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On the pole indicatrix curve of the spacelike Salkowski curve with timelike principal normal in Lorentzian 3-space

3-boyutlu Lorentz uzayında timelike asli normalli spacelike Salkowski eğrisinin pol gösterge eğrisi üzerine

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Abstract
In this study, we compute the Frenet frame, the curvatures, the Frenet derivative formulas, the Darboux vector, the arc length, the geodesic curvatures according to $\mathbb{E}_1^3$ and $\mathbb{S}_1^2$ (Lorentzian sphere) of the pole indicatrix curve of the spacelike Salkowski curve with the timelike principal normal in Lorentzian 3-space $\mathbb{E}_1^3$ and show the graph of the indicatrix curve on the Lorentzian sphere $\mathbb{S}_1^2$.

Keywords: Geodesic curvatures, Pole indicatrix curve, Salkowski curve

Öz
Bu çalışmada 3-boyutlu Lorentz uzayında timelike asli normalli spacelike Salkowski eğrisinin pol gösterge eğrisinin Frenet çatısı, eğrilikleri, Frenet türev formülleri, Darboux vektörü, yay uzunluğu, $\mathbb{E}_1^3$ *e ve* $\mathbb{S}_1^2$ *(Lorentzian küre) ye göre geodezik eğrilikleri hesaplanmıştır ve bu pol gösterge eğrisi Lorentzian küre* $\mathbb{S}_1^2$ *üzerinde gösterilmiştir.*

***Anahtar kelimeler**: Geodezik eğrilikler, Pol gösterge eğrisi, Salkowski eğrisi*

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1. Introduction

1. Giriş

Lorentz-Minkowski space is a field of study that is developing day by day, which is of interest to geometers as well as physicists. Einstein worked on this space in his theory of relativity. The fact that the theory is interesting since the first day increases the popularity of this space day by day. Although physicists work on quantum geometry in this space, geometers mostly work on differential geometry. Due to the definition of the inner product function in this space, many studies in Euclidean space can be updated in this space. Especially the theory of curves and surfaces is one of them. The basic information about Lorentz-Minkowski space can be found in sources (Akutagava & Nishikawa, 1990; Alluhaibi & Abdel-Baky, 2022; Beem et al., 2017; Birman & Nomizu, 1984; Li et al., accepted; O'Neill, 1983; Önder & Uğurlu, 2013; Ratcliffe, 1994; Uğurlu & Çalışkan, 2012; Uğurlu & Kocayiğit, 1996; Walrave, 1995; Woestijne, 1990). While working the geometry of a curve, any frame can be created at any point on the curve. One of these frames is the Frenet frame, (Hacısalihoğlu, 1983; Özdemir, 2020; Şenatalar, 1978; Struik, 1961). The Frenet frame is thought to rotate around a fixed axis at every moment s. This axis is called the Darboux axis and the unit vector in the direction of this axis is called the pole vector. The Frenet frame is an orthonormal frame derived from the derivatives of the curve. The curvature and torsion functions obtained from the elements of this frame play a role in determining the character of the curve. For example, the curvature and torsion ratio of a general helix is constant. Slant helices, which are a more special case of helices, make a fixed angle with a fixed direction, The studies on helices and some other special curves are available in related sources, (Ali, 2012; Ali & Lopez, 2011; Bulut & Bektaş, 2020; Bükcü & Karacan, 2007; Gür & Şenyurt, 2013; Izumiya & Tkeuchi, 2004; Kılıçoğlu & Hacısalihoğlu, 2008; Külahcı et al., 2009; Li et al., 2020; Li et al., 2018; Li & Pei, 2016; Lopez, 2014; Şenyurt & Gür, 2012; Uğurlu, 1997; Uğurlu & Topal, 1996, Yüksel et al., 2014, Yüksel et al., 2015). One of the best examples of these helices is the Salkowski curve, (Salkowski, 1909). Since those years, the various studies have been carried out on these curves both in Euclidean and Lorentzian 3-space. Monterde (2009) gave Frenet vectors and the parametric equation of the Salkowski curve. Gür and Şenyurt (2010) studied the on the Salkowski curve in Euclidean 3-space $\mathbb{E}^3$. Ali (2011) examined spacelike Salkowski and spacelike anti-Salkowski curves in his study. In addition, Şenyurt et al. (2015) studied on the timelike Salkowski curve. Other studies on the Salkowski curve are (Gür Mazlum et al., in pressed; Şenyurt & Gür, 2017; Şenyurt & Uzun, 2020; Şenyurt & Kemal, 2020; Şenyurt & Öztürk, 2018). The unit pole vector along a regular curve $\vec{\alpha}(t)$ in Lorentzian 3-space $\mathbb{E}_1^3$ generate a spherical curve on the unit Lorentzian sphere (or hyperbolic sphere). The spherical curve $(\vec{C})$ is called on the pole indicatrix curve of the curve $(\vec{\alpha})$ Let t_C be the parameter for the curve $(\vec{C})$. The parametric equation for this curve is $(\vec{C})=\vec{\alpha_C}(t_C)=\vec{C}(t)$, (Şenatalar, 1978; Uğurlu & Çalışkan, 2012). In this study, we work on the Frenet frame, the curvatures, the Frenet derivative formulas, the Darboux vector, the arc length, the geodesic curvatures of the pole indicatrix curve of the spacelike Salkowski curve with the timelike principal normal in Lorentzian 3-space $\mathbb{E}_1^3$ and show the graph of this indicatrix curve.

