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Homotopies of 2-Algebra Morphisms

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Abstract
In [1] it is defined the notion of 2-algebra as a categorification of algebras, and shown that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras. In this paper we define the notion of homotopy for 2-algebras and we explore the relations of crossed module homotopy and 2-algebra homotopy.

Keywords: 2-categories, Crossed modules, Homotopy
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1. Introduction

A crossed module [20] $\mathscr{A} = (\partial : C \longrightarrow R)$ of commutative algebras is given by an algebra morphism $\partial : C \longrightarrow R$ together with an action $\cdot$ of R on C such that the relations below hold for each $r \in R$ and each $c, c^{'} \in C$,

$$\begin{aligned} \partial(r \cdot c) &= r\partial(c) \\ \partial(c) \cdot c^{'} &= cc^{'}. \end{aligned}$$

Group crossed modules were firstly introduced by Whitehead in [21],[22]. They are algebraic models for homotopy 2-types, in the sense that [5],[15] the homotopy category of the model category [6],[9] of group crossed modules is equivalent to the homotopy category of the model category [11] of pointed 2-types: pointed connected spaces whose homotopy groups π_i vanish, if $i \geq 3$. The homotopy relation between crossed module maps $\mathscr{A} \longrightarrow \mathscr{A}^{'}$ was given by Whitehead in [22], in the contex of "homotopy systems" called free crossed complexes.

In [2] it is addressed the homotopy theory of maps between crossed modules of commutative algebras. It is proven that if $\mathscr{A}$ and $\mathscr{A}^{'}$ are crossed modules of algebras without any restriction on $\mathscr{A}$ and $\mathscr{A}^{'}$ then the crossed module maps $\mathscr{A} \longrightarrow \mathscr{A}^{'}$ and their homotopies give a groupoid.

In [1] it is defined the notion of 2-algebra as a categorification of algebras, and shown that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras. In this paper we define the notion of homotopy for 2-algebras. This definition is essentially a special case of 2-natural transformation due to Gray in [12]. And we explore the relations between the crossed module homotopies and 2-algebra homotopies. Similar results are given [13] by İçen for 2-groupoids.

2. Preliminaries

In [1] it is defined the notion of 2-algebra as a categorification of algebras, and shown that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras.

2.1 2-algebras

Definition 2.1. *A weak 2-algebra consists of*

· a 2-module A equipped with a functor $\bullet : A \times A \longrightarrow A$*, which is defined by* $(x,y) \mapsto x \bullet y$ *and bilinear on objects and defined by* $(f,g) \mapsto f \bullet g$ *on morphisms satisfying interchange law, i.e.,*

$$(f_1 \bullet g_1) \circ (f_2 \bullet g_2) = (f_1 \circ f_2) \bullet (g_1 \circ g_2)$$

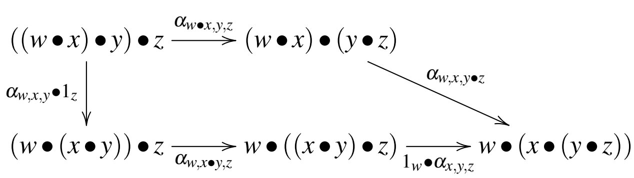
· $k-$*bilinear natural isomorphisms*

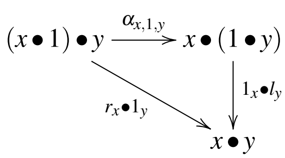
$$\alpha_{x,y,z} : (x \bullet y) \bullet z \longrightarrow x \bullet (y \bullet z)$$

$$l_x : 1 \bullet x \longrightarrow x$$

$$r_x : x \bullet 1 \longrightarrow x$$

such that the following diagrams commute for all objects $w,x,y,z \in A_0$.





A strict 2-algebra is the special case where $\alpha_{x,y,z}$*,* l_x*,* r_x *are all identity morphisms. In this case we have*

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

$$1 \bullet x = x, x \bullet 1 = x$$

Strict 2-algebra is called commutative strict 2-algebra if $x \bullet y = y \bullet x$ *for all objects* $x,y \in A_0$ *and* $f \bullet g = g \bullet f$ *for all morphisms* $f,g \in A_1$.

