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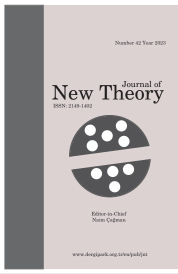

The Effect of the Additive Row Operation on the Permanent

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Abstract — The permanent function is not as stable as the determinant function under the elementary row operations. For example, adding a non-zero scalar multiple of a row to another row does not change the determinant of a matrix, but this operation changes its permanent. In this article, the variation in the permanent by applying the operation, which adds a scalar multiple of a row to another row, is examined. The relationship between the permanent of the matrix to which this operation is applied and the permanent of the initial matrix is given by a theorem. Finally, the paper inquires the need for further research.

Keywords *Permanent, Gaussian elimination, elementary row operations, Laplace expansion*

Mathematics Subject Classification (2020) 15A15, 03D15

1. Introduction

The permanent function was introduced first by Binet and Cauchy. According to Binet's definition, provided $m \leq 4$, the permanent of a matrix with order m by n is the sum of all possible products of m elements any two of which are not at the same column or row. Minc [1] emphasized that this definition given by Binet could be generalized for all finite values of m and n and gave the following definition.

Let $A = [a_{i,j}]$ be a matrix of order m by n. Then, per(A), the permanent of A, is defined by

$$\text{per}(A) = \sum_{\sigma} a_{1,\sigma_1} a_{2,\sigma_2} \dots a_{m,\sigma_m}$$

where the summation runs over on the set σ, which includes all one-to-one functions defined from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$ such that $m \leq n$ [1].

The permanent function can be interpreted as a kind of assessment using all matrix elements. This scalar-valued function of the matrix is best known for its relations with solutions to enumeration problems in combinatorics. For example, the Menage problem is a classical combinatorial enumeration problem, and it has been connected to the permanents of $(0, 1)-$matrices [2]. Another critical problem is computing the permanent of some kind of matrices, for example, the sparse and the circulant. This problem appears in various applications in mathematics, physics, computers, information systems, cryptography, and other fields. It has been studied to obtain various linear recurrence relations for permanents of certain sparse circulant matrices in [3], one of the recent studies in quantum computing.

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By solving the linear recursive system consisting of the obtained recurrences in [3], computing of permanent will be realized in linear time.

The definition of permanent is the same as the definition of determinant except for the factor ± 1 before terms in the summation. Thus, some properties of the permanent have direct analogs for the determinant. The following will give a brief summary of several fundamental properties related to both the permanent and the determinant.

The procedure of reducing anything to a simpler form is frequently used in both the determinant and the permanent. Laplace expansion is an important example of reduction, and it can be applied similarly to these two functions. Let $A(i,j)$ denote the matrix of size $m-1$ by $n-1$ obtained from the matrix $A=[a_{i,j}]$ by deleting row i and column j such that $i=1,2,\cdots,m$ and $j=1,2,\cdots,n$. It follows as a direct consequence of the definition of the permanent that

$$\operatorname{per}(A)=\sum_{j=1}^{n} a_{ij}\operatorname{per}(A(i,j))$$

called Laplace expansion of the permanent according to the i^{th} row of matrix A. Another procedure that can be used for reduction is Gaussian elimination. The determinant can be evaluated efficiently using Gaussian elimination (or row reduction) [4]. However, computation of the permanent in this way is much more complicated. The elementary row operations may differ in these functions because the permanent is not as stable as the determinant under the elementary row operations [2]. For example, adding a non-zero scalar multiple of a row to another row does not change the determinant of a matrix, but this operation changes its permanent. The determinant of a matrix with two equal rows is zero, but its permanent does not have to be zero. Multiplying a row by a scalar requires multiplying the determinant by the same scalar. This is also valid for the permanent. Interchanging two rows varies the sign of determinant, but permanent is invariant under this operation [5].