2. Preliminaries

2. Ön Bilgiler

Let $\vec{M}=(m_1,\ m_2,\ m_3)$ and $\vec{R}=(r_1,\ r_2,\ r_3)\in\mathbb{E}^3$ be two vectors. For these vectors, let it be defined as the inner product function

$$\langle,\rangle:\mathbb{E}_1^3\times\mathbb{E}_1^3\to\mathbb{E}_1^3,\qquad \langle\vec{M},\ \vec{R}\rangle=m_1r_1+m_2r_2-m_3r_3 \tag{1}$$

and the vectorial product function

$$\wedge:\ \mathbb{E}_1^3\times\mathbb{E}_1^3\to\mathbb{E}_1^3,\qquad \vec{M}\wedge\vec{R}=(m_3r_2-m_2r_3,\ m_1r_3-m_3r_1,\ m_1r_2-m_2r_1). \tag{2}$$

Here, the function $\langle,\rangle$ is called as Lorentzian metric. The space $\mathbb{E}^3$ with the Lorentzian metric is called the Lorentzian 3-space and is showed by $\mathbb{E}_1^3$ For $\vec{M}\in\mathbb{E}_1^3$, if $\langle\vec{M},\ \vec{M}\rangle>0$ or $\vec{M}=0$, $\vec{M}$ is called spacelike (sl) vector, if $\langle\vec{M},\ \vec{M}\rangle<0$, $\vec{M}$ is called timelike (tl) vector, if $\langle\vec{M},\ \vec{M}\rangle=0$ and $\vec{M}\neq 0$, $\vec{M}$ is called lightlike or null vector. If $\langle\vec{M},\ \vec{E}\rangle<0$, $\vec{M}$ is called future pointing timelike vector (fptl), if $\langle\vec{M},\ \vec{E}\rangle>0$, $\vec{M}$ is called past pointing timelike vector (pptl), where $\vec{E}=(0,\ 0,\ 1)$

The unit sphere $\mathbb{S}_1^2 = \left\{ \vec{M} \in \mathbb{E}_1^3 \mid \left\langle \vec{M}, \vec{M} \right\rangle = 1 \right\}$ is called unit Lorentzian sphere. For the vectors $\vec{M}$ and $\vec{R}$ in $\mathbb{E}_1^3$, if $\left\langle \vec{M}, \vec{R} \right\rangle = 0$, $\vec{M}$ and $\vec{R}$ are Lorentz orthogonal vectors. Let $\vec{M}$ and $\vec{R}$ be nonzero Lorentz orthogonal vectors in $\mathbb{E}_1^3$, if $\vec{M}$ is timelike, then $\vec{R}$ is spacelike, (Ratcliffe, 1994). Now, let's talk about the angle in between of any two vectors in $\mathbb{E}_1^3$:

i) If $\vec{M}$ and $\vec{R}$ be future (past) pointing timelike vectors in $\mathbb{E}_1^3$, the hyperbolic angle between of these vectors is as follows:

$$\left\langle \vec{M}, \vec{R} \right\rangle = -\left\| \vec{M} \right\| \left\| \vec{R} \right\| \cosh \varphi . \tag{3}$$

ii) If $\vec{M}$ be a spacelike vector and $\vec{R}$ be a positive timelike vector in $\mathbb{E}_1^3$, the hyperbolic angle between of these vectors is as follows:

$$\left| \left\langle \vec{M}, \vec{R} \right\rangle \right| = \left\| \vec{M} \right\| \left\| \vec{R} \right\| \sinh \varphi , \tag{4}$$

iii) If $\vec{M}$ and $\vec{R}$ be spacelike vectors in $\mathbb{E}_1^3$ that span a spacelike vector subspace, the reel angle between of these vectors is as follows:

$$\left\langle \vec{M}, \vec{R} \right\rangle = \left\| \vec{M} \right\| \left\| \vec{R} \right\| \cos \varphi , \tag{5}$$

(Ratcliffe, 1994). An curve $\left(\vec{\alpha}\right)$ in $\mathbb{E}_1^3$ is called the spacelike, timelike or null (lightlike) curve, if all of the velocity vector of the curve are the spacelike, timelike or null (lightlike), respectively. The norm of the vector $\vec{M} \in \mathbb{E}_1^3$ is $\left\| \vec{M} \right\| = \sqrt{\left| \left\langle \vec{M}, \vec{M} \right\rangle \right|}$ and if $\left\| \vec{M} \right\| = 1$, $\vec{M}$ is called a unit vector. Let $\{\vec{T}, \vec{N}, \vec{B}\}$, κ, τ, $\vec{F}$ and $\vec{C}$ be the Frenet trihedron, the curvature, the torsion, the Darboux vector and the unit vector of direction $\vec{F}$ of the differentiable curve $\left(\vec{\alpha}\right)$ in $\mathbb{E}_1^3$, respectively. Also, Let $\left\| \vec{\alpha}' \right\| = v$ for the curve $\left(\vec{\alpha}\right)$. For any spacelike curve $\left(\vec{\alpha}\right)$, there are the following equations between these elements:

i. Let the vectors $\vec{T}$ and $\vec{N}$ be spacelike vectors and $\vec{B}$ be timelike vector. There are the following relationships between these vectors:

$$\vec{T} \wedge \vec{N} = \vec{B}, \quad \vec{N} \wedge \vec{B} = -\vec{T}, \quad \vec{B} \wedge \vec{T} = -\vec{N}. \tag{6}$$

The Frenet derivative formulas and the Darboux vector for this curve is as follows, respectively:

$$\vec{T}' = v\kappa \vec{N}, \quad \vec{N}' = v\left(-\kappa \vec{T} + \tau \vec{B}\right), \quad \vec{B}' = v\tau \vec{N}, \tag{7}$$

$$\vec{F} = v\left(\tau \vec{T} + \kappa \vec{B}\right). \tag{8}$$

Let the hyperbolic angle between the timelike unit vector $\vec{B}$ and the vector $\vec{F}$ be φ.