In the rest of this paper, the term 2-algebra will always refer to a commutative strict 2-algebra. A homomorphism between 2-algebras should preserve both the 2-module structure and the • functor.

Definition 2.2. *Given 2-algebras A and* A'*, a homomorphism*

$$F : A \longrightarrow A'$$

consists of

· a linear functor F from the underlying 2-module of A to that of A'*, and*

· a bilinear natural transformation

$$F_2(x,y) : F_0(x) \bullet F_0(y) \longrightarrow F_0(x \bullet y)$$

· an isomorphism $F : 1' \longrightarrow F_0(1)$ *where* 1 *is the identity object of A and* $1'$ *is the identity object of* A'*,*
such that the following diagrams commute for $x,y,z \in A_0$*,*

$$\begin{array}{ccccc}
(F(x)\bullet F(y))\bullet F(z) & \xrightarrow{F_2\bullet 1} & F(x\bullet y)\bullet F(z) & \xrightarrow{F_2} & F((x\bullet y)\bullet z) \\
{\scriptstyle \alpha_{F(x),F(y),F(z)}}\downarrow & & & & \downarrow{\scriptstyle F(\alpha_{x,y,z})} \\
F(x)\bullet(F(y)\bullet F(z)) & \xrightarrow[1\bullet F_2]{} & F(x)\bullet F(y\bullet z) & \xrightarrow[F_2]{} & F(x\bullet(y\bullet z)).
\end{array}$$

$$\begin{array}{ccc}
1'\bullet F(x) & \xrightarrow{l'_{F(x)}} & F(x) \\
{\scriptstyle F_0\bullet 1}\downarrow & & \uparrow{\scriptstyle F(l_x)} \\
F(1)\bullet F(x) & \xrightarrow[F_2]{} & F(1\bullet x).
\end{array}$$

$$\begin{array}{ccc}
F(x)\bullet 1' & \xrightarrow{r'_{F(x)}} & F(x) \\
{\scriptstyle 1\bullet F_0}\downarrow & & \uparrow{\scriptstyle F(r_x)} \\
F(x)\bullet F(1) & \xrightarrow[F_2]{} & F(x\bullet 1).
\end{array}$$

Definition 2.3. *2-algebras and homomorphisms between them give the category of 2-algebras denoted by* ***2Alg*** *.*

Therefore if $A=(A_0,A_1,s,t,e,\circ,\bullet)$ is a 2-algebra, A_0 and A_1 are algebras with this $\bullet$ bilinear functor. Thus we can take that 2-algebra is a 2-category with a single object say $*$, and A_0 collections of its 1-morphisms and A_1 collections of its 2-morphisms are algebras with identity.

2.2 Crossed modules

Crossed modules have been used widely and in various contexts since their definition by Whitehead [23] in his investigations of the algebraic structure of relative homotopy groups. We recalled the definition of crossed modules of commutative algebras given by Porter [20].

Let R be a k-algebra with identity. A pre-crossed module of commutative algebras is an R-algebra C together with a commutative action of R on C and a morphism

$$\partial : C \longrightarrow R$$

such that for all $c\in C$, $r\in R$

CM1) $\partial(r\blacktriangleright c)=r\partial c.$

This is a crossed R-module if in addition for all $c,c'\in C$

CM2) $\partial c\blacktriangleright c'=cc'.$

The last condition is called the Peiffer identity. We denote such a crossed module by (C,R,∂).

A morphism of crossed modules from (C,R,∂) to (C',R',∂') is a pair of k-algebra morphisms $\phi : C\longrightarrow C'$, $\psi : R\longrightarrow R'$ such that

$$\partial'\phi=\psi\partial \qquad \text{and} \qquad \phi(r\blacktriangleright c)=\psi(r)\blacktriangleright\phi(c).$$

Thus we get a category $\mathbf{XMod}_k$ of crossed modules (for fixed k).

