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On the Geometry of Tangent Bundle and Unit Tangent Bundle with Deformed-Sasaki Metric

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

Let (M^m, g) be a Riemannian manifold and TM its tangent bundle equipped with a deformed Sasaki metric. In this paper, firstly we investigate all forms of Riemannian curvature tensors of TM (Riemannian curvature tensor, Ricci curvature, sectional curvature and scalar curvature). Secondly, we study the geometry of unit tangent bundle equipped with a deformed Sasaki metric, where we presented the formulas of the Levi-Civita connection and also all formulas of the Riemannian curvature tensors of this metric.

Keywords: *Tangent bundle, deformed-Sasaki metric, unit tangent bundle, curvature tensor.*
AMS Subject Classification (2020): *Primary: 53C20, 53A45; Secondary: 58B20, 53B20.*

1. Introduction

On the tangent bundle of a Riemannian manifold one can define natural Riemannian metrics. Their construction make use of the Levi-Civita parallelization. Among them, the so called Sasaki metric [11] is of particular interest. That is why the geometry of tangent bundle equipped with the Sasaki metric has been studied by many authors such as Yano K, Ishihara S. [14], Dombrowski P. [6], Salimov A A, Gezer A, Akbulut K. [10] etc. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on tangent bundle. This is the reason why they have attempted to search for different metrics on the tangent bundle which are different deformations of the Sasaki metric. Musso E, Tricerri F. has introduced the notion of Cheeger-Gromoll metric [9], this metric has been studied also by many authors (see [7, 12]). In this direction, N. Boussekkine and A. Zagane [5] propose a deformed Sasaki metric on tangent bundle. The deformations of the Sasaki metric on the tangent bundle are not limited to those mentioned above. Also, we refer to [1, 3, 10, 15, 16, 17].

In previous work [5], we proposed a "On deformed-Sasaki metric and harmonicity in tangent bundles", and as a supplement to these works, in this paper, firstly we present all Riemannian curvature tensors of the tangent bundle with a deformed-Sasaki metric (Theorem 4.1), Ricci curvature (Proposition 4.2), sectional curvature (Theorem 4.2 and Proposition 4.3) and scalar curvature(Theorem 4.3 and Proposition 4.4). Secondly, we introduce the unit tangent bundle equipped with a deformed-Sasaki metric, we investigate the formulas relating to the Levi-Civita connection of this metric (Theorem 5.1) and we establish all formulas of the Riemannian curvature tensors (Theorem 5.2).

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and $\pi : TM \to M$ be the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=\overline{1,m}}$ on TM. Denote by

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Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im_0^1(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM. \tag{2.1}$$

of the tangent bundle to TM at any $(x,u) \in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial u^i}|_{(x,u)},\ \xi^i \in \mathbb{R}\}, \tag{2.2}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)},\ \xi^i \in \mathbb{R}\}. \tag{2.3}$$

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The map $\xi \to {}^V\xi = \xi^i \frac{\partial}{\partial u^i}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. Similarly, the map $\xi \to {}^H\xi = \xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Obviously, each tangent vector Z to TM at (x,u) can be written in the form $Z = {}^HX + {}^VY$, where X and Y are uniquely determined tangent vectors to M at x.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$ {}^VX = X^i \frac{\partial}{\partial u^i}, \tag{2.4}$$

$$ {}^HX = X^i(\frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}). \tag{2.5}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial u^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If U is a vector field such that $(U_x = u)$, $(x,u) \in TM$, the vertical lift ${}^VU$ is called the canonical vertical vector field or Liouville vector field on TM.

The bracket operation of vertical and horizontal vector fields is given by the formulas: [6, 14]

$$\begin{cases} [{}^HX, {}^HY] = {}^H[X,Y] - {}^V(R(X,Y)u), \\ [{}^HX, {}^VY] = {}^V(\nabla_X Y), \\ [{}^VX, {}^VY] = 0, \end{cases} \tag{2.6}$$

for any vector fields X and Y on M, where R is the Riemannian curvature of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

3. Deformed-Sasaki metric.

Definition 3.1. [5] Let (M^m, g) be a Riemannian manifold and $f : M \to [0, +\infty[$ be a positive smooth function on M. On the tangent bundle TM, we define a deformed-Sasaki metric noted g^f by

$$\begin{aligned} 1.\ g^f({}^HX, {}^HY)_{(x,u)} &= g_x(X,Y), \\ 2.\ g^f({}^HX, {}^VY)_{(x,u)} &= 0, \\ 3.\ g^f({}^VX, {}^VY)_{(x,u)} &= g_x(X,Y) + f(x) g_x(X,u) g_x(Y,u), \end{aligned}$$

for any vector fields X and Y on M, $(x,u) \in TM$. Here, f is called twisting function.

In the following, we consider $f \neq 0$, $\alpha = 1 + fr^2$ and $r^2 = g(u,u) = |u|^2$, where $|.|$ denotes the norm with respect to (M^m, g).

Lemma 3.1. *[1] Let (M^m, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function, we have:*

1. ${}^HX(\rho(r^2)) = 0,$

2. ${}^{V}X(\rho(r^2)) = 2\rho'(r^2)g(X,u)$,
3. ${}^{H}X(g(Y,u)) = g(\nabla_X Y, u)$,
4. ${}^{V}X(g(Y,u) = g(X,Y)$.

for any vector fields X and Y on M.

Lemma 3.2. *[5] Let (M^m, g) be a Riemannian manifold, we have the following*

$$\begin{array}{lcl} 1)\ {}^{H}Xg^f({}^{H}Y, {}^{H}Z) &=& Xg(Y,Z), \\ 2)\ {}^{V}Xg^f({}^{H}Y, {}^{H}Z) &=& 0, \\ 3)\ {}^{H}Xg^f({}^{V}Y, {}^{V}Z) &=& g^f((\nabla_X Y)^V, {}^{V}Z) + g^f({}^{V}Y, (\nabla_X Z)^V) + X(f)g(Y,u)g(Z,u), \\ 4)\ {}^{V}Xg^f({}^{H}Y, {}^{H}Z) &=& f\big(g(X,Y)g(Z,u) + g(Y,u)g(X,Z)\big), \end{array}$$

for any vector fields X, Y and Z on M.

