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Research Article

A criterion for nonzero multiplier for Orlicz spaces of an affine group $\mathbb{R}_+ \times \mathbb{R}$

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Abstract

Let $\mathbb{A} = \mathbb{R}_+ \times \mathbb{R}$ be an affine group with right Haar measure $d\mu$ and Φ_i, $i = 1, 2$, be Young functions. We show that there exists an isometric isomorphism between the multiplier of the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ and $(L^{\Psi_2}(\mathbb{A}), L^{\Psi_1}(\mathbb{A}))$ where Ψ_i are complementary pairs of Φ_i, $i = 1, 2$, respectively. Moreover we show that under some conditions there is no nonzero multiplier for the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$, i.e., for an affine group $\mathbb{A}$ only the spaces $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$, with a concrete condition, are of any interest.

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1. Introduction

Orlicz spaces were first proposed by Z. W. Birnbaum and W. Orlicz ([3]) and after that developed by Orlicz himself. These spaces represent a significant class of Banach function spaces in analysis and applications (see [14]). The concept of Orlicz space extends the classical concept of L^p Lebesgue space for $p \geq 1$. A more general convex function $\Phi(x)$ is used for the function x^p appearing in the definition of L^p spaces. This function Φ is called a Young function. In addition to L^p spaces, several function spaces are consisted of Orlicz spaces. For instance, $L \log^+ L$ Zygmund spaces, which are Banach spaces with regard to Hardy-Littlewood maximal functions. Moreover, Sobolev spaces could be also contained in Orlicz spaces as subspaces (see [6]). Most of the linear properties of Orlicz spaces have been investigated thoroughly (see [18], for example). Furthermore Orlicz spaces determined on measure spaces have been considered thoroughly (see for example [7, 9, 12, 18]). In the recent years, Orlicz spaces and the weighted case are examined as Banach algebras over a locally compact group. Moreover their several properties are also studied (see [1, 15–17, 19, 20]).

On the other hand one of the basic problems in harmonic analysis is the description of multipliers. Multipliers have also been considered in several contexts, for example Banach algebras and Banach modules theories, partial differential equations, the existence of invariant means etc. Our interest is to analyze the existence of an isometric isomorphism between the multiplier of the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ and $(L^{\Psi_2}(\mathbb{A}), L^{\Psi_1}(\mathbb{A}))$ where Ψ_i are complementary pairs of Φ_i, $i = 1, 2$, and determine the relation between Φ_1 and Φ_2 to get $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) = \{0\}$.

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For L^p spaces, Wendel [21] showed that $M(G)$, the set of all complex regular Borel measures on G, is the space of multipliers when $p = 1$ and G is a locally compact group. Moreover if $1 \leq p \leq \infty$, Brainerd and Edwards characterized the elements of $M(L^1(G), L^p(G))$ (see [5]). In [11] Lau investigated the convex, closed subsets of Lebesgue spaces when $1 \leq p < \infty$; especially, he showed that the convex, compact, translation invariant (right or left) subset of $L^p(G)$ is the origin, i.e. $\{0\}$. Moreover in [19] these standard results were extended to $L^\Phi(G)$ spaces by Üster and Öztop. In particular, they achieved some results for the multipliers in $L^\Phi(G)$ spaces.

In the current paper, we study some of the basic multiplier problems for Orlicz spaces over an affine group $\mathbb{A}$.

The paper is organized as follows. In Section 2 we give essential definitions and recall some basic properties. In Section 3, we deduce the existence of an isometric isomorphism between the multiplier of the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ and $(L^{\Psi_2}(\mathbb{A}), L^{\Psi_1}(\mathbb{A}))$ where Ψ_i are complementary pairs of Φ_i, $i = 1, 2$, respectively. We also give a criterion to get $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) = \{0\}$. Thus for an affine group $\mathbb{A}$ only the spaces $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$, with some concrete conditions, are any of interest. Moreover we give some concrete Young function examples that can be applied to the results. Thus we generalize some of the standard conclusions for Lebesgue spaces to Orlicz spaces.