Examples of Crossed Modules

1. Any ideal I in R gives an inclusion map, $inc : I\longrightarrow R$ which is a crossed module. Conversely given an arbitrary R-module $\partial : C\longrightarrow R$ one easily sees that the Peiffer identity implies that ∂C is an ideal in R.

2. Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ sending everything in M to the zero element of R is a crossed module. Conversely: If (C,R,∂) is a crossed module, $\partial(C)$ acts trivially on $\ker\partial$, hence $\ker\partial$ has a natural $R/\partial(C)$-module structure.

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

3. Let be $\mathscr{M}(C)$ multiplication algebra. Then $(C,\mathscr{M}(C),\mu)$ is multiplication crossed module. $\mu : C \to \mathscr{M}(C)$ is defined by $\mu(r) = \delta_r$ with $\delta_r(r') = rr'$ for all $r,r' \in C$, where δ is multiplier $\delta : C \to C$ such that for all $r,r' \in C$, $\delta(rr') = \delta(r)r'$. Also $\mathscr{M}(C)$ acts on C by $\delta \blacktriangleright r = \delta(r)$.(See [3] for details).

In [20] Porter states that there is an equivalence of categories between the category of internal categories in the category of k-algebras and the category of crossed modules of commutative k-algebras. In the following theorem, it is given a categorical presentation of this equivalence.

Theorem 2.4. *[1] The category of crossed modules* $\mathbf{XMod}_k$ *is equivalent to that of 2-algebras,* ***2Alg****.*

Proof. Let $A = (A_0,A_1,s,t,e,\circ,\bullet)$ be a 2-algebra consisting of a single object say $*$ and an algebra A_0 of 1-morphisms and an algebra A_1 of 2-morphisms and $\partial = t|_{Kers}$ algebra homomorphism by $\partial : Kers \longrightarrow A_0, \partial(h) = t(h)$. Then $(Kers,A_0,\partial)$ is a crossed module.

Let $\mathscr{A} = (A_0,A_1,s,t,e,\circ,\bullet)$ and $\mathscr{A}' = (A_0',A_1',s',t',e',\circ',\bullet')$ be 2-algebras and $F = (F_0,F_1) : \mathscr{A} \longrightarrow \mathscr{A}'$ be a 2-algebra morphism. Then $F_0 : A_0 \longrightarrow A_0'$ and $F_1 : A_1 \longrightarrow A_1'$ are the k-algebra morphisms. For $f_1 = F_1|_{Kers} : Kers \longrightarrow Kers'$ and $f_0 = F_0 : A_0 \longrightarrow A_0'$, (f_1, f_0) map is a crossed module morphism $(Kers,A_0,\partial) \longrightarrow (Kers',A_0',\partial')$. So it is got a functor

$$\Gamma : \mathbf{2Alg} \longrightarrow \mathbf{XMod}_k.$$

Conversely, let (G,C,∂) be a crossed module of algebras. For $s,t : G \rtimes C \to C$ and $e : C \to G \rtimes C$ by $s(g,c) = c, t(g,c) = \partial(g)+c, e(c) = (0,c)$ and
the compositions

$$(g,c) \bullet (h,d) = (c \blacktriangleright h + d \blacktriangleright g + gh, cd)$$

$$(g,c) \circ (g',\partial(g)+c) = (g+g',c)$$

such that $t(g,c) = s(g',\partial(g)+c) = \partial(g)+c$, it is constructed a 2-algebra $\mathscr{A} = (C,G \rtimes C,s,t,e,\circ,\bullet)$ consists of the single object say $*$ and the k-algebra C of 1-morphisms and the k-algebra $G \rtimes C$ of 2-morphisms. Let (G,C,∂) and (G',C',∂') be crossed modules and $f = (f_1, f_0) : (G,C,\partial) \longrightarrow (G',C',\partial')$ be a crossed module morphism. For

$$\begin{array}{rccc} F_1 : & G \rtimes C & \longrightarrow & G' \rtimes C' \\ & (g,c) & \longmapsto & F_1(g,c) = (f_1(g), f_0(c)) \end{array}$$

and

$$\begin{array}{rccc} F_0 : & C & \longrightarrow & C' \\ & c & \longmapsto & F_0(c) = f_0(c). \end{array}$$