Theorem 3.1. *[5] Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. If ∇ (resp ∇^f) denote the Levi-Civita connection of (M^m, g) (resp (TM, g^f)), then we have:*

$$\begin{array}{rcl} (i)\ (\nabla^f_{{}^{H}X}{}^{H}Y)_p &=& {}^{H}(\nabla_X Y)_p - \frac{1}{2}{}^{V}(R_x(X,Y)u), \\ (ii)\ (\nabla^f_{{}^{H}X}{}^{V}Y)_p &=& {}^{V}(\nabla_X Y)_p + \frac{1}{2\alpha}X_x(f)g_x(Y,u){}^{V}U_p + \frac{1}{2}{}^{H}(R_x(u,Y)X), \\ (iii)\ (\nabla^f_{{}^{V}X}{}^{H}Y)_p &=& \frac{1}{2\alpha}Y_x(f)g_x(X,u){}^{V}U_p + \frac{1}{2}{}^{H}(R_x(u,X)Y), \\ (iv)\ (\nabla^f_{{}^{V}X}{}^{V}Y)_p &=& \frac{-1}{2}g_x(X,u)g_x(Y,u){}^{H}(grad\, f)_p + \frac{f}{\alpha}g_x(X,Y){}^{V}U_p, \end{array}$$

for any vector fields X and Y on M and $p = (x,u) \in TM$, where R denotes the curvature tensor of (M^m, g).

If we denote the horizontal and vertical projections by $\mathcal{H}$ and $\mathcal{V}$, respectively, then we can state the followings:

(i) The vertical distribution VTM is totally geodesic in TTM if $\mathcal{H}\nabla^f_{{}^{V}X}{}^{V}Y = 0$,
(ii) The horizontal distribution HTM is totally geodesic in TTM if $\mathcal{V}\nabla^f_{{}^{H}X}{}^{H}Y = 0$.[3]

Hence, we can state the following result.

Proposition 3.1. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. Then*

(i) The vertical distribution VTM is totally geodesic in TTM if and only if f is a constant function,
(ii) The horizontal distribution HTM is totally geodesic in TTM if and only if (M^m, g) is flat.

Proof. The results come immediately from (i) and (iv) of Theorem 3.1. □

As a consequence of Theorem 3.1, we get the following Lemma.

Lemma 3.3. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then we have:*

$$\begin{array}{rcl} 1.\ (\nabla^f_{{}^{H}X}{}^{V}U) &=& \frac{\alpha-1}{2f\alpha}X(f){}^{V}U, \\ 2.\ (\nabla^f_{{}^{V}U}{}^{H}X) &=& \frac{\alpha-1}{2f\alpha}X(f){}^{V}U, \\ 3.\ (\nabla^f_{{}^{V}X}{}^{V}U) &=& {}^{V}X + \frac{1-\alpha}{2f}g(X,u){}^{H}(grad\, f) + \frac{f}{\alpha}g(X,u){}^{V}U, \\ 4.\ (\nabla^f_{{}^{V}U}{}^{V}X) &=& \frac{1-\alpha}{2f}g(X,u)(grad\, f)^H_p + \frac{f}{\alpha}g(X,u){}^{V}U, \\ 5.\ (\nabla^f_{{}^{V}U}{}^{V}U) &=& -\frac{(1-\alpha)^2}{2f^2}{}^{H}(grad\, f) + \frac{2\alpha-1}{\alpha}{}^{V}U, \end{array}$$

for any vector field X on M.

Definition 3.2. Let (M^m, g) be a Riemannian manifold and $K : TM \longrightarrow TM$ be a smooth bundle endomorphism of TM. Then the vertical and horizontal vector fields VK and HK are defined on TM by

$$\begin{array}{rcl} VK : TM & \to & TTM \\ (x,u) & \mapsto & {}^V(K(u)), \end{array}$$

$$\begin{array}{rcl} HK : TM & \to & TTM \\ (x,u) & \mapsto & {}^H(K(u)), \end{array}$$

locally we have

$$VK = y^i K_i^j \frac{\partial}{\partial y^j} = y^{i\,V}(K(\frac{\partial}{\partial x^i})), \tag{3.1}$$

$$HK = y^i K_i^j \frac{\partial}{\partial x^j} - y^i y^k K_i^j \Gamma_{jk}^s \frac{\partial}{\partial y^s} = y^{i\,H}(K(\frac{\partial}{\partial x^i})). \tag{3.2}$$

Proposition 3.2. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. If ∇ (resp ∇^f) denote the Levi-Civita connection of (M^m, g) (resp (TM, g^f)) and K is a tensor field of type $(1,1)$ on M, then:*

$$\begin{aligned} \nabla^f_{^HX} HK &= H(\nabla_X K) - \frac{1}{2}{}^V(R(X, K(u))u), \\ \nabla^f_{^HX} VK &= V(\nabla_X K) + \frac{1}{2\alpha}X(f)g(K(u),u)^V U + \frac{1}{2}{}^H(R(u, K(u))X), \\ \nabla^f_{^VX} HK &= (K(X))^H + \frac{1}{2\alpha}g(K(u), grad\, f)g(X,u)^V U + \frac{1}{2}{}^H(R(u,X)K(u)), \\ \nabla^f_{^VX} VK &= (K(X))^V - \frac{1}{2}g(X,u)g(K(u),u)^H(grad\, f) + \frac{f}{\alpha}g(X, K(u))^V U, \end{aligned}$$

for any vector field X on M.

Proof. The results come immediately from Theorem 3.1. ☐

4. Curvatures of deformed-Sasaki metric.

Theorem 4.1. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. If R (resp R^f) denote the Riemann curvature tensor of (M^m, g) (resp (TM, g^f)), then we have the following formulas*

$$\begin{aligned} R^f({}^HX, {}^HY)^HZ &= {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(u, R(X,Y)u)Z) + \frac{1}{4}{}^H(R(u, R(X,Z)u)Y) - \frac{1}{4}{}^H(R(u, R(Y,Z)u)X) \\ &\quad + \frac{1}{2}{}^V((\nabla_Z R)(X,Y)u). \end{aligned} \tag{4.1}$$

$$\begin{aligned} R^f({}^HX, {}^VY)^VZ &= -\frac{1}{2}{}^H(R(Y,Z)X) + \frac{\alpha-1}{4f\alpha}X(f)g(Y,u)g(Z,u)^H(grad\, f) - \frac{1}{2}g(Y,u)g(Z,u)^H(\nabla_X grad\, f) \\ &\quad - \frac{1}{4}{}^H(R(u,Y)R(u,Z)X) + \frac{1}{4}g(Y,u)g(Z,u)^V(R(X, grad\, f)u) - \frac{1}{2\alpha}X(f)g(Z,u)^V Y \\ &\quad - \frac{1}{4\alpha}g(Y,u)g(R(u,Z)X, grad\, f)^V U + \frac{1}{2\alpha^2}X(f)g^f({}^VY, {}^VZ)^V U. \end{aligned} \tag{4.2}$$