➢ If $|\kappa| < |\tau|$, $\vec{F}$ is a spacelike. So, the unit pol vector in the direction of $\vec{F}$ are as follows:

$$\vec{C} = \cosh \varphi \vec{T} - \sinh \varphi \vec{B} . \tag{9}$$

➢ If $|\kappa| > |\tau|$, then $\vec{F}$ is a timelike vector. In this case, the unit pol vector in the direction of the timelike vector $\vec{F}$ is as follows:

$$\vec{C} = \sinh \varphi \vec{T} - \cosh \varphi \vec{B} . \tag{10}$$

ii. Let the vectors $\vec{T}$ and $\vec{B}$ are spacelike vectors, $\vec{N}$ is timelike vector. There are the following relationships between these vectors:

$$\vec{T} \wedge \vec{N} = -\vec{B} \ , \ \vec{N} \wedge \vec{B} = -\vec{T} \ , \ \vec{B} \wedge \vec{T} = \vec{N} . \tag{11}$$

The Frenet derivative formulas and the Darboux vector for this curve are as follows, respectively:

$$\vec{T}' = \kappa \vec{N}, \quad \vec{N}' = \kappa \vec{T} + \tau \vec{B}, \quad \vec{B}' = \tau \vec{N}, \tag{12}$$

$$\vec{F} = v\left(-\tau \vec{T} + \kappa \vec{B}\right). \tag{13}$$

Let the reel angle between the timelike unit vector $\vec{B}$ and the vector $\vec{F}$ be φ. Here, since $\left\langle \vec{F}, \vec{F} \right\rangle = \kappa^2 + \tau^2 > 0$, the unit pol vector in the direction of $\vec{F}$ is as follows:

$$\vec{C} = -\sin \varphi \vec{T} + \cos \varphi \vec{B} . \tag{14}$$

Supposing $a \le t \le b$, the real number

$$\int_a^b \left\| \frac{d\vec{\alpha}}{dt} \right\| dt = \int_a^b v \, dt \tag{15}$$

is called the arc length between the points $\vec{\alpha}(a)$ and $\vec{\alpha}(b)$ of the curve $(\vec{\alpha})$ in $\mathbb{E}_1^3$. Let $\vec{C}(t)$ be the pole vector of curve $(\vec{\alpha})$ at the point of $\vec{\alpha}(t)$. Then, the geodesic curvature with respect to $\mathbb{E}_1^3$ is

$$k_g = \left\| \overrightarrow{D_C C} \right\|. \tag{16}$$

Let the normal vector of the surface be $\vec{C}(t)$. If $\vec{C}(t)$ is the spacelike vector, the geodesic curvature γ_g with respect to $\mathbb{S}_1^2$ of the curve $(\vec{\alpha})$ is

$$\gamma_g = \left\| \overline{D}_C \vec{C} \right\|,\ \overrightarrow{D_C C} = \overline{D}_C \vec{C} + \left\langle S(\vec{C}),\ \vec{C} \right\rangle \vec{C}. \tag{17}$$

$\overline{D}$ is the connection and $\overline{S}$ is the shape operator of $\mathbb{S}_1^2$, (Uğurlu & Çalışkan, 2012).

Definition 2.1. For $m \in \mathbb{R}$, $m > 1$ or $m < -1$ and $n = \dfrac{m}{\sqrt{m^2 - 1}}$, the curve whose parametric expression is given below is called the spacelike Salkowski curve with the timelike principal normal in $\mathbb{E}_1^3$, Figure 1:

$$\overrightarrow{\gamma_m}(t) = \Bigg(2\sin t - \frac{1+n}{1-2n}\sin\left[(1-2n)t\right] - \frac{1-n}{1+2n}\sin\left[(1+2n)t\right],$$
$$2\cos t - \frac{1+n}{1-2n}\cos\left[(1-2n)t\right] - \frac{1-n}{1+2n}\cos\left[(1+2n)t\right],\ \frac{1}{m}\cos(2nt) \Bigg).$$

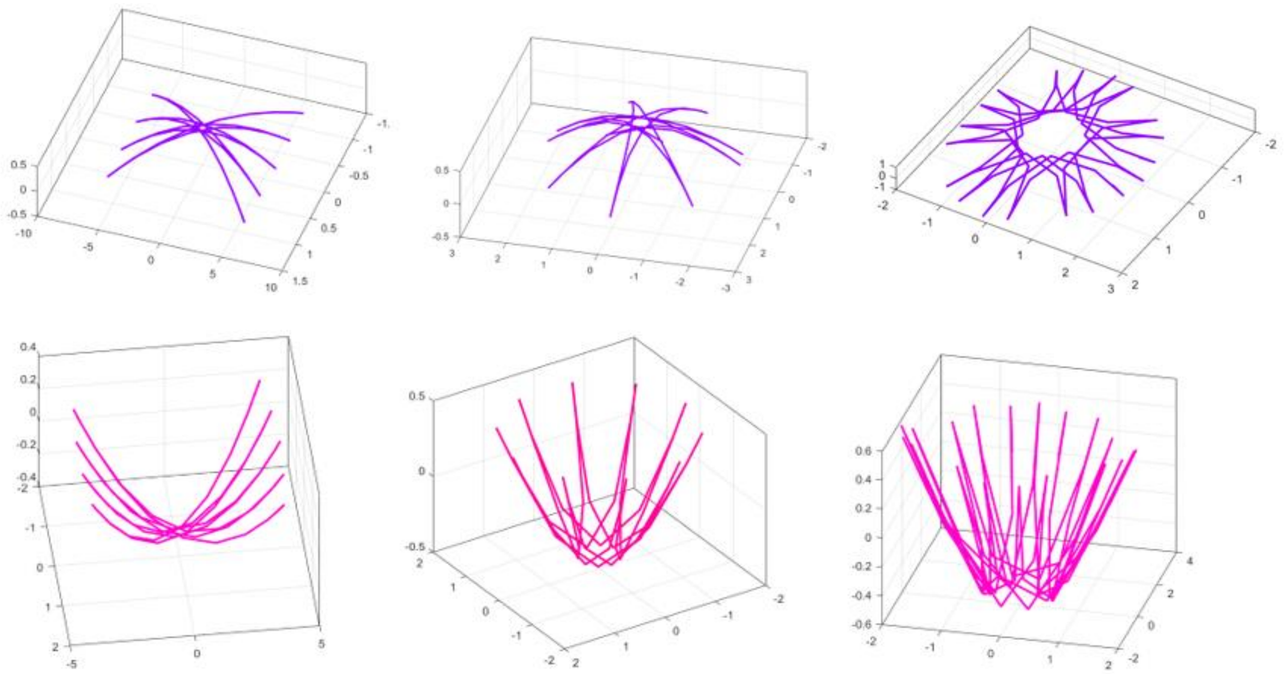

Figure 1. The curve $\overrightarrow{\gamma_m}(t)$ for $m = -6, -2, -\frac{10}{9}, 4, \frac{5}{4}, \frac{10}{9}$ respectively.