$F = (F_1,F_0)$ is a 2-algebra morphism from $(C,G \rtimes C,s,t,e,\circ,\bullet)$ to $(C',G' \rtimes C',s',t',e',\circ',\bullet')$. Thus it is got a functor

$$\Psi : \mathbf{XMod}_k \longrightarrow \mathbf{2Alg}.$$

□

3. Homotopies of Crossed Modules and 2-Algebras

The notion of homotopy for morphisms of crossed modules over commutative algebras is given in [2]. In this section, we explain the relation between homotopies for crossed modules over commutative algebras and homotopies for 2-algebras. The formulae given below are playing important role in our study.

Definition 3.1. *[2] Let $\mathscr{A} = (E,R,\partial)$ and $\mathscr{A}' = (E',R',\partial')$ be crossed modules and $f_0 : R \longrightarrow R'$ be an algebra morphism. An f_0-derivation $s : R \longrightarrow E'$ is a k-linear map satisfying for all $r,r' \in R$,*

$$s(rr') = f_0(r) \blacktriangleright s(r') + f_0(r') \blacktriangleright s(r) + s(r)s(r').$$

Let $f=(f_1,f_0)$ be a crossed module morphism $\mathscr{A}\longrightarrow\mathscr{A}^{'}$ and s be an f_0-derivation. If $g=(g_1,g_2)$ is defined as (where $e\in E$ and $r\in R$)

$$\begin{aligned} g_0(r) &= f_0(r)+(\partial^{'}s)(r) \\ g_1(e) &= f_1(e)+(s\partial)(e), \end{aligned}$$

then g is also crossed module morphism $\mathscr{A}\longrightarrow\mathscr{A}^{'}$. In such a case we write $f\overset{(f_0,s)}{\longrightarrow}g$, and say that (f_0,s) is a homotopy connecting f to g.

If (f_0,s) and $(g_0,s^{'})$ are homotopies connecting f to g and g to u respectively, then $(f_0,s+s^{'})$ is a homotopy connecting f to u, where $s+s^{'}:R\longrightarrow E^{'}$ is an f_0-derivation defined by $(s+s^{'})(r)=s(r)+s^{'}(r)$.

The notion of homotopy for 2-algebras is essentially a special case of 2-natural transformation due to Gray in [12].

Definition 3.2. *Let $\mathbf{A}=(A_0,A_1,s,t,e,\circ,\bullet)$ and $\mathbf{A}^{'}=(A_0^{'},A_1^{'},s^{'},t^{'},e^{'},\circ^{'},\bullet^{'})$ be 2-algebras and let $F=(F_1,F_0)$ and $G=(G_1,G_0)$ be 2-algebra morphisms $\mathbf{A}\longrightarrow\mathbf{A}^{'}$. A k-algebra morphism $\mu:A_0\longrightarrow A_1^{'}$ satisfying the following conditions is called a homotopy connecting F to G:*
1) $s^{'}\mu=F_0$
2) $t^{'}\mu=G_0$
3) $F_1\circ^{'}\mu t=\mu s\circ^{'}G_1$. In such a case we write $F\overset{\mu}{\longrightarrow}G$.

Theorem 3.3. *Let $\mathscr{A}=(A_0,A_1,s,t,e,\circ,\bullet)$, $\mathscr{A}^{'}=(A_0^{'},A_1^{'},s^{'},t^{'},e^{'},\circ^{'},\bullet^{'})$ be 2-algebras, $F=(F_1,F_0)$, $G=(G_1,G_0)$ and $U=(U_1,U_0)$ be 2-algebra morphisms $\mathscr{A}\longrightarrow\mathscr{A}^{'}$ and μ be a homotopy connecting F to G, $\mu^{'}$ be a homotopy connecting G to U. Then the map $\mu*\mu^{'}:A_0\longrightarrow A_1$ defined by $(\mu*\mu^{'})(x)=\mu(x)+\mu^{'}(x)-e^{'}(t^{'}\mu)(x)$ is a homotopy connecting F to U.*

