mu)$ for each $i = 1, ..., n$ (see [12], or Lemma 3.2 of [6]). Then by a similar argument of Lemma 3.4 of [25], one can show that $\mathcal{M}_k$ has finite co-dimension by showing that

$$\mathcal{M}_k^{\perp} = \bigvee \{k^\mu_{\zeta_i}\}_{i=1}^{n} \cup \{k^\mu_{\alpha_{k_i}}\}_{i=1}^{t_k},$$

where the symbol $\bigvee$ denotes the closed linear span, $k^\mu_{\zeta_i}$ and $k^\mu_{\alpha_{k_i}}$ are the reproducing kernels at ζ_i and α_{k_i}, respectively. □

If $\theta \equiv 1$ in Theorem 2.1, then $\mathcal{M} = \mathcal{D}_E(\mu)$. Hence one can take $\mathcal{M}_k = \mathcal{D}_E(\mu)$. The easy way to prove Theorem 3.3 for $\mathcal{D}_E(\mu)$ is the observation that we have reproducing kernels k^μ_ζ for $\zeta \in \mathbb{T}$. But one can do more. In fact, next we show that $\mathcal{D}_E(\mu)$ admits strong approximate spectral cosynthesis by a sequence of zero-based invariant subspaces for some zero set in the unit disc. We can prove this by an argument that is similar to Theorem 3.6 of [25]. For the sake of completeness, we provide the details here.

Without loss of generality let $E = \{\zeta_1, ..., \zeta_n\}$. For $r < 1$ define $rE := \{r\zeta_1, ..., r\zeta_n\}$, and set $I(rE) = \{f \in \mathcal{D}(\mu) : f(r\zeta_j) = 0,\ \forall j = 1, ..., n\}$. Then it is clear that $I(rE) \in \mathrm{Lat}(S, \mathcal{D}(\mu))$. Note that this kind of invariant subspaces are called zero-based subspaces, and we reserve this notation for them. By choosing an increasing sequence $r_k < 1$ with $\sum_{k=1}^{\infty}(1 - nr_k) < \infty$, let the Blaschke product $B(z)$ be given by

$$B(z) = \prod_{j=1}^{\infty} \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z},$$

where $\{z_j\}_{j=1}^{\infty} = \bigcup_{k=1}^{\infty} r_k E = \{r_1\zeta_1, ..., r_1\zeta_n, r_2\zeta_1, ..., r_2\zeta_n, ...\}$. Set $f_0(z) := B(z)F(z)$, where $F(z) = \prod_{j=1}^{n}(z - \zeta_j)$ is the outer function defined in Proposition 3.2.

Proposition 3.4. *With the above notation, we have* $f_0 \in \mathcal{D}(\mu)$ *and* $f_0 \in \bigcap_{k=1}^{\infty} I(r_k E)$.

Proof. Proof of $f_0 \in \mathcal{D}(\mu)$ is similar to the Proposition 3.1, and by construction it is clear that $f_0 \in \bigcap_{k=1}^{\infty} I(r_k E)$. □

Theorem 3.5. *Let $\mathcal{D}_E(\mu) \in Lat(S, \mathcal{D}(\mu))$ be nontrivial. Then $\mathcal{D}_E(\mu)$ admits strong approximate spectral cosynthesis by a sequence of zero based invariant subspaces that has a finite zero set in the unit disc.*

Proof. We assume that $E = \{\zeta_1, ..., \zeta_n\}$. Note that in this case $E = \mathcal{Z}_{\mathbb{T}}(\mathcal{M})$.

For each $j \in \{1, ..., n\}$, let $F_j(z) = z - \zeta_j$ and $F(z) = \prod_{j=1}^{n} F_j(z)$. Then by Proposition 3.2, F_j and F are outer functions with $F_j, F \in \mathcal{D}(\mu)$. Then by Theorem 5.9 of [12] $\mathcal{D}_E(\mu) = [F]_{\mathcal{D}(\mu)}$.

Let $r_k < 1$ be an increasing sequence such that $\sum_{k=1}^{\infty} (1 - nr_k) < \infty$, and by reindexing let $\{z_j\}_{j=1}^{\infty} = \bigcup_{k=1}^{\infty} r_k E = \{r_1\zeta_1, ..., r_1\zeta_n, r_2\zeta_1, ..., r_2\zeta_n, ...\}$ and consider the Blaschke product $B(z)$ with zeroes $\{z_j\}_{j=1}^{\infty}$. Let $f_0(z) := B(z)F(z)$. Then by Proposition 3.4, $f_0 \in \mathcal{D}(\mu)$ and $f_0 \in \bigcap_{k=1}^{\infty} I(r_k E)$, where $I(r_k E)$ is the zero-based invariant subspace on $r_k E$. Next, we claim that

$$\mathcal{D}_E(\mu) = \bigvee_{t=1}^{\infty} \bigcap_{k=t}^{\infty} I(r_k E).$$

For each t, let $f \in \bigcap_{k=t}^{\infty} I(r_k E)$. Then $f(r_k \zeta_j) = 0$ for all $j = 1, ..., n$ and $k \geq t$. As $k \to \infty$, $r_k \to 1$, we have the radial limits $f(\zeta_j) = 0$ for all $j = 1, ..., n$. This implies $f \in \mathcal{D}_E(\mu)$. Hence $\bigvee_{t=1}^{\infty} \bigcap_{k=t}^{\infty} I(r_k E) \subseteq \mathcal{D}_E(\mu)$. To show the reverse inclusion, since $\mathcal{D}_E(\mu) = [F]_{\mathcal{D}(\mu)}$, it is enough to show that $F \in \bigvee_{t=1}^{\infty} \bigcap_{k=t}^{\infty} I(r_k E)$.