One of the fundamental rules of the determinant is $\det(AB)=\det(A)\det(B)$. This rule is clearly false for permanent. However, in [6], it has been proved that the equality $\operatorname{per}(AB)=\operatorname{per}(A)\operatorname{per}(B)$ holds for the generalized complementary basic (GCB) matrices which have many remarkable properties such as permanental, graph-theoretic, spectral, and inheritance properties [7, 8].

As mentioned above, for a square matrix A, adding a non-zero scalar multiple of a row to another row varies its permanent. To the best of our knowledge, there is no discussion on the effect of this operation on the permanent, in related literature. In this paper, the variation in the permanent has been studied when the operation “adding a scalar multiple of one row to another row” is applied to a square matrix. An equality that gives a relationship between the permanent of the original matrix and the permanent of its changed form is presented. In addition, an algorithm that calculates the variation is also given.

2. Main Results

Let

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & m & n \end{bmatrix}$$

be a square matrix of order 3×3 and $|A|$ denote the determinant of the matrix A. The determinant of a matrix remains unchanged when adding a non-zero scalar k multiple of a row to another row. As an example of this property, the following equations can be written for the matrix A:

$$\begin{vmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g & m & n \end{vmatrix} - |A| = 0$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g+ak & m+bk & n+ck \end{vmatrix} - |A| = 0$$

and

$$\begin{vmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g+ak & m+bk & n+ck \end{vmatrix} - |A| = 0$$

These situations for the permanent function, in contrast to the determinant, are illustrated by the equations below:

$$\text{per}\left(\begin{bmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g & m & n \end{bmatrix}\right) - \text{per}\,(A) = x \tag{1}$$

$$\text{per}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g+ak & m+bk & n+ck \end{bmatrix}\right) - \text{per}\,(A) = y \tag{2}$$

and

$$\text{per}\left(\begin{bmatrix} a & b & c \\ d+ak & e+bk & f+ck \\ g+ak & m+bk & n+ck \end{bmatrix}\right) - \text{per}\,(A) = z \tag{3}$$

where $x \neq y \neq z \neq 0$. The following theorem proposes obtaining x and y using $(n-2)$-ordered submatrices of a matrix A with order n by n. Namely, with the following theorem, we express the variation in the permanents for which additive row operation is applied only once, as in x and y. We note that the variation notion used in this study corresponds to x and y in Equalities 1 and 2. We also note that the calculation of variation we suggest is necessary twice to calculate the z value in Equality 3.

Let the notations used in this study clarify before giving on to the theorem. Let $A = [a_{i,j}]$ be a matrix of order n by n. The notation

$$\widetilde{A}_{r|t}$$

denotes the submatrix obtained by deleting the r^{th} row and the t^{th} column of the matrix A. The notation

$$\widetilde{A}_{i,r|j,t}$$

denotes the submatrix obtained by deleting i^{th} row, r^{th} row, j^{th} column, and t^{th} column of the matrix A. The submatrix $\widetilde{A}_{r|t}$ is of order $(n-1)\times(n-1)$ and the submatrix $\widetilde{A}_{i,r|j,t}$ is of order $(n-2)\times(n-2)$. As an example, if we consider the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

then

$$\widetilde{B}_{1|4} = \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \quad \text{and} \quad \widetilde{B}_{1,2|3,4} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

are some submatrices obtained from the matrix B.

Theorem 2.1. Let $A = [a_{i,j}]$ be a matrix of order $n \times n$ and B be the matrix obtained by adding k times of the i^{th} row to the r^{th} row of the matrix A. Then,

$$\operatorname{per}(B) - \operatorname{per}(A) = 2k \sum_{(j,t)\in\Omega} a_{i,j} a_{i,t} \operatorname{per}(\widetilde{A}_{i,r|j,t})$$

where the summation extends over the set $\Omega = \{(j,t) \in S \times S \mid j < t\}$ such that $S = \{1, 2, ..., n\}$.