Proof. We first show that $\mu*\mu^{'}$ is an algebra morphism. Since μ and $\mu^{'}$ are algebra morphisms, $\mu(x\bullet x^{'})=\mu(x)\bullet^{'}\mu(x^{'})$ and $\mu^{'}(x\bullet x^{'})=\mu^{'}(x)\bullet^{'}\mu^{'}(x^{'})$ for all $x,x^{'}\in A_0$. Then we get

$$\begin{aligned} (\mu*\mu^{'})(x\bullet x^{'}) &= \mu(x\bullet x^{'})+\mu^{'}(x\bullet x^{'})-e^{'}(t^{'}\mu)(x\bullet x^{'}) \\ &= \mu(x\bullet x^{'})+\mu^{'}(x\bullet x^{'})-e^{'}(G_0)(x\bullet x^{'}) \\ &= \mu(x\bullet x^{'})\circ^{'}\mu^{'}(x\bullet x^{'}) \\ &= (\mu(x)\bullet^{'}\mu(x^{'}))\circ^{'}(\mu^{'}(x)\bullet^{'}\mu^{'}(x^{'})) \\ &= (\mu(x)\circ^{'}\mu^{'}(x))\bullet^{'}(\mu(x^{'})\circ^{'}\mu^{'}(x^{'})) \qquad \text{(interchange law)} \\ &= (\mu(x)+\mu^{'}(x)-e^{'}(G_0)(x))\bullet^{'}(\mu(x^{'})+\mu^{'}(x^{'})-e^{'}(G_0)(x^{'})) \\ &= (\mu*\mu^{'})(x)\bullet^{'}(\mu*\mu^{'})(x^{'}). \end{aligned}$$

For all $x\in A_0$

$$\begin{aligned} s^{'}(\mu*\mu^{'})(x) &= s^{'}(\mu(x)+\mu^{'}(x)-e^{'}G_0(x)) \\ &= s^{'}\mu(x)+s^{'}\mu^{'}(x)-s^{'}e^{'}G_0(x) \\ &= F_0(x)+G_0(x)-G_0(x) \\ &= F_0(x), \end{aligned}$$

$$\begin{aligned} t^{'}(\mu*\mu^{'})(x) &= t^{'}(\mu(x)+\mu^{'}(x)-e^{'}G_0(x)) \\ &= t^{'}\mu(x)+t^{'}\mu^{'}(x)-t^{'}e^{'}G_0(x) \\ &= G_0(x)+U_0(x)-G_0(x) \\ &= U_0(x), \end{aligned}$$

and since $F_1\circ^{'}\mu t=\mu s\circ^{'}G_1$ and $G_1\circ^{'}\mu^{'}t=\mu^{'}s\circ^{'}U_1$, we get

$$\begin{aligned} F_1\circ^{'}\mu t\circ^{'}\mu^{'}t &= \mu s\circ^{'}G_1\circ^{'}\mu^{'}t \\ &= \mu s\circ^{'}\mu^{'}s\circ^{'}U_1. \end{aligned}$$

Thus, we get

$$\begin{aligned} F_1\circ^{'}(\mu*\mu^{'})t &= F_1\circ^{'}(\mu t\circ^{'}\mu^{'}t) \\ &= (\mu s\circ^{'}\mu^{'}s)\circ^{'}U_1 \\ &= (\mu*\mu^{'})s\circ^{'}U_1. \end{aligned}$$

Therefore $\mu*\mu^{'}:A_0\longrightarrow A_1$ is a homotopy connecting F to U. □

Theorem 3.4. *Let $\Gamma :: \mathbf{2Alg} \longrightarrow: \mathbf{XMod_k}$ be the functor as mentioned in Teorem 1.4 and μ be homotopy connecting F to G. Then*

$$\begin{array}{rcccc} \Gamma(\mu) = h & : & A_0 & \longrightarrow & Kers^{'} \\ & & x & \longmapsto & h(x) = \mu(x) - e^{'}(s^{'}\mu)(x) \end{array}$$

is a homotopy of corresponding crossed module morphisms.