$$\begin{aligned} R^f({}^VX, {}^VY)^HZ &= \frac{1}{4}{}^H(R(u,X)R(u,Y)Z) - \frac{1}{4}{}^H(R(u,Y)R(u,X)Z) + {}^H(R(X,Y)Z) \\ &\quad + \frac{1}{2\alpha}Z(f)g(Y,u)^V X - \frac{1}{2\alpha}Z(f)g(X,u)^V Y + \frac{1}{4\alpha}g(X,u)g(R(u,Y)Z, grad\, f)^V U \\ &\quad - \frac{1}{4\alpha}g(Y,u)g(R(u,X)Z, grad\, f)^V U. \end{aligned} \tag{4.3}$$

$$R^f({}^HX, {}^VY)^HZ = \frac{1}{2}{}^H((\nabla_X R)(u,Y)Z) - \frac{1}{4}{}^V(R(X,R(u,Y)Z)u) + \frac{1}{2\alpha}g(Y,u)Hess_f(X,Z)^VU + \frac{f}{2\alpha}g(R(X,Z)u,Y)^VU + \frac{1-\alpha}{4f\alpha^2}X(f)Z(f)g(Y,u)^VU + \frac{1}{2}{}^V(R(X,Z)Y). \quad (4.4)$$

$$R^f({}^HX, {}^HY)^VZ = \frac{1}{2}{}^H((\nabla_X R)(u,Z)Y)) - \frac{1}{2}{}^H((\nabla_Y R)(u,Z)X) - \frac{1}{4}{}^V(R(X,R(u,Z)Y)u) + \frac{1}{4}{}^V(R(Y,R(u,Z)X)u) + {}^V(R(X,Y)Z) + \frac{f}{\alpha}g(R(X,Y)u,Z)^VU. \quad (4.5)$$

$$R^f({}^VX, {}^VY)^VZ = \frac{1}{4}g(X,u)g(Z,u)^H(R(u,Y)grad\, f) - \frac{1}{4}g(Y,u)g(Z,u)^H(R(u,X)grad\, f) + \frac{1}{2\alpha}\big(g(X,u)g(Y,Z) - g(Y,u)g(X,Z)\big)^H(grad\, f) + \frac{f}{\alpha}g(Y,Z)^VX - \frac{f}{\alpha}g(X,Z)^VY - \frac{f^2}{\alpha^2}\big(g(X,u)g(Y,Z) - g(Y,u)g(X,Z)\big)^VU. \quad (4.6)$$

for any vector fields X, Y and Z on M, where $Hess_f(X,Z) = g(\nabla_X gradf, Z)$.

Proof. In the proof, we will use the Definition 3.1, Lemma 3.2, Theorem 3.1 and Proposition 3.2
1) Let K be the bundle endomorphism given by $K : u \mapsto R(Y,Z)u$, from direct calculation we get:

$$\begin{aligned}\nabla^f_{{}^HX}\nabla^f_{{}^HY}{}^HZ &= \nabla^f_{{}^HX}{}^H(\nabla_Y Z) - \frac{1}{2}\nabla^f_{{}^HX}VK \\ &= {}^H(\nabla_X\nabla_Y Z) - \frac{1}{2}{}^V(R(X,\nabla_Y Z)u) - \frac{1}{4\alpha}{}^H(R(u,R(Y,Z)u)X) \\ &\quad -\frac{1}{2}{}^V(\nabla_X(R(Y,Z)u) - R(Y,Z)(\nabla_X U)),\end{aligned} \quad (4.7)$$

from which, with permutation of X by Y in the formula (4.7) we get

$$\nabla^f_{{}^HY}\nabla^f_{{}^HX}{}^HZ = {}^H(\nabla_Y\nabla_X Z) - \frac{1}{2}({}^VR(Y,\nabla_X Z)u) - \frac{1}{4\alpha}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{2}{}^V(\nabla_Y(R(X,Z)u) - R(X,Z)(\nabla_Y U)), \quad (4.8)$$

Also, we find

$$\begin{aligned}\nabla^f_{[{}^HX,{}^HY]}{}^HZ &= \nabla^f_{{}^H[X,Y]}{}^HZ - \nabla^f_{{}^V(R(X,Y)u)}{}^HZ \\ &= {}^H(\nabla_{[X,Y]}Z) - \frac{1}{2}{}^V(R([X,Y],Z)u) \\ &\quad -\frac{1}{2}{}^H(R(u,R(X,Y)u)Z).\end{aligned} \quad (4.9)$$

From (4.7), (4.8) and (4.9) we get

$$\begin{aligned}R^f({}^HX, {}^HY)^HZ &= {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(u,R(X,Y)u)Z) \\ &\quad +\frac{1}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X) \\ &\quad +\frac{1}{2}{}^V((\nabla_Y R)(X,Z)u) - \frac{1}{2}{}^V((\nabla_X R)(Y,Z)u),\end{aligned}$$

Using the second Bianchi identity,

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$$

we obtain the formula (4.1).

2) From direct calculation we get:

$$\begin{aligned}
\nabla^f_{{}^HX}\nabla^f_{{}^VY}{}^VZ &= \nabla^f_{{}^VY}\big(\frac{-1}{2}g(Y,u)g(Z,u)^H(grad\, f)+\frac{f}{\alpha}g(Y,Z)^VU\big)\\
&= -\frac{1}{2}\big(g(\nabla_XY,u)g(Z,u)+g(Y,u)g(\nabla_XZ,u)\big)^H(grad\, f)\\
&\quad -\frac{1}{2}g(Y,u)g(Z,u)\big({}^H(\nabla_X grad\, f)-\frac{1}{2}{}^H(R(X,grad\, f)u)\big)\\
&\quad +\frac{\alpha+1}{2\alpha^2}X(f)g(Y,Z)^VU+\frac{f}{\alpha}\big(g(\nabla_XY,Z)+g(Y,\nabla_XZ)\big)^VU.
\end{aligned}$$

Let $K:TM\to TM$ be the bundle endomorphism given by $K(u)=R(u,Z)X$, from direct calculation we get,

$$\begin{aligned}
\nabla^f_{{}^VY}\nabla^f_{{}^HX}{}^VZ &= \nabla^f_{{}^VY}\big({}^V(\nabla_XZ)+\frac{1}{2\alpha}X(f)g(Z,u)^VU+\frac{1}{2}HK\big)\\
&= \frac{-1}{2}g(Y,u)g(\nabla_XZ,u)^H(grad\, f)+\frac{f}{\alpha}g(Y,\nabla_XZ)^VU\\
&\quad +\big(\frac{-f}{2\alpha^2}X(f)g(Y,u)g(Z,u)+\frac{1}{2\alpha}X(f)g(Y,Z)\big)^VU\\
&\quad +\frac{1}{2\alpha}X(f)g(Z,u)^VY+\frac{1-\alpha}{4f\alpha}X(f)g(Y,u)g(Z,u)^H(grad\, f)\\
&\quad +\frac{1}{2}{}^H(R(Y,Z)X)+\frac{1}{4}{}^H(R(u,Y)R(u,Z)X)\\
&\quad +\frac{1}{4\alpha}g(R(u,Z)X,grad\, f)^VU.
\end{aligned}$$

Also,

$$\nabla^f_{[{}^HX,{}^VY]}{}^VZ = \frac{-1}{2}g(\nabla_XY,u)g(Z,u)^H(grad\, f)+\frac{f}{\alpha}g(\nabla_XY,Z)^VU,$$

which gives the formula (4.2).