PROOF.
By using the Laplace expansion with respect to r^{th} row of the matrix B, we obtain

$$\operatorname{per}(B) = (a_{r,1} + ka_{i,1}) \operatorname{per}(\widetilde{B}_{r|1}) + \cdots + (a_{r,n} + ka_{i,n}) \operatorname{per}(\widetilde{B}_{r|n}) \tag{4}$$

where $\widetilde{B}_{r|j}$ denotes the submatrices obtained by deleting r^{th} row and j^{th} column of the matrix B. Equality 4 can be arranged as the form

$$\operatorname{per}(B) = \left(a_{r,1} \operatorname{per}(\widetilde{B}_{r|1}) + \cdots + a_{r,n} per(\widetilde{B}_{r|n})\right) + k\left(a_{i,1} \operatorname{per}(\widetilde{B}_{r|1}) + \cdots + a_{i,n} \operatorname{per}(\widetilde{B}_{r|n})\right) \tag{5}$$

By applying the Laplace expansion along by the r^{th} row of the matrix A, it is easily seen that

$$a_{r,1} \operatorname{per}(\widetilde{B}_{r|1}) + \cdots + a_{r,n} per(\widetilde{B}_{r|n}) = \operatorname{per}(A)$$

Therefore, from Equality 5, $\operatorname{per}(B) = \operatorname{per}(A) + kV_A$ where

$$V_A = a_{i,1} \operatorname{per}(\widetilde{B}_{r|1}) + a_{i,2} \operatorname{per}(\widetilde{B}_{r|2}) + \cdots + a_{i,n} \operatorname{per}(\widetilde{B}_{r|n}) \tag{6}$$

At this point, the process will be continued by applying the Laplace expansion to every permanent in V_A seen by Equality 6, respectively. Firstly, by expanding the permanent, which in the first term of V_A seen by Equality 6, with respect to i^{th} row, the following equality is obtained:

$$\begin{aligned}\operatorname{per}(\widetilde{B}_{r|1}) =& a_{i,2} \operatorname{per}\left(\begin{bmatrix} a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,3} & a_{i-1,4} & \cdots & a_{i-1,n} \\ a_{i+1,3} & a_{i+1,4} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,3} & a_{r-1,4} & \cdots & a_{r-1,n} \\ a_{r+1,3} & a_{r+1,4} & \cdots & a_{r+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,4} & \cdots & a_{n,n} \end{bmatrix}\right) + a_{i,3} \operatorname{per}\left(\begin{bmatrix} a_{1,2} & a_{1,4} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,4} & \cdots & a_{i-1,n} \\ a_{i+1,2} & a_{i+1,4} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,2} & a_{r-1,4} & \cdots & a_{r-1,n} \\ a_{r+1,2} & a_{r+1,4} & \cdots & a_{r+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,4} & \cdots & a_{n,n} \end{bmatrix}\right) \\ &+ \cdots + a_{i,n} \operatorname{per}\left(\begin{bmatrix} a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n-1} \\ a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,2} & a_{r-1,3} & \cdots & a_{r-1,n-1} \\ a_{r+1,2} & a_{r+1,3} & \cdots & a_{r+1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \end{bmatrix}\right)\end{aligned} \tag{7}$$

Equality 7 can be written briefly as

$$\operatorname{per}(\widetilde{B}_{r|1}) = a_{i,2}\operatorname{per}(\widetilde{A}_{i,r|1,2}) + a_{i,3}\operatorname{per}(\widetilde{A}_{i,r|1,3}) + \cdots + a_{i,n}\operatorname{per}(\widetilde{A}_{i,r|1,n})$$