Şekil 1. *Sırasıyla* $m = -6, -2, -\frac{10}{9}, 4, \frac{5}{4}, \frac{10}{9}$ *değerleri için* $\overrightarrow{\gamma_m}(t)$ *eğrisi*

Here $\left\| \gamma_m' \right\| = v = \dfrac{\sin(nt)}{\sqrt{m^2 - 1}}$. From the expressions (11) and (12), the Frenet vectors are this curve are

$$\begin{cases} \vec{T}(t) = \left(\cos t \sin(nt) - n \sin t \cos(nt)\ ,\ -\sin t \sin(nt) - n \cos t \cos(nt)\ ,\ -\dfrac{n}{m}\cos(nt) \right) & \text{sl}, \\ \vec{N}(t) = \dfrac{n}{m}(\sin t, \cos t, m) & \text{tl}, \\ \vec{B}(t) = \left(-\cos t \cos(nt) - n \sin t \sin(nt),\ \sin t \cos(nt) - n \cos t \sin(nt),\ -\dfrac{n}{m}\sin(nt) \right) & \text{sl}. \end{cases} \tag{18}$$

Also the curvature is $\kappa(t)=1$, the torsion is $\tau(t)=-\cot(nt)$, the arc length is $s=-\dfrac{\cos(nt)}{m}$ and the interval is $\left[-\dfrac{\pi}{2n},\ t\right]$ of the curve $\overrightarrow{\gamma_m}(t)$, (Ali, 2011).

3. On The Pole Spherical Indicatrix Curves of the Spacelike Salkowski Curve with Timelike Principal Normal in Lorentzian 3-Space

3. 3-boyutlu Lorentz uzayında timelike asli normalli spacelike Salkowski eğrisinin pol gösterge eğrisi üzerine

First of all, let’s get the Darboux vector and unit pole vector in direction of the Darboux vector of the curve $\overrightarrow{\gamma_m}(t)$ and make sense of the angle nt.

Theorem 3.1. The Darboux vector $\vec{F}(t)$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ is as follows:

$$\vec{F}(t) = \left(-\frac{n^2}{m}\sin t, -\frac{n^2}{m}\cos t, -\frac{n^2}{m^2} \right). \tag{19}$$

Proof. If we take the derivative of the vector $\vec{N}(t)$ in the expression (18) according to t, we get

$$\vec{N'}(t) = \left(\frac{n}{m}\cos t, -\frac{n}{m}\sin t, 0 \right). \tag{20}$$

From the expressions (2), (18) and (20), if we consider the equations $\vec{F}(t) = \vec{N}(t) \wedge \vec{N'}(t)$, we obtain the expression (19). On the other hand, let’s compute the Darboux vector of $\overrightarrow{\gamma_m}(t)$ with the expression (13). Here, if we use the vectors $\vec{T}(t)$ and $\vec{B}(t)$ from the expression (18), the functions $\kappa(t)=1$, $\tau(t)=-\cot(nt)$ and $v=\dfrac{\sin(nt)}{\sqrt{m^2-1}}$, we again get the expression (19).

Corollary 3.2. The unit pole vector $\vec{C}(t)$ in the direction of the vector $\vec{F}(t)$ of the spacelike Salkowski curve with the timelike principal normal in $\mathbb{E}_1^3$ is as follows:

$$\vec{C}(t) = \left(-n \sin t, -n \cos t, -\frac{n}{m} \right). \tag{21}$$

Proof. If we use the equatility $\vec{C}(t) = \dfrac{\vec{F}(t)}{\left\| \vec{F}(t) \right\|}$, from the expression (19), we get the expression (21). Or let's show that expression (14) works. Let’s find the real angle φ between of the spacelike vectors $\vec{F}(t)$ and $\vec{B}(t)$ of the curve $\overrightarrow{\gamma_m}(t)$, Figure 2. From the expressions (5), (18) and (19), we get

$$\left\langle \vec{F}(t),\ \vec{B}(t) \right\rangle = \left\| \vec{F}(t) \right\| \left\| \vec{B}(t) \right\| \cos\varphi = \frac{n}{m}\cos\varphi. \tag{22}$$

On the other hand, the Lorentzian inner product of $\vec{F}(t)$ and $\vec{B}(t)$ from the expression (1) is obtained as

$$\left\langle \vec{F}(t),\ \vec{B}(t) \right\rangle = \frac{n}{m}\sin(nt). \tag{23}$$

Thus, from the equality of the expressions (22) and (23), we obtain

$$\sin(nt) = \cos\varphi. \tag{24}$$

Now, the real angle between of the spacelike vectors $\vec{F}(t)$ and $\vec{T}(t)$ in the expressions (5), (18) and (19) is as follows:

$$\left\langle \vec{F}(t),\ \vec{T}(t) \right\rangle = \left\| \vec{F}(t) \right\| \left\| \vec{T}(t) \right\| \cos\left(\frac{\pi}{2} + \varphi \right),$$

$$\left\langle \vec{F}(t),\ \vec{T}(t) \right\rangle = \frac{n}{m}(-\sin\varphi). \tag{25}$$