3) Applying formula (4.2) and 1^{st} Bianchi identity.

$$R^f({}^VX,{}^VY)^HZ=R^f({}^HZ,{}^VY)^VX-R^f({}^HZ,{}^VX)^VY,$$

we get

$$\begin{aligned}
R^f({}^HZ,{}^VY)^VX &= -\frac{1}{2}{}^H(R(Y,X)Z)+\frac{\alpha-1}{4f\alpha}Z(f)g(Y,u)g(X,u)^H(grad\, f)-\frac{1}{2}g(Y,u)g(X,u)^H(\nabla_Z grad\, f)\\
&\quad -\frac{1}{4}{}^H(R(u,Y)R(u,X)Z)+\frac{1}{4}g(Y,u)g(X,u)^V(R(Z,grad\, f)u)-\frac{1}{2\alpha}Z(f)g(X,u)^VY\\
&\quad -\frac{1}{4\alpha}g(Y,u)g(R(u,X)Z,grad\, f)^VU+\frac{1}{2\alpha^2}Z(f)g^f({}^VY,{}^VX)^VU.
\end{aligned}$$

and

$$\begin{aligned}
R^f({}^HZ,{}^VX)^VY &= -\frac{1}{2}{}^H(R(X,Y)Z)+\frac{\alpha-1}{4f\alpha}Z(f)g(X,u)g(Y,u)^H(grad\, f)-\frac{1}{2}g(X,u)g(Y,u)^H(\nabla_Z grad\, f)\\
&\quad -\frac{1}{4}{}^H(R(u,X)R(u,Y)Z)+\frac{1}{4}g(X,u)g(Y,u)^V(R(Z,grad\, f)u)-\frac{1}{2\alpha}Z(f)g(Y,u)^VX\\
&\quad -\frac{1}{4\alpha}g(X,u)g(R(u,Y)Z,grad\, f)^VU+\frac{1}{2\alpha^2}Z(f)g^f({}^VX,{}^VY)^VU.
\end{aligned}$$

which gives the formula (4.3).

The other formulas are obtained by a similar calculation. We omit them to avoid repetition. □

Proposition 4.1. *Let (M^m,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with a deformed-Sasaki metric. If (TM,g^f) is flat, then (M^m,g) is flat.*

Proof. It is easy to see from (4.1), if we assume that $R^f = 0$ and calculate the Riemann curvature tensor for three horizontal vector fields at $(x, 0)$ we get

$$R^f_{(x,0)}({}^HX, {}^HY){}^HZ = {}^H(R_x(X,Y)Z) = 0.$$

□

Now let $(x,u) \in TM$ with $u \neq 0$ and $(E_i)_{i=\overline{1,m}}$ a local orthonormal frame on M at x, such that $E_m = \dfrac{u}{|u|}$, then,

$$\{{}^HE_i, {}^VE_j, \frac{1}{\sqrt{\alpha}}{}^VE_m\}_{i=\overline{1,m}, j=\overline{1,m-1}}. \tag{4.10}$$

is a local orthonormal frame on TM at (x,u).

Proposition 4.2. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. If Ric (resp., Ric^f) denote the Ricci curvature of (M^m, g) (resp. (TM, g^f)), then we have*

$$\begin{aligned} Ric^f({}^HX, {}^HY) &= Ric(X,Y) - \frac{1}{2}\sum_{a=1}^{m} g(R(E_a,X)u, R(E_a,Y)u) + \frac{1-\alpha}{2f\alpha}Hess_f(X,Y) \\ &\quad + \frac{(1-\alpha)^2}{4f^2\alpha^2}X(f)Y(f), \end{aligned} \tag{4.11}$$

$$Ric^f({}^HX, {}^VY) = \frac{1}{2}\sum_{a=1}^{m} g((\nabla_{E_a}R)(u,Y)X, E_a) - \frac{m-1}{2\alpha}X(f)g(Y,u) + \frac{\alpha-1}{4f\alpha}g(R(u,Y)X, grad f), \tag{4.12}$$

$$\begin{aligned} Ric^f({}^VX, {}^VY) &= \frac{1}{4}\sum_{a=1}^{m} g(R(u,X)E_a, R(u,Y)E_a) - \frac{1}{2}g(X,u)g(Y,u)\Delta(f) + \frac{\alpha-1}{4f\alpha}g(X,u)g(Y,u)|grad f|^2 \\ &\quad + \frac{f^2}{\alpha^2}g(X,u)g(Y,u) + \frac{f(m\alpha-2\alpha+1)}{\alpha^2}g(X,Y). \end{aligned} \tag{4.13}$$

for any vector fields X and Y on M, where $\Delta(f)$ is the Laplacian of f.

Proof. Using the local orthonormal frame 4.10 on TM.
$i)$ From the formula (4.1), we have

$$\begin{aligned} Ric^f({}^HX, {}^HY) &= \sum_{a=1}^{m} g^f(R^f({}^HE_a, {}^HX){}^HY, {}^HE_a) + \sum_{a=1}^{m-1} g^f(R^f({}^VE_a, {}^HX){}^HY, {}^VE_a) + \frac{1}{\alpha}g^f(R^f({}^VE_m, {}^HX){}^HY, {}^VE_m) \\ &= \sum_{a=1}^{m}\Big(g(R(E_a,X)Y,E_a) + \frac{1}{2}g(R(u,R(E_a,X)u)Y,E_a) + \frac{1}{4}g(R(u,R(E_a,Y)u)X,E_a)\Big) \\ &\quad + \frac{1}{4}\sum_{a=1}^{m-1} g^f({}^V(R(X,R(u,E_a)Y)u), {}^VE_a) - \frac{1}{2\alpha^2}Hess_f(X,Y)g(E_m,u)g^f({}^VU, {}^VE_m) \\ &\quad - \frac{1-\alpha}{4f\alpha^3}X(f)Y(f)g(E_m,u)g^f({}^VU, {}^VE_m) \\ &= Ric(X,Y) - \frac{3}{4}\sum_{a=1}^{m} g(R(E_a,X)u, R(E_a,Y)u) + \frac{1-\alpha}{2f\alpha}Hess_f(X,Y) + \frac{(1-\alpha)^2}{4f^2\alpha^2}X(f)Y(f) \\ &\quad + \frac{1}{4}\sum_{a=1}^{m} g(R(u,E_a)X, R(u,E_a)Y), \end{aligned}$$