Proof. We first show that h is an f_0-derivation where $f_0 : A_0 \longrightarrow A_0^{'}$ defined by $f_0(x) = F_0(x)$. For $x, x^{'} \in A_0$,

$$\begin{array}{rcl} f_0(x) \blacktriangleright h(x^{'}) & & \\ +f_0(x^{'}) \blacktriangleright h(x) + h(x) \bullet^{'} h(x^{'}) & = & F_0(x) \blacktriangleright (\mu(x^{'}) - e^{'}(s^{'}\mu)(x^{'})) \\ & & +F_0(x^{'}) \blacktriangleright (\mu(x) - e^{'}(s^{'}\mu)(x)) \\ & & +(\mu(x) - e^{'}(s^{'}\mu)(x)) \bullet^{'} (\mu(x^{'}) - e^{'}(s^{'}\mu)(x^{'})) \\ & = & e^{'}(F_0(x)) \bullet^{'} (\mu(x^{'}) - e^{'}F_0(x^{'})) \\ & & +e^{'}(F_0(x^{'})) \bullet^{'} (\mu(x) - e^{'}F_0(x)) + \mu(x) \bullet^{'} \mu(x^{'}) \\ & & -\mu(x) \bullet^{'} e^{'}F_0(x^{'}) - e^{'}F_0(x) \bullet^{'} \mu(x^{'}) + e^{'}F_0(x) \bullet^{'} e^{'}F_0(x^{'}) \\ & = & \mu(x \bullet x^{'}) - e^{'}(s^{'}\mu)(x \bullet x^{'}) \\ & = & h(x \bullet x^{'}). \end{array}$$

Therefore h is an f_0-derivation.

Now we show that

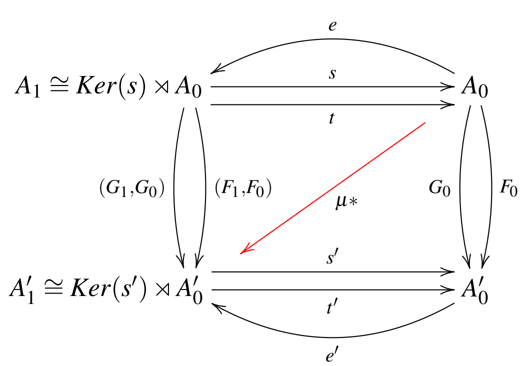
$$\begin{array}{rcl} g_0(x) & = & f_0(x) + \partial^{'} h(x) \\ g_1(n) & = & f_1(n) + h\partial(n) \end{array}$$

for $x \in A_0$ and $n \in Kers$.

$$\begin{array}{rcl} \partial^{'} h(x) & = & \partial^{'}(\mu(x) - e^{'}f_0(x)) \\ & = & \partial^{'}(\mu(x)) - \partial^{'}(e^{'}f_0(x)) \\ & = & (t^{'}\mu)(x) - (t^{'}e^{'})f_0(x) \\ & = & g_0(x) - f_0(x) \end{array}$$

and we get $g_0(x) = f_0(x) + \partial^{'} h(x)$.

Since $A_1 \simeq Kers \rtimes A_0$, we take $a = (n, x)$ for $a \in A_1$ where $n = a - es(a) \in Kers$ and $x = s(a) \in A_0$. We define $\mu^* : A_0 \longrightarrow Kers^{'} \rtimes A_0^{'}$, as $\mu^*(x) = (\mu(x) - e^{'}s^{'}(\mu(x)), s^{'}\mu(x))$ and $h^* : A_0 \longrightarrow Kers^{'} \rtimes A_0^{'}$, as $h^*(x) = (h(x), F_0(x))$. Therefore



for $(F_1, F_0)(n, x), (\mu^* t)(n, x) \in A_1 \simeq Kers^{'} \rtimes A_0^{'}$ such that $t(F_1, F_0)(n, x) = s(\mu^* t)(n, x)$, we have $(F_1, F_0)(n, x) \circ^{'} \mu^* t(n, x) = (F_1(n) + \mu t(n), F_0(x))$ and $-(F_1, F_0)(n, x) = (-F_1(n), t^{'}F_1(n) + F_0(x))$ and then, since