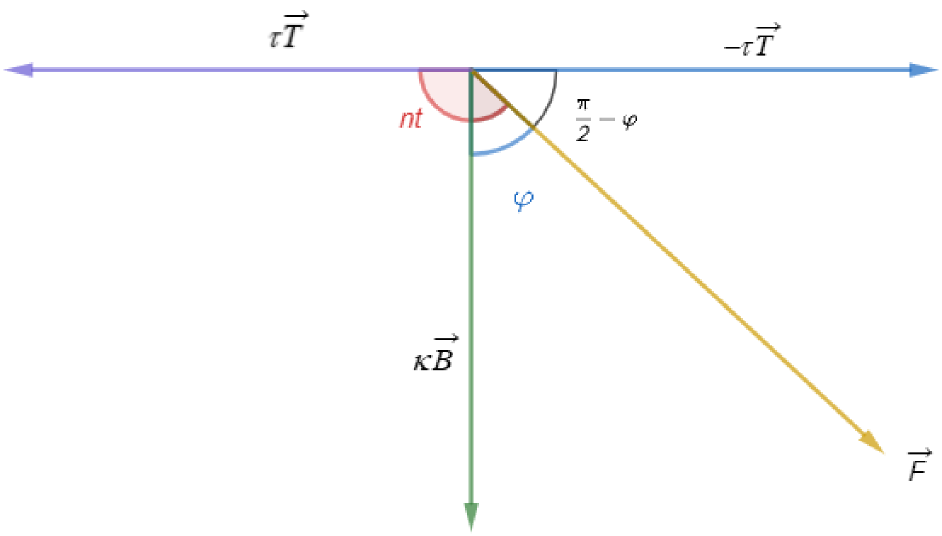

Figure 2. The Darboux vector of the curve $\overrightarrow{\gamma_m}(t)$

Şekil 2. *$\overrightarrow{\gamma_m}(t)$ eğrisinin Darboux vektörü*

On the other hand, from the inner product of $\vec{F}(t)$ and $\vec{T}(t)$, we get

$$\left\langle \vec{F}(t),\ \vec{T}(t)\right\rangle = \frac{n}{m}\cos(nt). \tag{26}$$

Thus, from the equality of the expressions (25) and (26), we obtain

$$\cos(nt) = -\sin\varphi. \tag{27}$$

So, if we substitute the expressions (18), (24), and (27) in the expression (14), we get the expression (21) again.

Corollary 3.3. Let φ be the real angle between of the spacelike vectors $\vec{F}(t)$ and $\vec{B}(t)$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$. There is following relationship between of the angles nt and φ,

$$nt = \varphi + \frac{\pi}{2}. \tag{28}$$

Proof. If we square the expressions (24) and (27),

$$\begin{aligned}
&\cos(nt) + \sin\varphi = 0\\
&\Rightarrow\ \cos^2(nt) + 2\cos(nt)\sin\varphi + \sin^2\varphi = 0,\\
&\sin(nt) - \cos\varphi = 0\\
&\Rightarrow\ \sin^2(nt) - 2\sin(nt)\cos\varphi + \cos^2\varphi = 0,
\end{aligned}$$

then add them side by side we do the necessary operations, we get

$$\sin(nt-\varphi) = 1. \tag{29}$$

From the expression (29), we obtain the expression (28).

Theorem 3.4. The Frenet frame $\left\{\overrightarrow{T_C}(t),\ \overrightarrow{N_C}(t),\ \overrightarrow{B_C}(t)\right\}$ of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ is as follows, respectively:

$$\begin{cases}
\overrightarrow{T_C}(t) = (-\cos t,\ \sin t,\ 0) & \text{sl},\\
\overrightarrow{N_C}(t) = (\sin t,\ \cos t,\ 0) & \text{sl},\\
\overrightarrow{B_C}(t) = (0,\ 0,\ -1) & \text{tl}.
\end{cases} \tag{30}$$

Proof. Let the parameter of the curve $(\vec{C})$ be t_C. If we take the derivative of the vector $\vec{C}(t)$ in the expression (21) according to the parameter t, we get

$$\vec{C}'(t) = (-n\cos t,\ n\sin t,\ 0). \tag{31}$$

Also, if we use the equation

$$\overrightarrow{T_C}(t) = \frac{d\vec{C}(t)}{dt_C} = \frac{d\vec{C}(t)}{dt}\frac{dt}{dt_C} = \vec{C}'(t)\frac{dt}{dt_C}. \tag{32}$$

From the expression (31) and since $\left\|\overrightarrow{T_C}(t)\right\|=1$, from the expression (21), we get

$$v_C=\frac{dt_C}{dt}=\left\|\overrightarrow{C'}(t)\right\|=n. \tag{33}$$

From the expressions (32) and (33), write $\overrightarrow{T_C}(t)=\dfrac{\overrightarrow{C'}(t)}{\left\|\overrightarrow{C'}(t)\right\|}$. Here, if we consider the expressions (31) and (33), we get the spacelike vector $\overrightarrow{T_C}(t)$ in the expression (30). Also, if we take the first and second derivatives of the vector $\vec{C}(t)$ in the expression (21) according to t and use the expression (2), we have

$$\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t)=\left(0,\ 0,\ -n^2\right),$$
$$\left\|\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t)\right\|=n^2. \tag{34}$$

If we consider the equation $\overrightarrow{B_C}(t)=\dfrac{\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t)}{\left\|\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t)\right\|}$, we get the timelike vector $\overrightarrow{B_C}(t)$ in the expression (30). Last, from the expression (6), if we use the equation $\overrightarrow{N_C}(t)=-\overrightarrow{B_C}(t)\wedge\overrightarrow{T_C}(t)$, we get the spacelike vector $\overrightarrow{N_C}(t)$ in the expression (30). Thus, the proof is completed.