Let $I_t := \bigcap_{k=t}^{\infty} I(r_k E)$. Observe that if $f \in I_t$, then $f(z) = 0$ for all $z \in \bigcup_{k=t}^{\infty} r_k E =: Z_t$. Also note that $I_t \subseteq I_{t+1}$. Let $B_t(z)$ be the Blaschke product with the zero set Z_t, and let $f_t(z) := B_t(z)F(z)$. Then by a similar argument to that in Proposition 3.4 one can show that $f_t \in \mathcal{D}(\mu)$ and trivially $f_t \in I_t$. Since $B_t(z) \to 1$ pointwise as $t \to \infty$, and $||f_t||_{\mathcal{D}(\mu)} \leq C$ for some constant C, we have $f_t \to F$ weakly by Theorem 2.3. Hence $F \in \bigvee_{t=1}^{\infty} \bigcap_{k=t}^{\infty} I(r_k E)$ since weak and norm closure are equivalent. Therefore $\mathcal{D}_E(\mu) = \bigvee_{t=1}^{\infty} I_t$, which implies that $\mathcal{D}_E(\mu)^{\perp} = \bigcap_{t=1}^{\infty} I_t^{\perp}$, and $I_t \subseteq I_{t+1}$ implies that $I_t^{\perp} \supseteq I_{t+1}^{\perp}$. Let P and P_t be the orthogonal projections onto $\mathcal{D}_E(\mu)$ and I_t, respectively. Then by Problem 120 of [13] $I - P_t \to I - P$ in SOT, and hence $P_t \to P$ in SOT. To finish the proof we need to construct the sequence of invariant subspaces $\mathcal{M}_m$ with finite codimension such that the corresponding projections converge to P in SOT.

Recall that $Z_t = \bigcup_{k=t}^{\infty} r_k E$. For each t, let $A_{m,t} = \{w_1, ..., w_m\} \subseteq Z_t$ be such that $A_{m,t} \subseteq A_{m+1,t}$ and $\bigcup_{m=1}^{\infty} A_{m,t} = Z_t$. Let $\mathcal{M}_m = I(A_{m,t})$ be the zero-based invariant subspace on $A_{m,t}$. Then it is clear that $\mathcal{M}_m \supseteq \mathcal{M}_{m+1}$ and $I_t = \bigcap_{m=1}^{\infty} \mathcal{M}_m$. Let $P_{m,t}$ be the orthogonal projection onto $\mathcal{M}_m$. Then for each t, $P_{m,t} \to P_t$ in SOT as $m \to \infty$. The SOT is metrizable on bounded subsets of all bounded linear opetators (see Proposition 1.3 in Chapter 9 of [4]), so we can get a subsequence $P_{m_t,t}$ such that $P_{m_t,t} \to P$ in SOT. It is left to show that $\mathcal{M}_m = I(A_{m,t})$ has finite codimension. This can be done as in Theorem 3.3. $\square$

4. Final remark

Let T be a bounded linear operator on a separable infinite dimensional Hilbert space $\mathcal{H}$. An invariant subspace $\mathcal{M}$ of T is called *stable* if whenever $\{T_n\}$ is a sequence of bounded linear operators on $\mathcal{H}$ such that $||T_n - T|| \to 0$, there is a sequence of invariant subspaces $\mathcal{M}_n$ of T_n such that $P_n \to P$ in the strong operator topology (SOT), where P_n and P are orthogonal projections onto $\mathcal{M}_n$ and $\mathcal{M}$, respectively. If the projections converge in the norm, then $\mathcal{M}$ is called a *norm-stable* invariant subspace. Following the notation in [5], let $\mathrm{Lat}_s(T, \mathcal{H})$ and $\mathrm{Lat}_{ns}(T, \mathcal{H})$ denote the collection of stable and norm-stable invariant subspaces of T, respectively. Conway and Hadwin asked the following question: Is $\mathrm{Lat}_s(T, \mathcal{H})$ the closure in the strong operator topology of $\mathrm{Lat}_{ns}(T, \mathcal{H})$ for any operator T? They showed that this question has affirmative answer when T is normal operator or the unweighted unilateral shift of finite multiplicity. A general result of Borichev et al. [2] implies that if T is any weighted unilateral shift operator, then an invariant subspace is norm-stable if and only if its co-dimension is finite. Let $\mathrm{Lat}_{fc}(S, \mathcal{D}(\mu))$ denotes the finite co-dimensional invariant subspaces. Then our main result Theorem 1.1 implies that the strong closure of $\mathrm{Lat}_{fc}(S, \mathcal{D}(\mu))$ is $\mathrm{Lat}(S, \mathcal{D}(\mu))$. On the other hand, $\mathrm{Lat}_{fc}(S, \mathcal{D}(\mu)) \subset \mathrm{Lat}_{ns}(S, \mathcal{D}(\mu)) \subset \mathrm{Lat}_s(S, \mathcal{D}(\mu))$; for the first inclusion see [2] and the second one is trivial. Then we have a positive answer to the above mentioned question of Conway and Hadwin.

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