$$(F_1, F_0)(n, x) \circ^{'} \mu^* t(n, x) = \mu^* s(n, x) \circ^{'} (G_1, G_0)(n, x)$$

we have

$$\begin{array}{rcl} \mu^* t(n, x) & = & -(F_1, F_0)(n, x) \circ^{'} \mu^* s(n, x) \circ^{'} (G_1, G_0)(n, x) \\ & = & (-F_1(n) + h(x) + G_1(n), t^{'}F_1(n) + F_0(x)) \end{array}$$

and

$$-e'F_0t(n,x) = (0, t'f_1(n) + f_0(x)).$$

Hence we get

$$\begin{array}{rcl} \mu^*t(n,x) - e'F_0t(n,x) & = & (I_{t'F_1(n)+F_0(x)} \circ \mu t)(n,x) \\ & = & \mu^*t(n,x). \end{array}$$

Then

$$\begin{array}{rcl} h^*(t(n,x)) & = & \mu^*(t(n,x)) - e'(s'\mu^*)(t(n,x)) \\ & = & \mu^*t(n,x) - e'F_0t^*(n,x) \\ & = & \mu^*t(n,x) \\ & = & (-F_1(n) + h(x) + G_1(n), t'F_1(n) + F_0(x)) \qquad (1) \end{array}$$

and

$$\begin{array}{rcl} h^*(t(n,x)) & = & h^*(\partial(n) + x)) \\ & = & (h(\partial(n) + x)), f_0(\partial(n) + x)) \\ & = & (h(\partial(n)) + h(x), f_0(\partial(n)) + f_0(x)) \\ & = & (h(\partial(n)) + h(x), t'F_1(n) + F_0(x)). \qquad (2) \end{array}$$

Therefore from (1) and (2) we have

$$h(\partial(n)) + h(x) = -F_1(n) + h(x) + G_1(n)$$

and

$$h(\partial(n)) = -F_1(n) + G_1(n).$$

Then

$$g_1(n) = f_1(n) + h\partial(n).$$

Hence

$$\begin{array}{rccc} h: & A_0 & \longrightarrow & Kers' \\ & x & \longmapsto & h(x) = \mu(x) - e'F_0(x) \end{array}$$

is a homotopy connecting $f = (f_1, f_0) : (Kers \xrightarrow{\partial} A_0) \longrightarrow (Kers' \xrightarrow{\partial'} A_0')$ to $g = (g_1, g_0) : (Kers \xrightarrow{\partial} A_0) \longrightarrow (Kers' \xrightarrow{\partial'} A_0')$.

Let $F \xrightarrow{\mu} G$ and $G \xrightarrow{\mu'} H$. Then we have

$$\begin{array}{rcl} \Gamma(\mu * \mu')(x) & = & (\mu * \mu')(x) - e'(s'\mu * \mu')(x) \\ & = & \mu(x) + \mu'(x) - e'(t'\mu)(x) - e'(s'\mu)(x) \\ & = & \mu(x) + \mu'(x) - e'(s'\mu')(x) - e'(s'\mu)(x) \\ & = & (\mu(x) - e'(s'\mu)(x)) + (\mu'(x) - e'(s'\mu')(x)) \\ & = & \Gamma(\mu)(x) + \Gamma(\mu')(x) \end{array}$$

for all $x \in A_0$. □

Theorem 3.5. *Let* $\Psi : \mathbf{XMod_k} \longrightarrow \mathbf{2Alg}$ *be the functor as mentioned in Theorem 1.4 and h be homotopy connecting* $f : (G, C, \partial) \longrightarrow (G', C', \partial')$ *to* $g : (G, C, \partial) \longrightarrow (G', C', \partial')$*. Then*

$$\begin{array}{rcccl} \Psi(h) = \mu & : & C & \longrightarrow & G' \rtimes C' \\ & & x & \longmapsto & \mu(x) = (h(x), f_0(x)) \end{array}$$

is a homotopy of corresponding 2-algebra morphisms.