2. Preliminaries

In this section, we give some basic definitions and state some technical results that will be used in this paper.

We start with introducing some basic facts for the affine group and essential constructions on it.

Let $\mathbb{A} := (\mathbb{R}_+ \times \mathbb{R}, \cdot_{\mathbb{A}})$. One can equip $\mathbb{A}$ with the multiplication

$$(a, b) \cdot_{\mathbb{A}} (x, y) = (ax, ay + b), \tag{2.1}$$

for (a, b), $(x, y) \in \mathbb{A}$. Note that $(1, 0) \cdot_{\mathbb{A}} (a, b) = (a, b) \cdot_{\mathbb{A}} (1, 0) = (a, b)$ and $(a, b) \cdot_{\mathbb{A}} (a^{-1}, -a^{-1}b) = (a^{-1}, -a^{-1}b) \cdot_{\mathbb{A}} (a, b) = (1, 0)$. Thus $\mathbb{A}$, endowed with the multiplication (2.1), becomes a group and this group is called an affine group.

Since a mapping of the real line can be defined by $F_{a,b} : \mathbb{R} \to \mathbb{R}$ such that

$$F_{a,b}(x) = (a, b) \cdot x = ax + b, \ x \in \mathbb{R}$$

for any $(a, b) \in \mathbb{A}$, the affine group is also called $ax + b$ group. $F_{a,b}$ is the affine mapping of the real line $\mathbb{R}$ and this operation is coherent with (2.1).

We can represent the affine group $\mathbb{A}$ with the matrix form

$$\mathbb{A} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, \, b \in \mathbb{R} \right\}.$$

The inverse and the identity elements are given by

$$\begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

One can decompose the affine group $\mathbb{A}$ such that $\mathbb{A} = H \circ A$ and here H is the closed normal subgroup which is given by $\{(1, b) : b \in \mathbb{R}\}$ that can be identify with $\mathbb{R}$ by means of the relation $(1, b) \leftrightarrow b$ and $A = \{(a, 0) : a > 0\}$ determine with $\mathbb{R}_+$ where $(a, 0) \leftrightarrow a$ (see [8]).

The operations of the multiplication and inversion are continuous for the product topology. Thus an affine group $\mathbb{A}$ is a locally compact group and

$$d\nu(x, y) = \frac{dx}{x^2} \, dy$$

$$d\mu(x, y) = \frac{dx}{x} \, dy$$

are the left (right) Haar measures, respectively (see [8]). Now since

$$d\nu(x,y) = \frac{dx}{x^2},\ dy = \frac{1}{x}d\mu(x,y),$$

the affine group is not unimodular. The modular function on the affine group is $\Delta(x,y) = x^{-1}$.

Throughout the study we use the affine group $\mathbb{A} := (\mathbb{R}_+ \times \mathbb{R}, \cdot_{\mathbb{A}})$ and the right Haar measure $d\mu$.

Let $f : \mathbb{A} \to \mathbb{C}$ and $(a,b) \in \mathbb{A}$. We use $L_{(a,b)}$ for the left translation and $R_{(a,b)}$ for the right translation given by

$$(L_{(a,b)}f)(x,y) := f((a,b)^{-1} \cdot_{\mathbb{A}} (x,y)) \quad \text{and} \quad (R_{(a,b)}f)(x,y) := f((x,y) \cdot_{\mathbb{A}} (a,b)^{-1}).$$

We recall that if f is a measurable function and $a > 0$ the dilation operator D_a is given by

$$D_a f(x) := f(ax).$$

Next we give some notions regarding Orlicz spaces and Young functions. Our main references are [7] and [18].

Definition 2.1. Let $\Phi : [0,\infty) \to [0,\infty]$ be a convex, non-zero function. Then Φ is called a Young function if

(1) $\Phi(0) = 0$,
(2) $\lim\limits_{x\to\infty} \Phi(x) = +\infty$.

If Φ is a Young function, its complementary function Ψ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\} \quad (y \geq 0).$$

By definition of Ψ, it can be seen that Ψ is also a Young function and (Φ,Ψ) is called a complementary Young functions pair.