and from,

$$\sum_{a=1}^{m} g(R(E_a,X)u, R(E_a,Y)u) = \sum_{a=1}^{m} g(R(u,E_a)X, R(u,E_a)Y).$$

we get

$$Ric^f({}^HX, {}^HY) = Ric(X,Y) - \frac{1}{2}\sum_{a=1}^{m} g(R(E_a,X)u, R(E_a,Y)u) + \frac{1-\alpha}{2f\alpha}Hess_f(X,Y) + \frac{(1-\alpha)^2}{4f^2\alpha^2}X(f)Y(f).$$

The other formulas are obtained by a similar calculation. □

Corollary 4.1. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then (TM, g^f) be an Einstein manifold, if and only if (M^m, g) is flat and $f = 0$.*

Proof. If we suppose that (TM, g^f) is λ-Einstein, then

$$Ric^f(\widetilde{X}, \widetilde{Y}) = \lambda g^f(\widetilde{X}, \widetilde{Y}).$$

Using 4.13, we obtain $f = \lambda = R = R^f = 0$. □

In the following, we consider the sectional curvature K^f on (TM, g^f) of the plane spanned by V and W is given by,

$$K^f(V,W) = \frac{g^f(R^f(V,W)W,V)}{Q^f(V,W)}, \tag{4.14}$$

where V, W are vector fields on TM and

$$Q^f(V,W) = g^f(V,V)g^f(W,W) - g^f(V,W)^2. \tag{4.15}$$

Let $K^f({}^HX, {}^HY)$, $K^f({}^HX, {}^VY)$ and $K^f({}^VX, {}^VY)$ denote the sectional curvature of the plane spanned by $\{{}^HX, {}^HY\}$, $\{{}^HX, {}^VY\}$ and $\{{}^VX, {}^VY\}$ on (TM, g^f) respectively, where X, Y are vector fields on M^m.

Lemma 4.1. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then we have*

$$\begin{aligned} i)\ Q^f({}^HX, {}^HY) &= Q(X,Y), \\ ii)\ Q^f({}^HX, {}^VY) &= |X|^2\big(|Y|^2 + fg(Y,u)^2\big), \\ iii)\ Q^f({}^VX, {}^VY) &= Q(X,Y) + f|X|^2 g(Y,u)^2 + f|Y|^2 g(X,u)^2 - 2fg(X,Y)g(X,u)g(Y,u). \end{aligned}$$

where $Q(X,Y) = |X|^2|Y|^2 - g(X,Y)^2$.

Proof. The result follows immediately from the formula (4.15). □

Lemma 4.2. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then we have*

$$\begin{aligned} g^f(R^f({}^HX, {}^HY){}^HY, {}^HX) &= g(R(X,Y)Y,X) - \frac{3}{4}|R(X,Y)u|^2, \\ g^f(R^f({}^HX, {}^VY){}^VY, {}^HX) &= \frac{1}{4}|R(u,Y)X|^2 - \frac{1}{2}g(\nabla_X grad\, f, X)g(Y,u)^2 + \frac{\alpha-1}{4f\alpha}X(f)^2 g(Y,u)^2, \\ g^f(R^f({}^VX, {}^VY){}^VY, {}^VX) &= \frac{f}{\alpha}Q(X,Y). \end{aligned}$$

Proof. Using (4.1), (4.2) and (4.6) we get immediately the result. □

Theorem 4.2. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then the sectional curvature K^f satisfy the following equations:*

$$\begin{aligned} K^f({}^HX, {}^HY) &= K(X,Y) - \frac{3}{4Q(X,Y)}|R(X,Y)u|^2, \\ K^f({}^HX, {}^VY) &= \frac{1}{|X|^2\big(|Y|^2 + fg(Y,u)^2\big)}\Big(\frac{1}{4}|R(u,Y)X|^2 - \frac{1}{2}g(\nabla_X grad\, f, X)g(Y,u)^2 + \frac{\alpha-1}{4f\alpha}X(f)^2 g(Y,u)^2\Big), \\ K^f({}^VX, {}^VY) &= \frac{f}{\alpha}\frac{Q(X,Y)}{Q^f({}^VX, {}^VY)}. \end{aligned}$$

where K denotes the sectional curvature of (M^m, g).

Proof. The division of $g^f(R^f({}^hX, {}^kY){}^kY, {}^hX))$ by $Q^f({}^hX, {}^kY)$ for $h, k \in \{H, V\}$ gives the result. □

Corollary 4.2. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then we have*

$$\begin{aligned} K^f({}^HX, {}^HY) &= K(X,Y) - \frac{3}{4}|R(X,Y)u|^2, \\ K^f({}^HX, {}^VY) &= \frac{1}{1+fg(Y,u)^2}\Big(\frac{1}{4}|R(u,Y)X|^2 - \frac{1}{2}g(\nabla_X grad\, f, X)g(Y,u)^2 + \frac{\alpha-1}{4f\alpha}X(f)^2 g(Y,u)^2\Big), \\ K^f({}^VX, {}^VY) &= \frac{f}{\alpha\big(1+f(g(X,u)^2+g(Y,u)^2)\big)}. \end{aligned}$$

where, X, Y are orthonormal vector fields on M^m.

Proposition 4.3. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature λ and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then we have*

$$\begin{aligned} K^f({}^HX, {}^HY) &= \lambda - \frac{3\lambda^2}{4}\big(g(X,u)^2 + g(Y,u)^2\big), \\ K^f({}^HX, {}^VY) &= \frac{1}{1+fg(Y,u)^2}\Big(\frac{\lambda^2}{4}g(X,u)^2 - \frac{1}{2}g(\nabla_X grad\, f, X)g(Y,u)^2 + \frac{\alpha-1}{4f\alpha}X(f)^2 g(Y,u)^2\Big), \\ K^f({}^VX, {}^VY) &= \frac{f}{\alpha\big(1+f(g(X,u)^2+g(Y,u)^2)\big)}. \end{aligned}$$

where, X, Y are orthonormal vector fields on M^m.