Similarly, if the permanents in other terms of Equality 6 are expanded along by their i^{th} row, then

$$\operatorname{per}(\widetilde{B}_{r|2}) = a_{i,1}\operatorname{per}(\widetilde{A}_{i,r|2,1}) + a_{i,3}\operatorname{per}(\widetilde{A}_{i,r|2,3}) + \cdots + a_{i,n}\operatorname{per}(\widetilde{A}_{i,r|2,n})$$

$$\operatorname{per}(\widetilde{B}_{r|3}) = a_{i,1}\operatorname{per}(\widetilde{A}_{i,r|3,1}) + a_{i,2}\operatorname{per}(\widetilde{A}_{i,r|3,2}) + a_{i,4}\operatorname{per}(\widetilde{A}_{i,r|3,4}) + \cdots + a_{i,n}\operatorname{per}(\widetilde{A}_{i,r|3,n})$$

$$\vdots$$

$$\operatorname{per}(\widetilde{B}_{r|n}) = a_{i,1}\operatorname{per}(\widetilde{A}_{i,r|n,1}) + a_{i,2}\operatorname{per}(\widetilde{A}_{i,r|n,2}) + \cdots + a_{i,n-1}\operatorname{per}(\widetilde{A}_{i,r|n,n-1})$$

If we plug the equalities obtained, for all $\operatorname{per}(\widetilde{B}_{r|\alpha})$, where $\alpha \in S$, into V_A seen by Equality 6, then we get

$$V_A = \sum_{(j,t)\in\Delta} a_{i,j}a_{i,t}\operatorname{per}(\widetilde{A}_{i,r|j,t}) \tag{8}$$

where $\Delta = \{(j,t) \in S \times S \mid j \neq t\}$. There are two of each term in the summation in Equality 8 because the terms of the form

$$a_{i,j}a_{i,t}\operatorname{per}(\widetilde{A}_{i,r|j,t})$$

equal to the terms of the form

$$a_{i,t}a_{i,j}\operatorname{per}(\widetilde{A}_{i,r|t,j})$$

Thus, we can write Equality 8 as

$$V_A = 2\sum_{(j,t)\in\Omega} a_{i,j}a_{i,t}\operatorname{per}(\widetilde{A}_{i,r|j,t})$$

where $\Omega = \{(j,t) \in S \times S \mid j < t\}$. □

According to Theorem 2.1, $\operatorname{per}(B) - \operatorname{per}(A)$ which we called as the variation, can be calculated by the following algorithm.

Algorithm 1 Calculation of the variation kV_A

```
INPUT: Matrix A = [a_{i,j}]_{n×n} and k, i, and r values.
OUTPUT: Result of the variation kV_A.
Step 1: row_set = The set of row numbers of the matrix A except the rows i and r
Step 2: for j = 1, 2, ··· , n − 1 do
            for t = j + 1, ··· , n do
                column_set = The set of column numbers of the matrix A except the columns j and t
                A_tilda = Form a submatrix of the matrix A using row_set and column_set
                Per(A_tilda) = Calculate the permanent of A_tilda
                summation = summation + a_{i,j} * a_{i,t}*Per(A_tilda)
            end for
        end for
Step 3: result = 2 * k* summation
```

3. Conclusion

It is important to note that this study does not propose any permanent calculation method. Instead, it provides a theoretical analysis of the variation that results from an additive row operation on the permanent of a square matrix. Moreover, it formulates the variation that occurs in the permanent of a square matrix in which the additive row operation is applied. This formula, called the variation in the permanent, proposes utilizing matrices of order $(n-2)\times(n-2)$ instead of matrices of order $n\times n$. Besides, this paper presents an algorithm to calculate this variation formula. The proposed algorithm needs

$$\frac{(n-2)n!}{2}$$

arithmetic operations if the permanent is calculated by the Naive algorithm, and

$$n(n-1)(n-2)2^{n-3}$$

arithmetic operations if the permanent is calculated by the Ryser-NW algorithm. For the numbers of arithmetic operations of the Naive and the Ryser-NW algorithms, see [9].

This article's findings can be extended to non-square matrices for further investigation. Furthermore, the variation formula suggested herein can be used to study the calculation of any square matrix permanents via the Gaussian elimination process.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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