Recall that complementary Young functions satisfy

$$xy \leq \Phi(x) + \Psi(y)$$

for every $x, y \geq 0$. This is called Young's inequality.

By definition of the Young function one can observe that it can be discontinuous at a point. However, unless otherwise specified, we only consider the real-valued Young functions. Thus, Φ is necessarily continuous and $\lim_{x\to\infty}\Phi(x) = \infty$. Note that the continuity of Φ may not require the continuity of Ψ.

Definition 2.2. Let Φ_i, $i = 1,2$, be two Young functions. Then Φ_1 is called stronger than Φ_2, if

$$\Phi_2(x) \leq \Phi_1(ax) \quad (x \geq 0)$$

for some $a > 0$. We denote $\Phi_1 \succ \Phi_2$ (or $\Phi_2 \prec \Phi_1$) for stronger Young functions. Moreover if $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$, they are called strongly equivalent and we write $\Phi_1 \approx \Phi_2$.

Note that if $\Phi_1 \approx \Phi_2$, Φ_1 and Φ_2 generate the same Orlicz spaces, i.e. they consist of the same functions and the norms are equivalent.

Let us recall some facts concerning Orlicz spaces. Let (Φ,Ψ) be complementary Young functions. Then the Orlicz space $L^\Phi(\mathbb{A})$ is defined to be

$$L^\Phi(\mathbb{A}) = \{f : \mathbb{A} \to \mathbb{C} : \int\limits_{\mathbb{A}} \Phi(\alpha|f(x,y)|)\, d\mu(x,y) < \infty \text{ for some } \alpha > 0\}.$$

Here f is a member in equivalence class of measurable functions with respect to the Haar measure $d\mu$. Recall that Orlicz space is a Banach space with respect to the Orlicz norm which is defined by

$$\|f\|_\Phi = \sup\left\{ \int_{\mathbb{A}} |f(x,y)\nu(x,y)|\, d\mu(x,y) : \int_{\mathbb{A}} \Psi(|\nu(x,y)|)\, d\mu(x,y) \le 1 \right\}$$

for $f \in L^\Phi(\mathbb{A})$. Here (Φ, Ψ) are complementary Young functions.

Another norm on Orlicz space is the Luxemburg norm $N_\Phi(f)$ is defined by

$$N_\Phi(f) = \inf\left\{ k > 0 : \int_{\mathbb{A}} \Phi\left(\frac{|f(x,y)|}{k}\right) d\mu(x,y) \le 1 \right\}.$$

Note that the Orlicz and Luxemburg norms are equivalent; that is,

$$N_\Phi(f) \le \|\cdot\|_\Phi \le 2N_\Phi(f).$$

Given $\gamma > 0$ one can define

$$N_{\Phi,\gamma}(f) := \inf\{k > 0 : \int_{\mathbb{A}} \Phi\left(\frac{|f(x,y)|}{k}\right) d\mu(x,y) \le \ \gamma\}.$$

Here $N_{\Phi,1} = N_\Phi$ and these norms are equivalent on $L^\Phi(\mathbb{A})$:

$$\frac{\gamma_1}{\gamma_2} N_{\Phi,\gamma_1}(f) \le N_{\Phi,\gamma_2}(f) \le N_{\Phi,\gamma_1}(f)$$

for $0 < \gamma_1 \le \gamma_2$.

Throughout the paper we write

$$C_\Phi(\lambda) = \|D_\lambda\|_{\mathrm{op}}.$$

We can observe that $C_\Phi(\lambda)$ is submultiplicative, non-increasing and $C_\Phi(1) = 1$. Moreover notice that if $\Phi(x) = |x|^p$, $1 \le p < \infty$, then $C_\Phi(\lambda) = \lambda^{-1/p}$ (see [4]).