Proof. If M has constant curvature λ, then we have

$$R(X,Y)u = \lambda\big(g(Y,u)X - g(X,u)Y\big),$$

and a direct calculations

$$\begin{aligned} |R(X,Y)u|^2 &= \lambda^2\big(g(X,u)^2 + g(Y,u)^2\big), \\ |R(u,Y)X|^2 &= \lambda^2 g(X,u)^2, \end{aligned}$$

then the result. □

Theorem 4.3. *Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. If σ (resp., σ^f) denote the scalar curvature of (M^m, g) (resp., (TM, g^f)), then we have*

$$\sigma^f = \sigma - \frac{1}{4}\sum_{i,j=1}^{m}|R(E_i,E_j)u|^2 + \frac{1-\alpha}{f\alpha}\Delta(f) + \frac{(1-\alpha)^2}{2f^2\alpha^2}|grad\, f|^2 + \frac{f(m-1)(m\alpha-2\alpha+2)}{\alpha^2}.$$

where (E_i) is a local orthonormal frame of M and $\Delta(f)$ is the Laplacian of f.

Proof. Let $(x,u) \in TM$ with $u \neq 0$ and $(E_i)_{i=\overline{1,m}}$ a local orthonormal frame on M at x, such that $E_m = \dfrac{u}{|u|}$, then, $\{{}^HE_i, {}^VE_j, \frac{1}{\sqrt{\alpha}}{}^VE_m\}_{i=\overline{1,m}, j=\overline{1,m-1}}$ is a local orthonormal frame on TM at (x,u). The scalar curvature is written as

$$\sigma^f = \sum_{b=1}^{m} Ric^f({}^HE_b, {}^HE_b) + \sum_{b=1}^{m-1} Ric^f({}^VE_b, {}^VE_b) + \frac{1}{\alpha}Ric^f({}^VE_m, {}^VE_m) \tag{4.16}$$

Using (4.11), we have

$$\begin{aligned} \sum_{b=1}^{m} Ric^f({}^HE_b, {}^HE_b) &= \sum_{b=1}^{m}\Big(Ric(E_b,E_b) - \frac{1}{2}\sum_{a=1}^{m} g(R(E_a,E_b)u, R(E_a,E_b)u) + \frac{1-\alpha}{2f\alpha}Hess_f(E_b,E_b) \\ &\quad + \frac{(1-\alpha)^2}{4f^2\alpha^2}E_b(f)^2\Big) \\ &= \sigma - \frac{1}{2}\sum_{a,b=1}^{m}|R(E_a,E_b)u|^2 + \frac{1-\alpha}{2f\alpha}\Delta(f) + \frac{(1-\alpha)^2}{4f^2\alpha^2}|grad f|^2. \end{aligned} \tag{4.17}$$

Using (4.13), we have

$$\begin{aligned}\sum_{b=1}^{m-1} Ric^f({}^V E_b, {}^V E_b) &= \sum_{b=1}^{m}\Big(\frac{1}{4}\sum_{a=1}^{m} g(R(u,E_b)E_a, R(u,E_b)E_a) + \frac{f(m\alpha-2\alpha+1)}{\alpha^2} g(E_b,E_b)\Big)\\ &= \frac{1}{4}\sum_{a,b=1}^{m} |R(u,E_b)E_a|^2 + \frac{f(m\alpha-2\alpha+1)(m-1)}{\alpha^2}, \qquad (4.18)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\alpha} Ric^f({}^V E_m, {}^V E_m) &= \frac{1}{\alpha}\Big(\frac{1}{4}\sum_{a=1}^{m} g(R(u,E_m)E_a, R(u,E_m)E_a) - \frac{1}{2} g(E_m,u)^2 \Delta(f)\\ &\quad + \frac{\alpha-1}{4f\alpha} g(E_m,u)^2 |grad f|^2 + \frac{f^2}{\alpha^2} g(E_m,u)^2 + \frac{f(m\alpha-2\alpha+1)}{\alpha^2} g(E_m,E_m)\Big)\\ &= \frac{1-\alpha}{2f\alpha}\Delta(f) + \frac{(1-\alpha)^2}{4f^2\alpha^2}|grad f|^2 + \frac{f(m-1)}{\alpha^2}. \qquad (4.19)\end{aligned}$$

In order to simplify the calculation, we use

$$\sum_{i,j=1}^{m} |R(u,E_j)E_i|^2 = \sum_{i,j=1}^{m} |R(E_i,E_j)u|^2. \qquad (4.20)$$

Substitution (4.17), (4.18), (4.19) and (4.20) into (4.16), we find

$$\sigma^f = \sigma - \frac{1}{4}\sum_{i,j=1}^{m} |R(E_i,E_j)u|^2 + \frac{1-\alpha}{f\alpha}\Delta(f) + \frac{(1-\alpha)^2}{2f^2\alpha^2}|grad\, f|^2 + \frac{f(m-1)(m\alpha-2\alpha+2)}{\alpha^2}.$$

□

Proposition 4.4. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature λ and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric, then we have*

$$\sigma^f = (m-1)\lambda\big(m - \frac{(\alpha-1)}{2f}\lambda\big) + \frac{f(m-1)(m\alpha-2\alpha+2)}{\alpha^2} + \frac{1-\alpha}{f\alpha}\Delta(f) + \frac{(1-\alpha)^2}{2f^2\alpha^2}|grad\, f|^2.$$

Proof. Using the formulas of curvature and scalar curvature of a Riemannian manifold with constant sectional curvature λ we have,

$$\sigma = m(m-1)\lambda,$$

and

$$\sum_{i,j=1}^{m} |R(E_i,E_j)u|^2 = 2(m-1)\lambda^2|u|^2 = 2\lambda^2(m-1)\frac{(\alpha-1)}{f},$$

From Theorem (4.3), we deduce the result. □

5. Deformed-Sasaki metric on unit tangent bundle T_1M

The tangent sphere bundle of radius $r > 0$ over a Riemannian manifold (M^m, g), is the hypersurface

$$T_rM = \big\{(x,u) \in TM,\ g(u,u) = r^2\big\}.$$

When $r = 1$, T_1M is called the unit tangent (sphere) bundle.

$$T_1M = \big\{(x,u) \in TM,\ g(u,u) = 1\big\}. \qquad (5.1)$$

If we set

$$\begin{array}{rcl} F : TM & \to & \mathbb{R} \\ (x,u) & \mapsto & F(x,u) = g(u,u) - 1, \end{array}$$

then the hypersurface T_1M is given by

$$T_1M = \{(x,u) \in TM, \quad F(x,u) = 0\},$$

and $\widetilde{grad}F$ (the gradient of F with respect to g^f) is a normal vector field to T_1M. From the Lemma 3.1, for any vector fields X on M, we get

$$g^f({}^HX, \widetilde{grad}F) = {}^HX(F) = {}^HX(g(u,u)-1) = 0,$$
$$g^f({}^VX, \widetilde{grad}F) = {}^VX(F) = {}^VX(g(u,u)-1) = 2g(X,u) = \frac{2}{1+f} g^f({}^VX, {}^VU).$$