Theorem 3.5. The curvature and torsion of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ are as follows:

$$\kappa_C(t)=\frac{1}{n} \quad\text{and}\quad \tau_C(t)=0. \tag{35}$$

Proof. If we consider the equation $\kappa_C(t)=\dfrac{\left\|\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t)\right\|}{\left\|\overrightarrow{C'}(t)\right\|^3}$, use the expressions (33) and (34), we get the curvature $\kappa_C(t)$ as the expression (35). And If we consider the equation $\tau_C(t)=\dfrac{\left\langle\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t),\ \overrightarrow{C'''}(t)\right\rangle}{\left\|\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t)\right\|^2}$ and use the expression (34), we get the torsion $\tau_C(t)$ in the expression (35), where

$$\overrightarrow{C'''}(t)=\left(n\cos t,\ -n\sin t,\ 0\right)$$

and

$$\left\langle\overrightarrow{C'}(t)\wedge\overrightarrow{C''}(t),\ \overrightarrow{C'''}(t)\right\rangle=0.$$

So, the proof is completed.

Corollary 3.6. The spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ is a planar circle of radius n on $\mathbb{S}_1^2$, Figure 3.

Proof. Since the torsion is $\tau_C(t)=0$, $(\vec{C})$ is a planar circle on $\mathbb{S}_1^2$ (on spacelike plane). Also the radius of this circle is $R_C=\dfrac{1}{\kappa_C(t)}=n$.

Theorem 3.7. The Frenet derivative formulas of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ are as follows:

$$\begin{aligned}
\overrightarrow{D_{T_C}T_C}(t)&=\frac{\overrightarrow{T_C'}(t)}{v_C}=\left(\frac{\sin t}{n},\frac{\cos t}{n},0\right)\quad\text{sl},\\
\overrightarrow{D_{T_C}N_C}(t)&=\frac{\overrightarrow{N_C'}(t)}{v_V}=\left(\frac{\cos t}{n},-\frac{\sin t}{n},0\right)\quad\text{sl},\\
\overrightarrow{D_{T_C}B_C}(t)&=\frac{\overrightarrow{B_C'}(t)}{v_C}=\left(0,\ 0,\ 0\right)\quad\text{sl}.
\end{aligned} \tag{36}$$

Proof. From the expression (7), we write

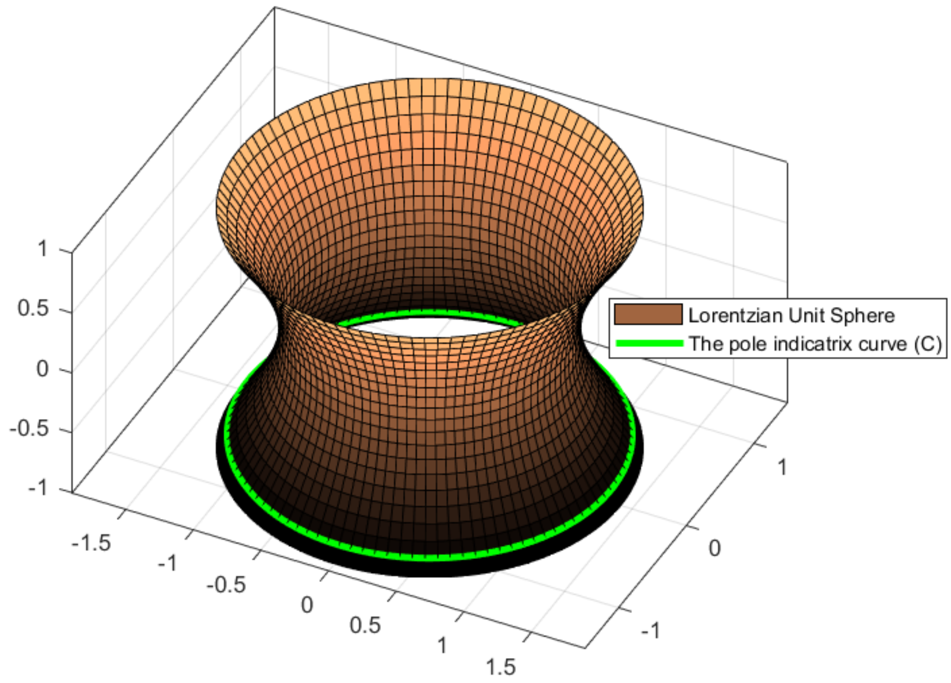

Figure 3. The pole indicatrix curve $\left(\vec{C}\right)$ of the curve $\vec{\gamma_m}(t)$ on $\mathbb{S}_1^2$

Şekil 3. *$\vec{\gamma_m}(t)$ eğrisinin $\mathbb{S}_1^2$ üzerinde $\left(\vec{C}\right)$ pol gösterge eğrisi*

$$\begin{bmatrix} \overrightarrow{D_{T_C}T_C}(t) \\ \overrightarrow{D_{T_C}N_C}(t) \\ \overrightarrow{D_{T_C}B_C}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_C(t) & 0 \\ -\kappa_C(t) & 0 & \tau_C(t) \\ 0 & \tau_C(t) & 0 \end{bmatrix} \begin{bmatrix} \vec{T_C}(t) \\ \vec{N_C}(t) \\ \vec{B_C}(t) \end{bmatrix}. \tag{37}$$

If we use the equation (37), from the expressions (30) and (35), we get the expression (36). On the other hand, we write

$$\overrightarrow{D_{T_C}T_C}(t) = \frac{d\vec{T_C}(t)}{dt_C} = \frac{d\vec{T_C}(t)}{dt}\frac{dt}{dt_C} = \frac{\vec{T_C}'(t)}{v_C}. \tag{38}$$

So, if we take the derivative of the vector $\vec{T_C}(t)$ in the expression (30) according to the parameter t_C and consider the expression (33), we get the vector $\overrightarrow{D_{T_C}T_C}(t)$ in the expression (36). Similar to the expression (38), we obtain the vectors $\overrightarrow{D_{T_C}N_C}(t)$ and $\overrightarrow{D_{T_C}B_C}(t)$. Thus, the proof is completed.