It has been said that Orlicz spaces extend the classical Lebesgue spaces. For $\Phi(x) = \frac{x^p}{p}$, $1 < p < \infty$, the complementary Young function is $\Psi(y) = \frac{y^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. In this case the space $L^\Phi(\mathbb{A})$ becomes the Lebesgue space $L^p(\mathbb{A})$ and the norm $\|\cdot\|_\Phi$ is equivalent to the norm $\|\cdot\|_p$. For $p = 1$, the complementary Young function of $\Phi(x) = x$ is

$$\Psi(y) = \begin{cases} 0, & 0 \le y \le 1 \\ \infty, & \text{otherwise} \end{cases}$$

and we have $L^\Phi(G) = L^1(G)$, $L^\Psi(G) = L^\infty(G)$. We can apply our results to different Young functions and we point out some of them below (see [13] and [18]):

(1) We can give the following examples which are called Zygmund spaces (see [2]): $L\log L$ Zygmund space is given by the Young function

$$x\log^+ x = \int_0^x \phi(s)ds$$

and here

$$\phi(s) = \begin{cases} 0, & 0 \le s \le 1 \\ 1 + \log s, & 1 < s < \infty \end{cases}$$

and $\log^+(x) = \max(\log x, 0)$.

(2) The exponential Zygmund space $\exp L$ is given by the next Young function

$$\Psi(x) = \int_0^x \psi(s)ds$$

where

$$\psi(s) = \begin{cases} 0, & s = 0 \\ 1, & 0 < s < 1 \\ e^{s-1}, & 1 < s < \infty. \end{cases}$$

Hence $\Psi(x) = x$ for $0 \leq x \leq 1$ and $\Psi(x) = e^{x-1}$ for $1 < x < \infty$.

(3) If $\Phi(x) = x\ln(1+x)$, then $\Psi(x) \approx \cosh x - 1$.
(4) If $\Phi(x) = \cosh x - 1$, then $\Psi(x) \approx x\ln(1+x)$.
(5) If $\Phi(x) = (1+x)\ln(1+x) - x$, then $\Psi(x) = e^x - x - 1$.

Next we define $S^\Phi(\mathbb{A})$ as the closure of the linear space of all step functions in $L^\Phi(\mathbb{A})$. Note that $S^\Phi(\mathbb{A})$ contains $C_c(\mathbb{A})$, the space of all continuous and compactly supported functions on $\mathbb{A}$, as a dense subspace and $S^\Phi(\mathbb{A})$ is a Banach space [18, Proposition 3.4.3]. Furthermore, we will mostly use Hölder's inequality for $L^\Phi(\mathbb{A})$. More specifically, if (Φ, Ψ) are complementary Young functions, it states that for any $f \in L^\Phi(\mathbb{A})$ and $g \in L^\Psi(\mathbb{A})$,

$$\|fg\|_1 \leq N_\Phi(f)\|g\|_\Psi.$$

In particular, $fg \in L^1(\mathbb{A})$.

Note that $S^\Phi(\mathbb{A})^*$, the dual of $S^\Phi(\mathbb{A})$, can be described by $L^\Psi(\mathbb{A})$ in a natural way (for general locally compact groups see [18, Theorem 4.1.6]).

For Orlicz spaces one of the important notion is Δ_2-condition. Let us recall the following definition.

Definition 2.3. Let Φ be a Young function. Then we say that Φ satisfies the Δ_2-condition, if

$$\Phi(2x) \leq C\Phi(x) \quad (x \geq 0)$$

for $C > 0$ and we write $\Phi \in \Delta_2$.

Note that if $\Phi \in \Delta_2$, then $L^\Phi(\mathbb{A}) = S^\Phi(\mathbb{A})$ and $L^\Phi(\mathbb{A})^* \cong L^\Psi(\mathbb{A})$ [18, Corollary 3.4.5]. Moreover if $\Psi \in \Delta_2$, then $L^\Phi(\mathbb{A})$ becomes a reflexive Banach space. For more detail in general case, see [9, 18].

3. Main results

In this section we give a characterization for the space of the multipliers of the pairs $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. We also give a criterion for Young functions Φ_1 and Φ_2 to get $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) = \{0\}$.

According to the definition of multipliers for topological linear spaces of functions [10, Chapter 3, p. 66], we give the next definition for the left multipliers of $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.