So

$$\widetilde{grad}F = \frac{2}{1+f} {}^VU.$$

Then the unit normal vector field to T_1M is given by

$$\mathcal{N} = \frac{\widetilde{grad}F}{\sqrt{g^f(\widetilde{grad}F, \widetilde{grad}F)}} = \frac{{}^VU}{\sqrt{g^f({}^VU, {}^VU)}} = \frac{1}{\sqrt{1+f}} {}^VU.$$

The tangential lift ${}^TX$ with respect to g^f of a vector $X \in T_xM$ to $(x,u) \in T_1M$ as the tangential projection of the vertical lift of X to (x,u) with respect to $\mathcal{N}$, that is

$${}^TX = {}^VX - g^f_{(x,u)}({}^VX, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} = {}^VX - g_x(X,u){}^VU_{(x,u)}.$$

For the sake of notational clarity, we will use $\bar{X} = X - g(X,u)U$, then ${}^TX = {}^V\bar{X}$.
From the above, we get the direct sum decomposition

$$T_{(x,u)}TM = T_{(x,u)}T_1M \oplus span\{\mathcal{N}_{(x,u)}\} = T_{(x,u)}T_1M \oplus span\{{}^VU_{(x,u)}\}, \tag{5.2}$$

where $(x,u) \in T_1M$.
Indeed, if $W \in T_{(x,u)}TM$, then they exist $X, Y \in T_xM$, such that

$$\begin{aligned} W &= {}^HX + {}^VY \\ &= {}^HX + {}^TY + g^f_{(x,u)}({}^VY, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} \\ &= {}^HX + {}^TY + g_x(Y,u){}^VU_{(x,u)}. \end{aligned} \tag{5.3}$$

From the (5.3) we can say that the tangent space $T_{(x,u)}T_1M$ of T_1M at (x,u) is given by

$$T_{(x,u)}T_1M = \{{}^HX + {}^TY \,/\, X, Y \in T_xM, Y \in u^\perp\}.$$

where $u^\perp = \{Y \in T_xM,\ g(Y,u) = 0\}$. Hence $T_{(x,u)}T_1M$ is spanned by vectors of the form ${}^HX$ and ${}^TY$.

Given a vector field X on M, the tangential lift ${}^TX$ of X is given by

$${}^TX_{(x,u)} = \left({}^VX - g^f({}^VX, \mathcal{N})\mathcal{N}\right)_{(x,u)} = {}^VX_{(x,u)} - g_x(X_x,u){}^VU_{(x,u)}. \tag{5.4}$$

Remark 5.1. For any vector field X on M, we have the followings

(1) $g^f({}^HX, \mathcal{N}) = 0$,

(2) $g^f({}^TX, \mathcal{N}) = 0$,

(3) ${}^TX = {}^VX$ if and only if $g(X,u) = 0$,

(4) ${}^{T}U = 0$,

(5) $g(\bar{X}, u) = 0$.

Definition 5.1. Let (M^m, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with a deformed-Sasaki metric. The Riemannian metric $\hat{g}^f$ on T_1M, induced by g^f, is completely determined by the identities

$$\begin{aligned}
\hat{g}^f({}^{H}X, {}^{H}Y) &= g(X, Y), \\
\hat{g}^f({}^{T}X, {}^{H}Y) &= \hat{g}^f({}^{H}X, {}^{T}Y) = 0, \\
\hat{g}^f({}^{T}X, {}^{T}Y) &= g(X, Y) - g(X, u)g(Y, u),
\end{aligned}$$

Note that the deformed-Sasaki metric induced on T_1M coincides with the Sasaki metric induced on T_1M[8].

We shall calculate the Levi-Civita connection $\widehat{\nabla}$ of T_1M with deformed-Sasaki metric $\hat{g}^f$.(Those may be calculated using a similar method as in [2, 4]). This connection is characterized by the formula:

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \nabla^f_{\widehat{X}}\widehat{Y} - g^f(\nabla^f_{\widehat{X}}\widehat{Y}, \mathcal{N})\mathcal{N}. \tag{5.5}$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on T_1M.

Theorem 5.1. *[4, 8] Let (M^m, g) be a Riemannian manifold and $(T_1M, \hat{g}^f)$ its unit tangent bundle equipped with a deformed-Sasaki metric, then we have the following formulas.*

$$\begin{aligned}
1.\ \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{T}(R(X, Y)u), \\
2.\ \widehat{\nabla}_{{}^{H}X}{}^{T}Y &= {}^{T}(\nabla_X Y) + \frac{1}{2}{}^{H}(R(u, Y)X), \\
3.\ \widehat{\nabla}_{{}^{T}X}{}^{H}Y &= \frac{1}{2}{}^{H}(R(u, X)Y), \\
4.\ \widehat{\nabla}_{{}^{T}X}{}^{T}Y &= -g(Y, u){}^{T}X,
\end{aligned}$$

for all vector fields X, Y on M, where ∇ is the Levi-Civita connection and R is its curvature tensor of (M^{2m}, φ, g).

Proof. In the proof, we will use the Theorem 3.1, Lemma 3.3 and the formula (5.5).
1. By direct calculation, we have

$$\begin{aligned}
\widehat{\nabla}_{{}^{H}X}{}^{H}Y &= \nabla^f_{{}^{H}X}{}^{H}Y - g^f(\nabla^f_{{}^{H}X}{}^{H}Y, \mathcal{N})\mathcal{N} \\
&= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{V}(R(X, Y)u) - g^f(-\frac{1}{2}{}^{V}(R(X, Y)u), \mathcal{N})\mathcal{N} \\
&= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{T}(R(X, Y)u).
\end{aligned}$$

2. We have $\widehat{\nabla}_{{}^{H}X}{}^{T}Y = \nabla^f_{{}^{H}X}{}^{T}Y - g^f(\nabla^f_{{}^{H}X}{}^{T}Y, \mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\nabla^f_{{}^{H}X}{}^{T}Y = {}^{T}(\nabla_X Y) + \frac{1}{2}{}^{H}(R(u, Y)X) \ \ \textit{and} \ \ g^f(\nabla^f_{{}^{H}X}{}^{T}Y, \mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^{H}X}{}^{T}Y = {}^{T}(\nabla_X Y) + \frac{1}{2}{}^{H}(R(u, Y)X).$$