Theorem 3.8. There are the following differential equations for the Frenet vectors of the spacelike pole indicatrix curve $\left(\vec{C}\right)$ of the spacelike Salkowski curve with the timelike principal normal $\vec{\gamma_m}(t)$ in $\mathbb{E}_1^3$.

$$\begin{bmatrix} \overrightarrow{D_{T_C}T_C}(t) & \overrightarrow{D^2{}_{T_C}T_C}(t) & \overrightarrow{D^3{}_{T_C}T_C}(t) \\ \overrightarrow{D_{T_C}N_C}(t) & \overrightarrow{D^2{}_{T_C}N_C}(t) & \overrightarrow{D^3{}_{T_C}N_C}(t) \\ \overrightarrow{D_{T_C}B_C}(t) & \overrightarrow{D^2{}_{T_C}B_C}(t) & \overrightarrow{D^3{}_{T_C}B_C}(t) \end{bmatrix} \begin{bmatrix} \lambda_C \\ \mu_C \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{39}$$

where

$$\lambda_C(t) = \frac{1}{n^2} \quad \text{and} \quad \mu_C(t) = 0.$$

Proof. If we take the first, second and third derivatives of the vector $\vec{T_C}(t)$ in the expression (30) according to the parameter t_C:

$$\overrightarrow{D_{T_C}T_C}(t)=\left(\frac{\sin t}{n},\ \frac{\cos t}{n},\ 0\right),$$
$$\overrightarrow{D^2{}_{T_C}T_C}(t)=\left(\frac{\cos t}{n^2},\ -\frac{\sin t}{n^2},\ 0\right),$$
$$\overrightarrow{D^3{}_{T_C}T_C}(t)=\left(-\frac{\sin t}{n^3},\ -\frac{\cos t}{n^3},\ 0\right).$$

If we substitute these vectors in the expression (39) and solve this differential equation, we get the values of $\lambda_C(t)$ and $\mu_C(t)$. Similar operations are also obtained the vector $\overrightarrow{D_{T_C}N_C}(t)$ and $\overrightarrow{D_{T_C}B_C}(t)$. So the proof is completed.

Theorem 3.9. The Darboux vector $\overrightarrow{F_C}(t)$ belong to the Frenet frame of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ is as follows:

$$\overrightarrow{F_C}(t)=(0,\ 0,\ 1) \qquad tl. \tag{40}$$

Proof. From the expressions (2), (30) and (36), we get

$$\overrightarrow{F_C}(t)=-\overrightarrow{N_C}(t)\wedge\overrightarrow{N_C'}(t)=(0,\ 0,\ 1).$$

Or from the expressions (8), (30), (33) and (35), we get again

$$\overrightarrow{F_C}(t)\ =\ \nu_C(t)\left(\tau_C(t)\overrightarrow{T_C}(t)-\kappa_C(t)\overrightarrow{B_C}(t)\right)=\ (0,\ 0,\ 1)$$

Theorem 3.10. The hyperbolic angle φ_C between of the unit spacelike vectors $\overrightarrow{T_C}(t)$ and timelike vectors $\overrightarrow{F_C}(t)$ of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal in $\mathbb{E}_1^3$ is as follows:

$$\varphi_C=0. \tag{41}$$

Proof. Since $\overrightarrow{T_C}(t)$ is spacelike vector and $\overrightarrow{F_C}(t)$ is timelike vector, from the expressions (4), (30) and (40), we have

$$\left|\left\langle\overrightarrow{F_C}(t),\ \overrightarrow{T_C}(t)\right\rangle\right|=\left\|\overrightarrow{F_C}(t)\right\|\left\|\overrightarrow{T_C}(t)\right\|\sinh\varphi_C=\sinh\varphi_C \tag{42}$$

On the other hand, from the Lorentzian inner product of $\overrightarrow{T_C}(t)$ and $\overrightarrow{F_C}(t)$ in the expressions (30) and (40), we get

$$\left\langle\overrightarrow{F_C}(t),\ \overrightarrow{T_C}(t)\right\rangle=0. \tag{43}$$

From the equality of the expressions (42) and (43), we get

$$\sinh\varphi_C=0. \tag{44}$$

Similarly, since the timelike vector $\overrightarrow{B_C}(t)$ is past pointing and the timelike vector $\overrightarrow{F_C}(t)$ is future pointing, Figure 4, from the inner product formula and the expressions (3), (30) and (40), we get

$$\left\langle\overrightarrow{F_C}(t),\ \overrightarrow{B_C}(t)\right\rangle=\cosh\varphi_C=1. \tag{45}$$

From the expressions (44) and (45), we obtain the expression (41).

Corollary 3.11. The unit pole vector $\overrightarrow{C_C}(t)$ in the direction of $\overrightarrow{F_C}(t)$ of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ is as follows:

$$\overrightarrow{C_C}(t)=(0,\ 0,\ 1). \tag{46}$$

Proof. We can easily see the expression (46) by using the expression (40) in the equation $\overrightarrow{C_C}(t)=\dfrac{\overrightarrow{F_C}(t)}{\left\|\overrightarrow{F_C}(t)\right\|}$, but we want to show that the expression (10) works. For the curve $(\vec{C})$, according to expression (10), we write

$$\overrightarrow{C_C}(t)=\sinh\varphi_C\overrightarrow{T_C}(t)-\cosh\varphi_C\overrightarrow{B_C}(t). \tag{47}$$

So, indeed, if we use the expressions (30), (44) and (45) into the expression (47), we get the expression (46).