Definition 3.1. Let Φ_1, Φ_2 be Young functions and T be a linear bounded operator from $L^{\Phi_1}(\mathbb{A})$ to $L^{\Phi_2}(\mathbb{A})$. Then T is called a left multiplier for the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ if

$$T(L_{(a,b)}f) = L_{(a,b)}(Tf)$$

for all $f \in L^{\Phi_1}(\mathbb{A})$ and $(a,b) \in \mathbb{A}$. We use the notation $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ for the set of all left multipliers of the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.

Theorem 3.2. *Let (Φ_i, Ψ_i) be complementary Young pair with $\Phi_i \in \Delta_2$, $\Psi_i \in \Delta_2$, $i = 1, 2$. Then there exists an isometric linear isomorphism from $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ onto $M(L^{\Psi_2}(\mathbb{A}), L^{\Psi_1}(\mathbb{A}))$.*

Proof. Let $T \in M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. Define the adjoint operator $T^* : L^{\Psi_2}(\mathbb{A}) \to L^{\Psi_1}(\mathbb{A})$ where the linear operator is determined by

$$< f, T^*g > = < Tf, g >$$

for all $f \in L^{\Phi_1}(\mathbb{A})$ and $g \in L^{\Psi_2}(\mathbb{A})$. Then we have that

$$\begin{aligned} < f, T^* L_{(a,b)} g > &=< Tf, L_{(a,b)} g > \\ &=< L_{(a,b)} Tf, g > \\ &=< T(L_{(a,b)} f), g > \\ &=< L_{(a,b)} f, T^* g > \\ &=< f, L_{(a,b)} T^* g > \end{aligned}$$

which implies $T^* L_{(a,b)} = L_{(a,b)} T^*$ for all $(a,b) \in \mathbb{A}$. Hence $T^* \in M(L^{\Psi_2}(\mathbb{A}), L^{\Psi_1}(\mathbb{A}))$ and $\|T\| = \|T^*\|$.

Since $\Phi_i, \Psi_i \in \Delta_2$, $i = 1, 2$, the reflexivity of $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$ shows that the mapping $T \to T^*$ is surjective. Thus this mapping defines an isometric linear isomorphism from $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ onto $M(L^{\Psi_2}(\mathbb{A}), L^{\Psi_1}(\mathbb{A}))$. □

Notice that for a general locally compact group G, Theorem 3.2 still holds.

Lemma 3.3. *Let Φ be a Young function satisfying the Δ_2 condition. If $f \in L^{\Phi}(\mathbb{A})$ then $\lim_{(a,b)\to(+\infty,+\infty)} N_\Phi(f + L_{(a,b)} f) = N_{\Phi,\frac{1}{2}}(f)$.*

Proof. Suppose that $g \in C_c(\mathbb{A})$ with compact support K. Then $(a,b)^{-1} \cdot_{\mathbb{A}} (x,y) \in K$ and hence $(x,y) \in (a,b).K$. Then if $(a,b) \notin KK^{-1}$ we have

$$\begin{aligned} &N_\Phi(g + L_{(a,b)} g) \\ &= \inf\{\lambda > 0 : \int_{\mathbb{A}} \Phi(\frac{|g(x,y) + L_{(a,b)} g(x,y)|}{\lambda}) \frac{dx}{x} dy \le 1\} \\ &= \inf\{\lambda > 0 : [\int_K \Phi(\frac{|g(x,y)|}{\lambda}) \frac{dx}{x} dy + \int_{(a,b)\cdot K} \Phi(\frac{|g((a,b)^{-1} \cdot_{\mathbb{A}} (x,y))|}{\lambda}) \frac{dx}{x} dy] \le 1\} \\ &= \inf\{\lambda > 0 : [\int_K \Phi(\frac{|g(x,y)|}{\lambda}) \frac{dx}{x} dy + \int_K \Phi(\frac{|g(x,y)|}{\lambda}) \frac{dx}{x} dy] \le 1\} \\ &= \inf\{\lambda > 0 : \int_K \Phi(\frac{|g(x,y)|}{\lambda}) \frac{dx}{x} dy \le \frac{1}{2}\} = N_{\Phi,\frac{1}{2}}(g). \end{aligned}$$