3. Also, we have $\widehat{\nabla}_{{}^{T}X}{}^{H}Y = \nabla^f_{{}^{T}X}{}^{H}Y - g^f(\nabla^f_{{}^{T}X}{}^{H}Y, \mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\nabla^f_{{}^{T}X}{}^{H}Y = \frac{1}{2}{}^{H}(R(u, Y)X) \ \ \textit{and} \ \ g^f(\nabla^f_{{}^{H}X}{}^{T}Y, \mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^{H}X}{}^{T}Y = \frac{1}{2}{}^{H}(R(u, Y)X).$$

4. In the same way above, we have $\widehat{\nabla}_{{}^TX}{}^TY = \nabla^f_{{}^TX}{}^TY - g^f(\nabla^f_{{}^TX}{}^TY, \mathcal{N})\mathcal{N}$,

$$\nabla^f_{{}^TX}{}^TY = -g(Y,u)^VX - \frac{1}{1+f}g(X,Y)^VU + \frac{2+f}{1+f}g(X,u)g(Y,u)^VU.$$

and

$$g^f(\nabla^f_{{}^TX}{}^TY, \mathcal{N})\mathcal{N} = -g(X,u)g(Y,u)^VU - \frac{1}{1+f}g(X,Y)^VU + \frac{2+f}{1+f}g(X,u)g(Y,u)^VU.$$

Hence

$$\widehat{\nabla}_{{}^TX}{}^TY = -g(Y,u)^TX.$$

□

Now, we shall calculate the Riemannian curvature tensor of T_1M with the deformed Sasaki metric $\hat{g}^f$. Note that the curvature of T_1M with Sasaki metric has already been considered by A. L.Yampol'skii, who calculates its curvature components in [13].

Denoting by $\widehat{R}^f$ the Riemannian curvature tensors of $(T_1M, \hat{g}^f)$, from the Gauss equation for hypersurfaces we deduce that $\widehat{R}^f(\widehat{X},\widehat{Y})\widehat{Z}$ satisfies

$$\widehat{R}^f(\widehat{X},\widehat{Y})\widehat{Z} = {}^t(R^f(\widehat{X},\widehat{Y})\widehat{Z}) - B(\widehat{X},\widehat{Z}).A_{\mathcal{N}}\widehat{Y} + B(\widehat{Y},\widehat{Z}).A_{\mathcal{N}}\widehat{X}, \tag{5.6}$$

for all $\widehat{X},\widehat{Y}$ and $\widehat{Z}$ vector fields on T_1M. where ${}^t(R^f(\widehat{X},\widehat{Y})\widehat{Z})$ is the tangential component of $R^f(\widehat{X},\widehat{Y})\widehat{Z}$ with respect to the direct sum decomposition (5.2), $A_{\mathcal{N}}$ is the shape operator of T_1M in (TM, g^f) derived from $\mathcal{N}$, and B is the second fundamental form of T_1M (as a hypersurface immersed in TM), associated to $\mathcal{N}$ on T_1M.

$A_{\mathcal{N}}\widehat{X}$ is the tangential component of $(-\nabla^f_{\widehat{X}}\mathcal{N})$ i.e.

$$A_{\mathcal{N}}\widehat{X} = -{}^t(\nabla^f_{\widehat{X}}\mathcal{N}), \tag{5.7}$$

$B(\widehat{X},\widehat{Y})$ is given by Gauss's formula, $\nabla^f_{\widehat{X}}\widehat{Y} = \widehat{\nabla}_{\widehat{X}}\widehat{Y} + B(\widehat{X},\widehat{Y}).\mathcal{N}$, so

$$B(\widehat{X},\widehat{Y}) = g^f(\nabla^f_{\widehat{X}}\widehat{Y}, \mathcal{N}). \tag{5.8}$$

Theorem 5.2. *Let (M^m, g) be a Riemannian manifold and $(T_1M, \hat{g}^f)$ its unit tangent bundle equipped with a deformed-Sasaki metric, then we have the following formulas.*

$$\begin{aligned}
\widehat{R}^f({}^HX,{}^HY)^HZ &= {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(u,R(X,Y)u)Z) + \frac{1}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X) \\
&\quad + \frac{1}{2}{}^T((\nabla_Z R)(X,Y)u), \\
\widehat{R}^f({}^HX,{}^TY)^TZ &= -\frac{1}{4}{}^H(R(u,Y)R(u,Z)X) - \frac{1}{2}{}^H(R(\bar{Y},\bar{Z})X), \\
\widehat{R}^f({}^TX,{}^TY)^HZ &= \frac{1}{4}{}^H(R(u,X)R(u,Y)Z) - \frac{1}{4}{}^H(R(u,Y)R(u,X)Z) + {}^H(R(\bar{X},\bar{Y})Z), \\
\widehat{R}^f({}^HX,{}^TY)^HZ &= \frac{1}{2}{}^H((\nabla_X R)(u,Y)Z) - \frac{1}{4}{}^T(R(X,R(u,Y)Z)u) + \frac{1}{2}{}^T(R(X,Z)\bar{Y}), \\
\widehat{R}^f({}^HX,{}^HY)^TZ &= \frac{1}{2}{}^H((\nabla_X R)(u,Z)Y)) - \frac{1}{2}{}^H((\nabla_Y R)(u,Z)X) \\
&\quad - \frac{1}{4}{}^T(R(X,R(u,Z)Y)u) + \frac{1}{4}{}^T(R(Y,R(u,Z)X)u) + {}^T(R(X,Y)\bar{Z}), \\
\widehat{R}^f({}^TX,{}^TY)^TZ &= g(\bar{Y},\bar{Z})^TX - g(\bar{X},\bar{Z})^TY.
\end{aligned}$$

for all vector fields X, Y and Z on M, where $\bar{X} = X - g(X,u)U$, see [4, 8].

Proof. Using Theorem 3.1, Lemma 3.3, (5.7) and (5.8), we obtain

$$A_{\mathcal{N}}{}^{H}X = 0\ ,\quad A_{\mathcal{N}}{}^{T}X = \frac{-1}{\sqrt{1+f}}{}^{T}X, \tag{5.9}$$

$$B({}^{H}X, {}^{H}Y) = B({}^{H}X, {}^{T}Y) = B({}^{T}X, {}^{H}Y) = 0, \tag{5.10}$$

and

$$B({}^{T}X, {}^{T}Y) = \frac{-1}{\sqrt{1+f}}\big(g(X,Y) - g(X,u)g(Y,u)\big) = \frac{-1}{\sqrt{1+f}}g(\bar{X},\bar{Y}). \tag{5.11}$$

It now suffices to using Theorem 4.1 and (5.6)-(5.11), we obtain the required formulae for the curvature tensor (see [2]). □

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Author's contributions

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