Theorem 3.12. The arc length s_C of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\overrightarrow{\gamma_m}(t)$ in $\mathbb{E}_1^3$ is as follows:

$$s_C=nt+\frac{\pi n}{2}. \tag{48}$$

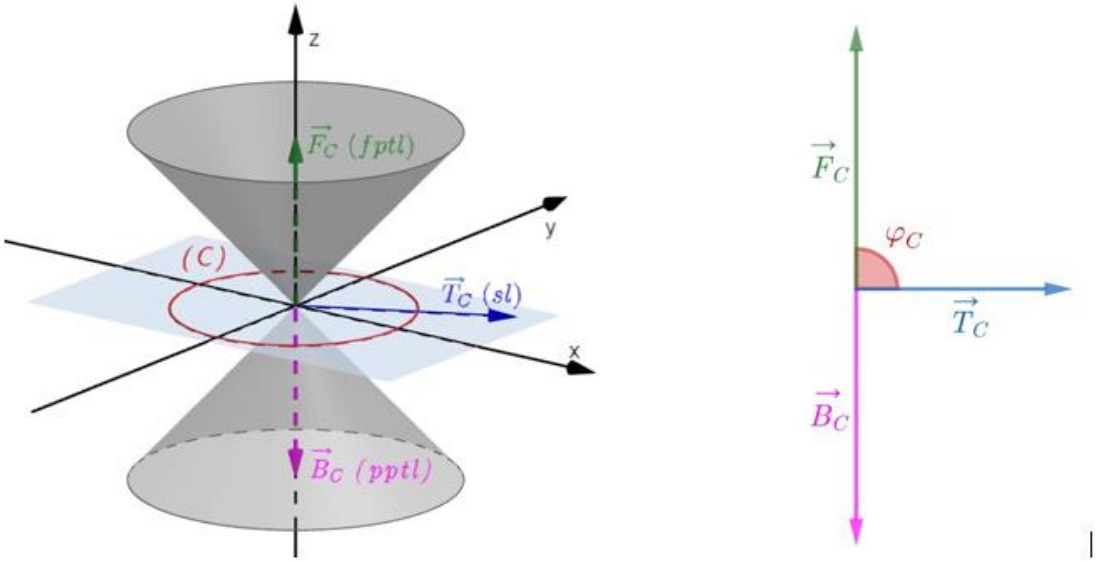

Figure 4. The Darboux vector $\vec{F_C}(t)$ of the pole indicatrix curve (C) of the curve $\vec{\gamma_m}(t)$ on $\mathbb{S}_1^2$

Şekil 4. *$\vec{\gamma_m}(t)$ eğrisinin $\mathbb{S}_1^2$ üzerinde $(\vec{C})$ pol gösterge eğrisinin $\vec{F_C}(t)$ Darboux vektörü*

Proof. Since $\vec{\gamma_m}(t)$ is in the interval $\left[-\frac{\pi}{2n},\, t\right]$, from the expression (15), (32) and (33), the arc length of $(\vec{C})$ is given as follows:

$$s_C = \int_{-\frac{\pi}{2n}}^{t} \left\| \frac{d\vec{C}(t)}{dt} \right\| dt = \int_{-\frac{\pi}{2n}}^{t} \left\| \vec{C}'(t) \right\| dt,$$

$$s_C = n \int_{-\frac{\pi}{2n}}^{t} dt = nt + \frac{\pi n}{2}.$$

From the epression (48), we get $t = \frac{s_C}{n} - \frac{\pi}{2}$. Indeed, if this value is substituted into curve $(\vec{C})$ in the expression (21), $\left\| \frac{d\vec{C}}{ds_C} \right\| = 1$ is obtained.

Theorem 3.13. The geodesic curvatures k_C and γ_C with respect to $\mathbb{E}_1^3$ and $\mathbb{S}_1^2$ of the spacelike pole indicatrix curve $(\vec{C})$ of the spacelike Salkowski curve with the timelike principal normal $\vec{\gamma_m}(t)$ in $\mathbb{E}_1^3$ are as follows:

$$k_C(t) = \frac{1}{|n|} \quad \text{and} \quad \gamma_C = \frac{1}{|m|}. \tag{49}$$

Proof. From the expressions (16) and (36), the geodesic curvature k_C with respect to $\mathbb{E}_1^3$ of the curve $(\vec{C})$ is given as follows:

$$k_C(t) = \sqrt{\left| \left\langle \overrightarrow{D_{T_C} T_C}(t),\ \overrightarrow{D_{T_C} T_C}(t) \right\rangle \right|} = \frac{1}{|n|}.$$

Also, from the expression (17) and (36), we get

$$\overrightarrow{\overline{D}_{T_C} T_C}(t) = \overrightarrow{D_{T_C} T_C}(t) + \vec{C}(t),$$

$$\overrightarrow{\overline{D}_{T_C} T_C}(t) = \left(\frac{(1-n^2)}{n} \sin t, \frac{(1-n^2)}{n} \cos t, -\frac{n}{m} \right). \tag{50}$$

So, from the expression (50), the geodesic curvature γ_C with respect to $\mathbb{S}_1^2$ of $(\vec{C})$ is obtained as follows:

$$\gamma_C = \sqrt{\left| \left\langle \overrightarrow{\overline{D}_{T_C} T_C}(t),\ \overrightarrow{\overline{D}_{T_C} T_C}(t) \right\rangle \right|} = \frac{1}{|m|}.$$

4. Discussion and Conclusions

Tartışma ve Sonuçlar

In this study, we calculate some properties on the pole indicatrix curve of the spacelike Salkowski curve with the timelike principal normal in Lorentzian 3-space $\mathbb{E}_1^3$. Similar work can be done for the other indicatrix curves of the same curve or for other types of Salkowski curves. All the equations and graphs used and clearly expressed in this study will be useful for new studies on curves and surfaces in Lorentzian 3-space.

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Author contribution
Yazar katkısı

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Etik beyanı
Declaration of ethical code

The authors of this article declare that the materials and methods used in this study do not require ethical committee approval and/or legal-specific permission.

Conflicts of interest
Çıkar çatışması beyanı

The authors declare that they have no conflict of interest.

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