For $f \in L^{\Phi}(\mathbb{A})$ and $\varepsilon > 0$ choose $g \in C_c(\mathbb{A})$ such that $N_\Phi(f - g) \le \frac{\varepsilon}{4}$. Let $\operatorname{supp}(g) = K$. Then by straightforward computations we have

$$\begin{aligned} |N_\Phi(f + L_{(a,b)} f) - N_{\Phi,\frac{1}{2}}(f)| &\le |N_\Phi(f + L_{(a,b)} f) - N_\Phi(g + L_{(a,b)} g)| \\ &\quad + |N_\Phi(g + L_{(a,b)} g) - N_{\Phi,\frac{1}{2}}(g)| + |N_{\Phi,\frac{1}{2}}(g) - N_{\Phi,\frac{1}{2}}(f)| \\ &\le N_\Phi(f - g) + N_\Phi(L_{(a,b)} f - L_{(a,b)} g) + 2N_\Phi(g - f) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{2\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore $\lim_{(a,b)\to(+\infty,+\infty)} N_\Phi(f + L_{(a,b)} f) = N_{\Phi,\frac{1}{2}}(f)$ for each $f \in L^{\Phi}(\mathbb{A})$. □

Now we can give the following condition on the Young functions Φ_1 and Φ_2 to get $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) = \{0\}$.

Theorem 3.4. *Let Φ_1, Φ_2 be Young functions satisfying Δ_2-condition. If $C_{\Phi_1}(\frac{1}{2}) C_{\Phi_2}(2) < 1$ then $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) = \{0\}$.*

Proof. Suppose that $T \in M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ such that $T \ne 0$. Then for each $f \in L^{\Phi_1}(\mathbb{A})$ and $(a,b) \in \mathbb{A}$ we have

$$N_{\Phi_2}(Tf + L_{(a,b)} Tf) = N_{\Phi_2}(T(f + L_{(a,b)} f)) \le \|T\| N_{\Phi_1}(f + L_{(a,b)} f)$$

From Lemma 3.3 we have

$$N_{\Phi_2,\frac{1}{2}}(Tf) \leq \|T\| N_{\Phi_1,\frac{1}{2}}(f)$$

which implies

$$N_{\Phi_2}(Tf) = N_{\Phi_2}(D_2(D_{\frac{1}{2}}Tf)) \leq C_{\Phi_2}(2) N_{\Phi_2}(D_{\frac{1}{2}}Tf)$$

and

$$C_{\Phi_2}^{-1}(2) N_{\Phi_2}(Tf) \leq N_{\Phi_2}(D_{\frac{1}{2}}Tf) \leq \|T\| C_{\Phi_1}(\frac{1}{2}) N_{\Phi_1}(f).$$

Then we have

$$C_{\Phi_1}(\frac{1}{2}) C_{\Phi_2}(2) \geq 1$$

which contradicts the hypothesis. Therefore $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) = \{0\}$. □

Thus for the affine group $\mathbb{A}$ only the spaces $M(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$, $C_{\Phi_1}(\frac{1}{2})C_{\Phi_2}(2) \geq 1$, are of any interest.

The above theorem also implies that if $\Phi_1(x) = |x|^p$ and $\Phi_2(x) = |x|^q$, $1 \leq p, q < \infty$, then $M(L^p(\mathbb{A}), L^q(\mathbb{A})) = \{0\}$, whenever $q < p$.

Remark 3.5. Besides the previous criterion for the nonzero multiplier on the affine group $\mathbb{A}$, we can give the following facts as an example.

Example 3.6. Let Φ_i, $i = 1, 2$, be Young functions.

(1) By using [19, Theorem 5.4], one can see that $T = 0$ is the only compact multiplier for the pair $(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.

(2) Let A be a weakly compact closed left invariant subspace of $L^{\Phi_1}(\mathbb{A})$ and $T : A \to L^1(\mathbb{A})$ be a bounded linear operator commuting with left translations. Then by [19, Theorem 5.5], $T = 0$.

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