Proof. We first show that μ is an algebra morphism. For $x,x^{'} \in C$

$$\begin{aligned}\mu(xx^{'}) &= (h(xx^{'}), f_0(xx^{'}))\\ &= (f_0(x) \blacktriangleright h(x^{'}) + f_0(x^{'}) \blacktriangleright h(x) + h(x)h(x^{'}),\ f_0(x)f_0(x^{'}))\\ &= (h(x), f_0(x))(h(x^{'}), f_0(x^{'}))\\ &= \mu(x)\mu(x^{'}).\end{aligned}$$

Now we show that

1) $s^{'}\mu = F_0$ 2) $t^{'}\ \mu = G_0$ 3) $(f_1, f_0) \circ' \mu t = \mu s \circ' (g_1, g_0)$

1)For all $x \in C$,

$$\begin{aligned}s^{'}\mu(x) &= s^{'}(h(x), f_0(x))\\ &= f_0(x) = F_0(x),\end{aligned}$$

2)For all $x \in C$,

$$\begin{aligned}t^{'}\mu(x) &= t^{'}(h(x), f_0(x))\\ &= t^{'}(h(x)) + f_0(x)\\ &= \partial^{'}h(x) + f_0(x)\\ &= g_0(x) = G_0(x),\end{aligned}$$

3)For all $x \in C, a \in G$, since $t^{'}(f_1(a), f_0(x)) = \partial^{'}f_1(a) + f_0(x)$,

$$\begin{aligned}s^{'}(\mu t(a,x)) &= s^{'}(\mu(\partial(a)+x))\\ &= s^{'}(h(\partial(a)+x), f_0(\partial(a)+x))\\ &= f_0(\partial(a)+x)\\ &= f_0(\partial(a)) + f_0(x)\\ &= \partial^{'}f_1(a) + f_0(x)\end{aligned}$$

then $t^{'}(f_1(a), f_0(x)) = s^{'}(\mu t(a,x))$ and (f_1, f_0) , μt are composable pairs. Also since

$$\begin{aligned}t^{'}(\mu s(a,x)) &= t^{'}(\mu(x)) = t^{'}(h(x), f_0(x))\\ &= \partial^{'}(h(x)) + f_0(x)\\ &= g_0(x)\end{aligned}$$

and $s^{'}(g_1(a), g_0(x)) = g_0(x)$ then $t^{'}(\mu s) = s^{'}(g_1, g_0)$ and $\mu s, (g_1, g_0)$ are composable pairs.

Therefore we get

$$(f_1(a), f_0(x)) \circ^{'} \mu t(a,x) = (f_1(a) + h(\partial(a)+x), f_0(x))$$

and

$$\mu s(a,x) \circ^{'} (g_1(a), g_0(x)) = (f_1(a) + h(\partial(a)+x), f_0(x)).$$

Then $(f_1, f_0) \circ^{'} \mu t = \mu s \circ^{'} (g_1, g_0)$. So

$$\begin{aligned}\mu :\ C &\longrightarrow G^{'} \rtimes C^{'}\\ c &\longmapsto \mu(x) = (h(x), f_0(x))\end{aligned}$$

is a homotopy connecting $F = ((f_1, f_0), f_0)$ to $G = ((g_1, g_0), g_0)$.

Let $f \xrightarrow{h} g$ and $g \xrightarrow{h^{'}} u$. Then we have

$$\begin{aligned}\Psi(h+h^{'})(x) &= ((h+h^{'})(x), f_0(x))\\ &= (h(x) + h^{'}(x), f_0(x))\\ &= (h(x), f_0(x)) + (h^{'}(x), g_0(x)) - (0, g_0(x))\\ &= \Psi(h)(x) + \Psi(h^{'})(x) - e^{'}(t^{'}(\Psi)(h))(x)\\ &= (\Psi(h) * \Psi(h))(x).\end{aligned}$$

□

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Author's contributions

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